A Two-level ADMM Algorithm for AC OPF with Convergence Guarantees

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Abstract—This paper proposes a two-level distributed algorithmic framework for solving the AC optimal power flow (OPF) problem with convergence guarantees. The presence of highly nonconvex constraints in OPF poses significant challenges to distributed algorithms based on the alternating direction method of multipliers (ADMM). In practice, such algorithms heavily rely on warm start or parameter tuning to display convergence behavior; in theory, convergence is not guaranteed for nonconvex network optimization problems like AC OPF. In order to overcome such difficulties, we propose a new distributed reformulation for AC OPF and a two-level ADMM algorithm that goes beyond the standard framework of ADMM. We establish the global convergence and iteration complexity of the proposed algorithm under mild assumptions. Extensive numerical experiments over some largest NESTA test cases (9000- and 13000-bus systems) demonstrate advantages of the proposed algorithm over existing ADMM variants in terms of convergence, scalability, and robustness. Moreover, under appropriate parallel implementation, the proposed algorithm exhibits fast convergence comparable to or even better than the state-of-the-art centralized solver.

Index Terms—Distributed optimization, optimal power flow, augmented Lagrangian method, alternating direction method of multipliers.

I. INTRODUCTION

The AC optimal power flow (OPF) is a basic building block in electric power grid operation and planning. It is a highly nonconvex optimization problem, due to nonlinear power flow equations, and is shown to be an NP-hard decision problem [1], [2]. Any computational method to be deployed in power system practice should meet the stringent requirement that the algorithm has guaranteed robust performance and requires minimal tuning and intervention in face of variations in system conditions. Moreover, to effectively coordinate multiple regions in a large power grid, distributed algorithms that do not require sharing critical private information between regions should be highly desirable. However, such a goal has remained challenging for solving large-scale AC-OPF problems. Many existing algorithms are only suited for centralized operation. Most distributed or decentralized algorithms for solving AC-OPF do not have guaranteed convergence performance and require extensive parameter tuning and experimentation.

In this paper, we develop a distributed algorithm for solving large-scale AC OPF problems, and the proposed algorithm is proven to have global convergence. This is the first time that guaranteed convergence performance under mild and realistic conditions is achieved and iteration complexity bounds are obtained for a distributed computation framework for solving AC OPF.

A. Literature Review

The research community has extensively studied local nonlinear optimization methods such as the interior point methods e.g. [3], [4]. Another line of research looks into convex relaxations of AC OPF and has drawn significant attentions in recent years. In particular, the semidefinite programming (SDP) relaxation is firstly applied to the OPF problem in [5], and sufficient conditions are studied to guarantee the exactness of SDP relaxations [6]. However, SDP suffers from expensive computation cost for large-scale problems, while the second-order cone programming (SOCP) relaxation initially proposed in [7] offers a favorable alternative. The strong SOCP relaxation proposed in [8] is shown to be close to or dominates the SDP relaxation, while the computation time of SOCP can be orders of magnitude faster than SDP. However, obtaining a primal feasible solution is a common challenge facing these convex relaxation methods.

The alternating direction method of multipliers (ADMM) offers a powerful framework for distributed computation. Sun et al. [9] apply ADMM to decompose the computation down to each individual bus, and observe the convergence is sensitive to initial conditions. Erseghe [10], [11] studies the case where the network is divided into overlapping subregions, and voltage information of shared buses are duplicated by their connected subregions. In [10], ADMM is directly applied to the underlying distributed reformulation, and convergence is established by assuming nonconvex OPF and ADMM subproblems have zero duality gaps. In [11], standard ALM techniques are adopted inside ADMM. Subsequential convergence is proved under the assumption that the penalty parameter stays finite, which is basically assuming the algorithm converges to a feasible solution. The assumption is quite strong as quadratic penalty in general does not admit an exact penalization for nonconvex problems [12]. A more recent work [13] applies ADMM to a component-based distributed reformulation of AC OPF. The ADMM penalty is adaptively changed in every iteration, and the resulting algorithm numerically converges for various networks under adaptive hyperparameter tuning. In summary, existing ADMM-based algorithms either directly apply ADMM or its variant as a heuristic, or rely on strong assumptions to establish asymptotic convergence. See [14] for a recent survey on distributed optimization techniques for OPF.
B. Contribution

In this paper, we address the convergence issues of ADMM by proposing a two-level distributed algorithmic framework. The proposed framework is motivated by our observation that some crucial structure necessary for the convergence of nonconvex ADMM is absent in traditional distributed reformulations of AC OPF, and we overcome such technical difficulty by embedding a three-block ADMM inside the classic ALM framework. We present the global convergence and iteration complexity results of the proposed framework, which rely on mild and realistic assumptions. We demonstrate the convergence, scalability, and robustness of the proposed algorithm over some largest NESTA test cases [15], on which existing ADMM variants may fail to converge. Generically, distributed algorithms can be slow due to limited access to global information and communication delay; however, we show that, with proper parallel implementation, the proposed algorithm achieves fast convergence close to or even better than centralized solver.

C. Notation and Organization

Throughout this paper, we use $\mathbb{R}^n$ to denote the n-dimensional real Euclidean space; the inner product of $x, y \in \mathbb{R}^n$ is denoted by $\langle x, y \rangle$; the Euclidean norm is denoted by $\|x\|$, and the $\ell^\infty$ norm is denoted by $\|x\|_\infty$. When $x$ consists of $p$ subvectors, we write $x = (x_1, \cdots, x_p)$. We use $\mathbb{Z}_{++}$ to denote the set of positive integers, and $[n] = \{1, \cdots, n\}$. For a closed set $C \subset \mathbb{R}^n$, the projection onto $C$ is denoted by $\text{Proj}_C(x)$, and the indicator function of $C$ is denoted by $\mathbb{I}_C(x)$, which takes value 0 if $x \in C$ and $+\infty$ otherwise.

The rest of this paper is organized as follows. In section II we review the AC OPF problem and nonconvex ADMM literature. Then in section III we present a new distributed reformulation and the proposed two-level algorithm. In section IV we state the main convergence results of the proposed two-level algorithm, and discuss related convergence issues. Finally, we present computational experiments in section V and conclude this paper in section VI.

II. BACKGROUND

A. AC OPF Formulation

Consider a power network $G(\mathcal{N}, \mathcal{E})$ where $\mathcal{N}$ denotes the set of buses and $\mathcal{E}$ denotes the set of transmission lines. Let $\delta(i)$ be the set of neighbours of $i \in \mathcal{N}$. Let $Y = G + jB$ denote the complex nodal admittance matrix, where $j = \sqrt{-1}$ and $G, B \in \mathbb{R}^{\mathcal{N} \times \mathcal{N}}$. Let $p_i^e, q_i^e$ (resp. $p_i^q, q_i^q$) be the real and reactive power produced by generator(s) (resp. loads) at bus $i$; if there is no generator (resp. load) attached to bus $i$, then $p_i^e, q_i^e$ (resp. $p_i^q, q_i^q$) are set to 0. The complex voltage $v_i$ at bus $i$ can be expressed by its real and imaginary parts as $v_i = e_i + jf_i$. The rectangular formulation of AC OPF is given as

$$\min_{x \in \mathbb{R}^n} \sum_{i \in \mathcal{N}} f_i(p_i^q)$$

s.t. $p_i^e - p_i^q = G_{ii}(e_i^2 + f_i^2) + \sum_{j \in \delta(i)} G_{ij}(e_i e_j + f_i f_j) - B_{ij}(e_i f_j - e_j f_i) \quad \forall i \in \mathcal{N}$

$$q_i^e - q_i^q = -B_{ii}(e_i^2 + f_i^2) + \sum_{j \in \delta(i)} B_{ij}(e_i e_j + f_i f_j) - G_{ij}(e_i f_j - e_j f_i) \quad \forall i \in \mathcal{N}$$

where

$$p_{ij} = -G_{ij}(e_i^2 + f_i^2 - e_i e_j - f_i f_j) - B_{ij}(e_i f_j - e_j f_i),$$

$$q_{ij} = B_{ij}(e_i^2 + f_i^2 - e_i e_j - f_i f_j) - G_{ij}(e_i f_j - e_j f_i).$$

In [12], the objective $f_i(p_i^q)$ represents the real power generation cost at bus $i$. Constraints (1b) and (1c) correspond to real and reactive power injection balance at bus $i$. The real and reactive power flow $p_{ij}, q_{ij}$ on line $(i, j)$ are given in (2), and (1d) restricts the apparent power flow on each transmission line. Constraints (1e)–(1f) limit voltage magnitude, real power output, and reactive power output at each bus to its physical capacity. Since the objective is typically linear or quadratic in real generation, formulation (1) is a nonconvex quadratically constrained quadratic program (QCQP) problem.

B. Nonconvex ADMM

ADMM was proposed in 1970s [16], [17] and regarded as a close variant of the augmented Lagrangian method (ALM) [18], [19]. The standard ADMM framework consists of a dual update using current primal residuals. The update of each individual block can be decomposed and carried out in parallel given certain separable structures are available. More recently, researchers realized that the ADMM framework can be used to solve more complicated nonconvex multi-block problems in the form

$$\min_{x=(x_1, \cdots, x_p)} \sum_{i=1}^{p} f_i(x_i) + g(x)$$

s.t. $\sum_{i=1}^{p} A_i x_i = b, \quad x_i \in \mathcal{X}_i \forall i \in [p],$

where the primal variable $x$ is partitioned into $p$ subvectors $x_i \in \mathbb{R}^{n_i}$ for $i \in [p]$, and $f_i$'s, $\mathcal{X}_i$'s, and $g$ can be potentially nonconvex. Different assumptions on the problem data are proposed to ensure global convergence to stationary solutions and in general an iteration complexity of $O(1/\epsilon^2)$ is expected [20–22]. Though motivated by different applications and adopting different analysis techniques, all these convergence results on nonconvex ADMM rely on the following three assumptions:

(a) All nonconvex subproblems need to be solved to global optimality;

(b) The functions $f_p$ and $g$ (if exist) are Lipschitz continuous differentiable, and $\mathcal{X}_p = \mathbb{R}^{n_p}$;

(c) The image space of $A_p$ is sufficiently large, i.e., $\text{Im}((A_1, \cdots, A_{p-1}, b)) \subseteq \text{Im}(A_p)$.

These three assumptions together restrict the application of ADMM on the OPF problem. No matter what reformulation of (1) is used, ADMM subproblems would still have highly nonconvex constraints, so Assumption (a) is unrealistic for
the OPF problem. Fortunately, this assumption can be relaxed to
finding a stationary solution with improved augmented
Lagrangian function value, which usually can be achieved at
the successful termination of some nonlinear solvers. It turns
out that Assumptions (b) and (c) cannot be satisfied simultane-
ously if ADMM were to achieve parallel computation among
different agents. Such limitation motivate us to go beyond
the framework of ADMM.

III. A NEW DISTRIBUTED REFORMULATION AND A
TWO-LEVEL ADMM ALGORITHM

A. A New Distributed Reformulation

Suppose the network $G$ is partitioned into $R$ subregions
$R_1, \cdots, R_R \subseteq \mathcal{N}$, each of which is connected to a local
central or operating agent. We say $(i, j) \in \mathcal{E}$ is a tie-line if $i \in R_r, j \in R_t, \text{and } r \neq i$. Agent $r$ controls variables $x_{ij} = (p_{e_i}^r, q_{e_i}^r, e_i, f_i)$
for all $i \in R_r$. We extend the notation $\delta(R_r)$ to denote the set
of all buses connected to (but not in) $R_r$ by some tie-lines.

The constraints of OPF can be satisfied simultaneously for
each agent, for example, suppose $(i,j)$ is a tie-line where $i \in R_r, j \in R_t$. Agent $r$ requires the information of variables $(e_i, f_i)$ to construct
constraints (1b)-(1d); however, agent $r$ cannot directly access
$(e_j, f_j)$ as they are controlled by agent $t$, and agent $t$ faces
the same situation. In order for these two agents to solve their
local problems in parallel, it is necessary to break the coupling
by introducing auxiliary variables. We let each agent $r$ keep additional variables $x_{ij}^t = (e_{ij}^t, f_{ij}^t)$ for all $j \in \delta(R_r)$. A direct consequence is that all constraints in formulation (1) are decomposed to local agents. For example, for each $i \in R_r$, constraints (1b)-(1d) can be rewritten as

\[
\begin{align*}
\sum_{j \in \delta(i) \cap R_r} G_{ij}(e_{ij} + f_{ij}^2) + & \quad \sum_{j \in \delta(i) \cap \delta(R_t)} G_{ij}(e_{ij} e_i + f_{ij} f_i) + B_{ij}(e_i f_i - e_j f_j) \\
& \quad \sum_{j \in \delta(i) \cap \delta(R_r)} G_{ij}(e_{ij} e_j + f_{ij} f_j) - B_{ij}(e_i f_i - e_j f_j) = 0, \\
(4a) \\
q_{e_i}^r - q_{e_i}^d = & \quad -B_{ij}(e_i e_j + f_i f_j) - G_{ij}(e_i e_j f_j - e_j e_j f_j), \\
(4b) \\
p_{e_i}^2 + q_{e_i}^2 & \leq 1 \\
(4c) \\
p_{f_i}^2 + q_{f_i}^2 & \leq 1 \\
(4d)
\end{align*}
\]

where $p_{ij}, q_{ij}$ are given in (2), and

\[
\begin{align*}
p_{ij} = & \quad -G_{ij}(e_{ij}^2 + f_{ij}^2 - e_{ij} e_{ij} - f_{ij} f_{ij}) - B_{ij}(e_{ij} f_{ij} - e_{ij} f_{ij}), \\
q_{ij} = & \quad B_{ij}(e_{ij}^2 + f_{ij}^2 - e_{ij} e_{ij} - f_{ij} f_{ij}) - G_{ij}(e_{ij} f_{ij} - e_{ij} f_{ij}).
\end{align*}
(5a)
(5b)
\]

Notice that all variables appeared in (4) are controlled by agent
$r$, and all such groups are grouped together and denoted by
$x^r = \{x_{ij}^r | i \in R_r, \{x_{ij}^r \} \in \delta(R_r)\}$. Moreover, local nonconvex constraints of $R_r$ can be conveniently expressed as

\[
X_r = \{x^r \mid (4a), (4b), (4c), (4d) \forall i \in R_r\}.
(6)
\]

Notice that for every $j \in \bigcup_{r=1}^{R} \delta(R_r)$, bus $j$ is connected
to some tie-line, and thus at least two regions need to keep a local
copy of $(e_{ij}, f_{ij})$. We use $R(j)$ to denote the subregion where
bus $j$ is located, and $(N(j) \setminus \{j\})$ to denote the set of subregions that
share a tie-line with $R(j)$ through bus $j$. Naturally we want to impose consensus on local copies of the same variables:

\[
e_{ij}^l = e_{ij}, f_{ij}^l = f_{ij} \quad \forall l \in N(j).
(7)
\]

If (7) is used, agents from $N(j)$ and agent $R(j)$ will need to
alternatively solve their subproblems in order to parallelize
the computation. However, ADMM does not guarantee
convergence when both subproblems carry nonconvex functional
constraints. In order to solve this issue, we follow the idea
proposed in [23] by using a global copy $\bar{x}_j = (\bar{e}_{ij}, \bar{f}_{ij})$ and
local slack variables $z_{ij}^l = (z_{eij}^l, z_{fij}^l)$ for $l \in N(j) \cup \{R(j)\}$. For notational consistency, we also write $x_{ij} R(j) = (e_{ij}, f_{ij} = (e_{ij}^{R(j)}, f_{ij}^{R(j)})$. The consensus is then achieved through

\[
e_{ij} - e_{ij}^l + z_{eij}^l = 0, f_{ij}^l - f_{ij} + z_{fij}^l = 0, \quad \forall l \in N(j) \cup \{R(j)\}
(8)
\]

Denote $x = \{x^r \in \mathbb{R}^n_r\}, \bar{x} = \{\bar{x}_j\} \in \cup_{j \in \mathbb{R}} \delta(R_r), \text{ and } z = \{z_{ij}^l\} \in (N(j) \cup \{R(j)\}) \cup \delta(R_r)\}$. Now we can now abstract the AC-OPF problem as:

\[
\begin{align*}
\min_{x,z} & \quad \sum_{r=1}^{R} c_r(x^r) := \sum_{r=1}^{R} \left( \sum_{i \in R_r} f_i(p_i^r) \right) \\
\text{s.t.} & \quad A x + B \bar{x} + z = 0, \quad z = 0, \quad x^r \in \mathcal{X}, \forall r \in [R], \quad \bar{x} \in \bar{X}. \quad (9)
\end{align*}
\]

The objective $c_r(x^r)$ is the sum of all generators’ costs in $R_r$. The linear coupling constraints (8) is compactly expressed as $A x + B \bar{x} + z = 0$ with matrices $A$ and $B$ of proper dimensions. Each local agent $r$ controls local OPF constraints $X_r$ defined in (6). Moreover, without changing the feasible region of (1), we may restrict $\bar{x}$ inside some convex set $\bar{X}$. For example, we can simply let $\bar{X} \subseteq \mathbb{R}^n$ be a hypercube

\[
\bar{X} = \prod_{j \in \mathbb{R}} X_j := \prod_{j \in \mathbb{R}} \{ x_j \mid \| x_j \|_{\infty} \leq \bar{v}_j \}
(10)
\]

which is compact and easy to project onto.

Next we define stationarity for problem (9). After projecting out
the slack variable $z$, the Lagrangian function of (9) is

\[
L(x, \bar{x}, y) = \sum_{r=1}^{R} c_r(x^r) + \lambda(x^r(x^r)) + \lambda(x, A x + B \bar{x} + y).
(11)
\]

We use $\partial f(\cdot)$ to denote the general subdifferential of a proper lower semi-continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ [24, Def 8.3], and $N_C(x)$ to denote the general normal cone of $C$ at $x \in C$ [24, Def 6.3].

**Definition 1.** We say $(x, \bar{x}, y)$ is an e-stationary point of problem (9) if there exist $(d_1, d_2, d_3)$ such that

\[
\max \{|d_1|, |d_2|, |d_3|\} \leq \epsilon
\]

where

\[
\begin{align*}
d_1 & \in \partial \left( \sum_{r=1}^{R} c_r(x^r) + \mathbb{I}(x^r) \right) + A^T y \quad (12a) \\
d_2 & \in N_X + B^T y \quad (12b) \\
d_3 & = A x + B \bar{x} \quad (12c)
\end{align*}
\]

or equivalently, $(d_1, d_2, d_3) \in O L(x, \bar{x}, y)$. We simply say
$(x, \bar{x}, y)$ is a stationary point if $0 \in \partial L(x, \bar{x}, y) \text{ or } \epsilon = 0$.

If cost functions are concatenated as $c(x) = \sum_{r=1}^{R} c_r(x^r)$, which is assumed to be continuously differentiable, and let $\mathcal{X} = \prod_{r=1}^{R} X_r$, then (12a) can be further reduced to

\[
0 = \nabla c(x) + N_X(x) + A^T y.
(13)
\]
Infeasibility
Generation Cost

\[ \min_{x \in \mathcal{X}, \bar{x} \in \bar{X}} c(x) \quad \text{s.t.} \quad Ax + B\bar{x} = 0, \]  
(14)

which was also directly used to develop distributed algorithms [10], [11]. We partition the IEEE case 30 available from [9] into three subregions, directly apply the vanilla version ADMM to the two-block formulation (14) with different penalty parameter \( \rho \), and plot the Infeasibility \( \|Ax^k + B\bar{x}^k\| \) and Generation Cost \( c(x^k) \) as in Figure 1.

![Divergent behaviour of vanilla ADMM](image)

As we can see, the primal residual and generation costs exhibit oscillating patterns for \( \rho \in \{1000, 2000, 3000, 4000\} \), and do not converge even when ADMM has performed 2000 iterations. For \( \rho = 5000 \), the primal residual converges to 0.0093, and the generation cost at termination is below the lower bound obtained from SOCP; these two facts indicate that ADMM indeed converges to an infeasible solution.

We believe such failures of ADMM to obtain feasible solutions result from the conflict between the aforementioned assumptions (b) and (c). To be more specific, it is straightforward to verify that \( \Im(B) \subset \Im(A) \): given a global copy \( \bar{x}_j \), every local agent \( l \in N(j) \cup R(j) \) can always keep the same value \( x^l_j = \bar{x}_j \), so that the constraint \( Ax + B\bar{x} = 0 \) is satisfied, but the converse is in general not true. However, there are functional constraints \( \mathcal{X} \) associated with \( x \), as agents need to coordinate local OPF constraints. As a result, the block with “larger” image \( A \) does not have a smooth objective \( (c(x) + \Im(\mathcal{X})(x)) \), so assumptions (b) and (c) cannot be satisfied simultaneously. Admittedly, we observe convergence when a even larger penalty \( \rho \) is used; however, we want to emphasize that despite the numerical success with adaptive parameter tuning [13], the traditional ADMM framework has no guarantee of convergence and could fail for nonconvex distributed OPF.

### C. A New Two-level ADMM Algorithm

In this section we give a full description of the two-level algorithm applied to the OPF problem. The key idea is to dualize and penalize the constraint \( z = 0 \) in (9), and apply three-block ADMM to solve the subproblem

\[ \min_{x \in \mathcal{X}, \bar{x} \in \bar{X}, z} \sum_{r=1}^{R} c_r(x^r) + \langle \lambda^k, z \rangle + \frac{\beta^k}{2} \|z\|^2 \]  
(15)

with some dual variable \( \lambda^k \) and penalty \( \beta^k \), to an approximate stationary solution in the following sense.

**Definition 2.** We say \( (x, \bar{x}, z, y) \) is an \( \epsilon \)-stationary point of problem (15) if there exist \( (d_1, d_2, d_3) \) such that

\[ \max \{\|d_1\|, \|d_2\|, \|d_3\|\} \leq \epsilon, \]  

where \( d_1 \) and \( d_2 \) satisfy (12a)-(12b), \( d_3 = Ax + B\bar{x} + z \), and \( \lambda^k + \beta^k z + y = 0 \).

Then at termination of ADMM, we update \( \lambda^{k+1} \) and \( \beta^{k+1} \) as in the classic ALM framework in order to drive \( z \) to 0. We say \( i \subseteq R_r \) is a boundary bus of \( R_r \) if \( i \) is connected to another subregion through a tie-line, and denote the set of boundary buses in \( R_r \) by \( B(R_r) \). Note that the inner-level ADMM dual variable and outer-level ALM dual variable \((y, \lambda)\) can be decomposed into subvectors \{\( \{y^i_j, \lambda^i_j\}_{j \in \delta(r)} \cup B(R_r) \}\) for \( r \in [R] \), so they are controlled and updated by different agents in parallel. See Algorithm 1 for a detailed description.

All agents simultaneously solve lower-dimensional nonconvex subproblems (16) by some nonlinear optimization solver. Then each agent \( r \in [R] \) sends the current local estimation of voltage \( (x^r_j)^{t+1} \) to \( R(j) \) for all neighboring buses \( j \in \delta(R_r) \). For each bus \( j \) connected to a tie-line, We let agent \( R(j) \) collect estimations \{\( (x_j^r)^{t+1} \)\} and update global copy \( (\bar{x}_j)^{t+1} \), though in practice any agents from \( N(j) \) can be assigned for this task. Notice that the global copy update (17) involves a projection evaluation, which in general does not admit a closed-form solution; however, if (10) is used for \( \mathcal{X} \), then \( \bar{x}_j^{t+1} \) is exactly the component-wise projection of the argument in (17) onto the box \( \mathcal{X} \). After agent \( R(j) \) broadcasts \( (\bar{x}_j)^{t+1} \) to agents from \( N(j) \), all agents are then able to update the slack variable and dual variables as in (18) and (19). When the inner-level iterates satisfy certain stopping criteria, all agents will update the outer-level dual variable \( \lambda^{k+1} \) and penalty \( \beta^{k+1} \), as in line 13 of Algorithm 1 which we will elaborate in the next section.

### IV. Convergence and Related Issues

In this section, we state the convergence results of the two-level ADMM framework for AC OPF, and discuss related issues. We require the following mild assumptions.

**Assumption 1.** (a) The objective \( c_r(\cdot) \) is continuous differentiable. The functional constraints \( \mathcal{X}_r \)'s and \( \bar{X} \) are compact, and \( \bar{X} \) is convex.

(b) For any \( t \in \mathbb{Z}_{++} \), every local agent \( r \in [R] \) is able to find a stationary solution \((x^r)^{t+1}\) of subproblem (16) such that \( F^1_r( (x^r)^{t+1} ) \leq F^1_r( (x^r)^t ) \).

For Assumption 1(a), the objective function \( c_r(\cdot) \) is usually linear or convex quadratic with respect to the argument; without loss of generality, we may assume the set \( \mathcal{X}_r \) defined in (6) also enforces bounds on local copies \{\( (e^r_j, f^r_j) \}_{j \in \delta(r)} \), and thus \( \mathcal{X}_r \) is ensured to be compact. The assumption on \( \mathcal{X} \) is justified in (10). Assumption 1(b) requires the nonconvex subproblem (16) to be solved to a stationary solution \((x^r)^{t+1}\) that has an improved objective value of \( F^1_r \) compared to the previous iterate \((x^r)^t\). Notice that \((x^r)^t\) is already feasible, i.e. \((x^r)^t \in \mathcal{X}_r \), so we believe this assumption is reasonable if
Algorithm 1: A Two-level ADMM Algorithm

1: Initialize starting points \((x^0, \tilde{x}^0) \in \mathcal{X} \times \mathcal{X}\) and \(\lambda^1 \in [\underline{\lambda}, \bar{\lambda}]\); \(\beta^1 > 0, k \leftarrow 1\);
2: while outer stopping criteria is not satisfied do
3: Initialize \((\bar{x}^0, z^0, y^0)\) such that \(\lambda^k + \beta^k z^0 + y^0 = 0\); \(\rho \leftarrow 2\beta^0\); \(t \leftarrow 1\);
4: while inner stopping criteria is not satisfied do
5: each agent \(r \in [R]\) updates \((x^r)^{t+1}\) by solving:
\[
\min_{x^r \in A_r} F^r_i(x^r) := c^r_i(x^r) + \frac{1}{2\rho} \|x^r_j - (\bar{x}_r^r)^t + (z^r_j)^t\|^2; \tag{16}
\]

6: each agent \(r \in [R]\) sends \((\bar{x}_r^r)^{t+1}\) to \(R(j)\) for \(j \in \delta(\mathcal{R}_r)\), and receives \((x^r_j)^{t+1}\) from each agent \(l \in N(i)\) for all \(i \in B(\mathcal{R}_r)\);
7: each agent \(r \in [R]\) updates global copy \(\bar{x}_r^{t+1} = \text{Proj}_{\mathcal{X}_r} \left( \sum_{\ell \in N(i) \cup R(i)} [(y^r_i)^t + \rho \|x^r_j|^{t+1} + (z^r_j)^t] \right) / (\|N(i)\| + 1)\rho \tag{17}
\]
for all \(i \in B(\mathcal{R}_r)\);
8: each agent \(r \in [R]\) sends \((\bar{x}_r^r)^{t+1}\) to agents in \(N(i)\) for all \(i \in B(\mathcal{R}_r)\), and receives \((\bar{x}_r^r)^{t+1}\) from agent \(R(j)\) for all \(j \in \delta(\mathcal{R}_r)\);
9: each agent \(r \in [R]\) update local slack variable
\[
(z^r_j)^{t+1} = \frac{-(\lambda^r_j)^t - (y^r_j)^t - \rho (x^r_j)^{t+1} - (\bar{x}_r^r)^t)}{\beta^r + \rho} \tag{18}
\]
and dual variable
\[
(y^r_j)^{t+1} = (y^r_j)^t + \rho \left[(x^r_j)^{t+1} - (\bar{x}_r^r)^t + (z^r_j)^{t+1}\right] \tag{19}
\]
for all \(j \in \delta(\mathcal{R}_r) \cup B(\mathcal{R}_r)\);
10: \(t \leftarrow t + 1\);
11: end while

12: denote the solution from the inner loop as \((x^k, \tilde{x}^k, z^k)\);
13: each agent \(r \in [R]\) updates outer-level dual variable \((\lambda^r_j)^{k+1}\) for all \(j \in \delta(\mathcal{R}_r) \cup B(\mathcal{R}_r)\) and penalty \(\beta^k + 1\);
14: \(k \leftarrow k + 1\);
15: end while

the nonlinear solver is warm-started. In addition, Assumption [1b] is weaker than assuming \((x^r)^{t+1}\) is a global minimizer of \((16)\), which is commonly adopted in nonconvex ADMM.

### A. Global Convergence

We consider two different rules for updating \((\lambda^{k+1}, \beta^{k+1})\). Let \(c > 1, \theta \in (0, 1)\), and \(\eta_k\) be a nonnegative sequence convergent to 0. Consider
\[
\lambda^{k+1} = \text{Proj}_{[\underline{\lambda}, \bar{\lambda}]}(\lambda^k + \beta^k z^k), \quad (20a)
\]
\[
\beta^{k+1} = \begin{cases} \beta^k & \text{if } \|z^k\| \leq \theta \|z^{k+1}\|, \\ c^k \beta^k & \text{otherwise}; \end{cases} \quad (20b)
\]
and
\[
(\lambda^{k+1}, \beta^{k+1}) = \begin{cases} (\lambda^k + \beta^k z^k, \beta^k) & \text{if } \|z^k\| \leq \eta_k, \\ (\lambda^k, c^k \beta^k) & \text{otherwise} \end{cases} \tag{21}
\]

**Theorem 1** (Global Convergence). Suppose Assumption [7] holds, and the \(k\)-th inner-level ADMM is solved to an \(\epsilon\)-stationary point \((x^k, \tilde{x}^k, z^k, y^k)\) of \((15)\) such that \(\epsilon_k \to 0\) as \(k \to +\infty\). Moreover, the outer-level dual variable \(\lambda^{k+1}\) and penalty \(\beta^{k+1}\) are updated according to either (20) or (21).

Then the following claims hold.

1) The sequence \(\{(x^k, \tilde{x}^k, z^k)\}_k\) is bounded, and therefore there exists at least one limit point \((x^*, \tilde{x}^*, z^*)\), where \(x^* \in \mathcal{X} = \bigcap_{k=1}^\infty \mathcal{X}_k\) and \(\tilde{x}^* \in \tilde{X}\).

2) Either \((x^*, \tilde{x}^*)\) is feasible, i.e., \(Ax^* + B\tilde{x}^* = 0\), or \((x^*, \tilde{x}^*)\) is a stationary point of the feasibility problem
\[
\min_{x, \tilde{x}} \frac{1}{2}\|Ax + B\tilde{x}\|^2 \quad \text{s.t. } x \in \mathcal{X}, \tilde{x} \in \tilde{X}. \tag{22}
\]

3) Suppose problem (9) is feasible and the set of stationary points is nonempty. Let \((x^*, \tilde{x}^*, z^*)\) be a limit point, and \(\{(x^{k_r}, \tilde{x}^{k_r}, z^{k_r})\}_r\) be the subsequence convergent to it. If \(\|y^{k_r}\|\) has a limit point \(y^\tau\), then \((x^*, \tilde{x}^*, y^\tau)\) is a stationary point of problem (9).

We note that in part 3 of Theorem 1, we make the assumption that the dual variable \(\{(\lambda^k)\}_k\) has a limit point. This is a standard sequentially bounded constraint qualification (SBQC) [25]. Proof of Theorem 1 is given in the Appendix.

### B. Iteration Complexity

**Theorem 2** (Iteration Complexity). Suppose Assumption [7] holds, and there exists \(0 < L < +\infty\) such that
\[
L \geq R \sum_{r=1}^R c_r \|x^r_0\| + (\lambda^k, z^0) + \frac{\beta}{2} \|z^0\|^2 + (y^0, Ax^0 + B\tilde{x}^0 + z^0) + \frac{\rho}{2} \|Ax + B\tilde{x}\|^2 \tag{23}
\]
for all \(k \in \mathbb{Z}_{++}\). Further assume each inner-level ADMM is solved to an \(\epsilon\)-stationary point \((x^k, \tilde{x}^k, z^k, y^k)\) of (15), and \(\beta^k = c^k \beta^0\) for some \(c > 1, \beta^0 > 0\). Define
- \(\tau = \max\{\|A\|, \|B\|, \frac{1}{\sqrt{\tau}}\}\),
- \(M = \max_{\lambda \in [\underline{\lambda}, \bar{\lambda}]} \|\lambda\|\),
- \(\hat{L} = \min_{x \in \mathcal{X}} c(x) - M^2 / \beta^0\),
- \(r_{max} = \max_{x \in \mathcal{X}, \tilde{x} \in \tilde{X}} \|Ax + B\tilde{x}\| \) (since \(\mathcal{X}, \tilde{X}\) compact),
- and for \(K \in \mathbb{Z}_{++}\),
\[
T(K) = \left( \frac{8\beta^0(\hat{L} - \hat{L}) r_{max}^2 c}{c - 1} \right) \left( \frac{\epsilon^K - 1}{\epsilon^2} \right) + K.
\]

Then Algorithm 1 finds an \(\epsilon\)-stationary solution of problem (9) in no more than
\[
K_1 = \log_c \left( \frac{2(L - L + M r_{max})}{\beta^0 c^2} \right)
\]
outer ALM iterations and \(T(K_1) = O(1/\epsilon^4)\) inner ADMM iterations. Moreover, if \(\|\lambda^k + \beta^k z^k\| \leq \Lambda\) for all outer index \(k\), then Algorithm 1 finds an \(\epsilon\)-stationary solution of problem (9) in no more than
\[
K_2 = \max \left\{ \log_c \left( \frac{1}{2\beta_0 \tau} \right), \|z^{k+1}\| \right\}
\]

outer ALM iterations and $T(K_2) = O(1/c^3)$ inner ADMM iterations.

The proof of Theorem 2 is given in the Appendix. We make some remarks.

1) The assumption 23 can be satisfied trivially, for example, if a feasible solution $(x^0, z^0)$ for (9) is known a priori and $(x^0, \bar{x}^0, z^0 = 0)$ is always used to start inner ADMM. In this case, we can choose $L = \max_{x \in \mathbb{R}} c(x)$.

2) In view of (8), we can calculate $\|A\|$ and $\|B\|$ directly. Each row of $A$ has exactly one non-zero entry, and each column of $A$ has at most one non-zero entry, so $A^T A$ is a diagonal matrix with either 0 or 1 on the diagonal, and thus $\|A\| = 1$. The number of non-zero entries in each column of $B$ specifies how many subregions keep a local copy of this global variable, so $B^T B$ is also a diagonal matrix, and we have

$$\|B\| = \left( \max_{j \in \mathbb{R}} |N(j)| + 1 \right)^{1/2} \leq \sqrt{R}.$$  

3) The complexity results suggest that a smaller $M$ is preferred, and $M = 0$ corresponds to the penalty method. However, we empirically observe that a relatively large range for $\lambda$ usually results in faster convergence, which is also better than the $O(1/c^3)$ or $O(1/c^4)$ iteration upper bound. We believe such phenomena can be explained by the local convergence properties of ALM.

V. Numerical Experiments

In this section, we demonstrate the performance of the proposed algorithmic framework. All codes are written in the Julia programming language 1.2.0, and implemented on a Red Hat Enterprise Linux Server 7.6 with 85 Intel 2.10GHz CPUs. All nonlinear constrained problems are modeled using the JuMP package [27] and solved by IPOPT [28] with linear solver MA57.

A. Network Information and Partition Generation

We experimented on the two largest networks from Nesta, namely, nesta_case9241pegase ("9K") and nesta_case13659pegase ("13K"). The centralization information are given in Table 1. We generate different partitions using the multilevel k-way partitioning algorithm [29] on the underlying graph, which is available from Metis.jl, the Julia wrapper of the Metis library. Each test case is partitioned into $R$ subregions, where $R$ ranges from 25 to 70 with an interval length of 5.

<table>
<thead>
<tr>
<th>Case</th>
<th>AC Obj.</th>
<th>AC Time(s)</th>
<th>SOCP Obj.</th>
<th>SOCP Time(s)</th>
<th>Gap (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9K</td>
<td>315913.26</td>
<td>51.13</td>
<td>310382.99</td>
<td>1224.81</td>
<td>1.76</td>
</tr>
<tr>
<td>13K</td>
<td>386117.10</td>
<td>160.10</td>
<td>380262.34</td>
<td>990.50</td>
<td>1.51</td>
</tr>
</tbody>
</table>

B. Three Acceleration Heuristics

It is known that the choice of penalty parameter $\rho$ significantly affects the convergence of ADMM and can potentially accelerate the algorithm [13]. Indeed, we observed that for some large cases, the inner-level ADMM may suffer from slow convergence to high accuracy when a constant penalty $\rho$ is used. As a result, given parameters $\theta \in (0, 1)$ and $\gamma > 1$, we propose three different heuristics to properly accelerate Algorithm 1.

1) Adaptive ADMM penalty: the inner-level ADMM penalty is indexed as $\rho^t$ and updated as follows: $\rho^{t+1} = \gamma \rho^t$ if $\|A\| \geq \theta \|Ax + B\bar{x} + z\|$ and $\rho^{t+1} = \rho^t$ otherwise; in words, we increase the ADMM penalty if the three-block residual $\|Ax + B\bar{x} + z\|$ does not decrease sufficiently.

2) Different ADMM penalties: we assign a different ADMM penalty for each row of the coupling constraint $Ax + B\bar{x} + z = 0$, and each penalty is updated according to the first heuristic, where the violation of each single constraint is measured, and the corresponding penalty is adjusted. Notice the ADMM penalty $\beta$ is a fixed constant for all components of $z$ during the inner ADMM.

3) Different ALM penalties: we assign a different ALM penalty $\beta^{t}_i$ for each component $z_i$ of the slack variable, and also update it inside ADMM iterations: $\beta^{t+1}_i = \gamma \beta^{t}_i$ if $|z^{t+1}_i| \geq \theta |z^{t}_i|$, and $\beta^{t+1}_i = \beta^{t}_i$ otherwise; the corresponding ADMM penalty $\rho^{t}_i$ is always assigned to be $2\beta^{t}_i$, as required in our analysis. When the $k$-th ADMM terminates, current values of $z^{k}_i$ and $\beta^{k}_i$ are used to update outer level dual variable $\lambda^{k+1}_i$.

We note that the first two heuristics have been used to accelerate ADMM, while the last heuristic also penalizes the slack variable $z$ adaptively in ADMM iterations.

C. Implementation Details

Parallelization of nonconvex subproblems: Each JuMP model carrying a subregion’s localized OPF constraints is initialized on a worker thread. During each (inner) iteration, these models are solved in parallel on worker threads by IPOPT, which consist of the major computation of the algorithm. Then current local solutions are gathered through the master thread, and auxiliary primal variables and dual variables in different subregions are updated in closed form.

Parameters and Initialization: For the heuristics introduced in the previous subsection, we set $\gamma = 6.0$, $\theta = 0.8$; for the first two heuristics, when ADMM terminates, the outer-level penalty is updated as $\beta^{k+1} = c\beta^k$ where $c = 6.0$. The initial value $\beta^0$ is set to 1000.0, and an upper bound of $1.0e24$ is imposed in all penalty updates. Each component of $\lambda$ is bounded between $\pm 1.0e12$. Flat start is used to initialize IPOPT: we choose $(e_i, f_i, p^{\ell}_i, q^{\ell}_i) = (1, 0, 0, 0)$ for all $i \in N$. Dual variables $y^0$ and $\lambda^0$ are initialized with zeros.

Scaling of IPOPT: The proposed algorithm inevitably needs to deal with potentially large penalties and dual variables, and we observe that IPOPT will encounter numerical failures or produce oscillating solutions. To overcome this problem, we manually scale the objective of the each JuMP model so that
We use "9K-R" to denote the *nesta_case9241_pegase* with *R* partitions, and similar for *nesta_case13659_pegase*.

the largest coefficient passed to the solver is in the order of 1.0e +8. This trick helps IPOPT output stable solutions efficiently without running into numerical troubles.

**Termination of Inner and Outer Iterations:** We stop the inner-level ADMM if: (1) $\| Ax^k + B \bar{x}^k + z^f \| \leq \sqrt{d}/(2500k)$ where $d$ is the dimension of the coupling constraint and $k$ is the current outer-level iteration index, or (2) $\| z^f - z^{f-1} \| \leq 1.0e-8$. The first condition ensures the ADMM primal residual is under certain tolerance, which also tends to 0 as $k \rightarrow \infty$. The second condition measures the dual residual of the last block in (15); if this quantity is small, then we believe $z^f$ has stabilized in the current inner iteration, which encourages us to terminate ADMM early and proceed to update outer-level dual variable. The outer level is terminated if the two-block residual satisfies $\| Ax^k + B \bar{x}^k \| \leq \sqrt{2}.0e-4$, which measures the infeasibility of the original OPF problem.

**D. Numerical Performance**

The results are displayed in Table [II] We use "TL-1", "TL-2", and "TL-3" to denote the two-level ADMM algorithm combined with the three heuristics mentioned in the previous section. The total number of tie-lines and the dimension of the coupling constraints of each case are recorded in the second and third columns. For each heuristic, we report the number of outer iterations ("Outer"), number of inner iterations ("Inner"), duality gap with SOCP lower bound ("Gap (%)"), and max violation of the coupling at termination ("$\| r \|_{\infty}$"). The three algorithms reach the desired infeasibility tolerance in all test instances, and the max constraint violation is in the order of $10^{-3}$. Overall the generation costs at termination are very close to the SOCP lower bound, indicating the algorithms converge to solutions with high qualities. Moreover, we emphasize the scalability of the algorithm by pointing out that, the number of inner and outer iterations are stable across instances, even though the dimension of the coupling ranges from 2000 to near 5000.

![Fig. 2: Average Time v.s. Number of Partitions](image)

We plot the averaged computation time for the proposed algorithm combined with three heuristics in Fig. 2 (left). During our experiments shown in Table II, the number of partitions is equal to the number of threads used for parallelization, so that each IPOPT model is assigned to a worker thread. The computation time drops significantly and becomes comparable to the centralized solver as the number of partitions increases: 735.50 to 97.04 seconds for 9K-bus system, and 1513.66 to 167.57 seconds for 13K-bus system. This is due to the OPF problem is decomposed into lower-dimensional subproblems, which are effectively solved on different threads in parallel. We further partition the networks into 80, 120, 160, 200, and 240 partitions, and investigate the average subproblem time in Fig. 2 (right). For the 9K-bus system, the averaged subproblem solution time drops from 2.61 to 1.15 seconds, while for the 13K-bus system, this metric drops more significantly from 22.13 to 0.7 seconds. Our results validate the feasibility of using parallelization to speed up computation.

**E. Comparison with One-level ADMM Variants**

To further illustrate the advantage of the proposed two-level ADMM algorithm, we implement two state-of-the-art one-level ADMM variants and display their results in Table [III]

All implementation issues mentioned above are considered; however, for all cases, both one-level ADMM algorithms fail
TABLE III: Comparison with One-level ADMM Variants

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td></td>
<td>Gap(%) Time (s)</td>
<td>Gap(%) Time (s)</td>
<td>Gap(%) Time (s)</td>
</tr>
<tr>
<td>13K-25</td>
<td>88.98 2689.48</td>
<td>-6.14 1946.21</td>
<td>1.52 1788.62</td>
</tr>
<tr>
<td>13K-30</td>
<td>90.64 2507.26</td>
<td>-6.21 1847.13</td>
<td>1.30 680.16</td>
</tr>
<tr>
<td>13K-35</td>
<td>91.86 2034.09</td>
<td>-6.97 1510.31</td>
<td>0.74 752.41</td>
</tr>
<tr>
<td>13K-40</td>
<td>92.79 2168.87</td>
<td>-8.09 1555.16</td>
<td>1.20 430.68</td>
</tr>
<tr>
<td>13K-45</td>
<td>93.54 1428.20</td>
<td>-8.06 1436.43</td>
<td>1.19 106.79</td>
</tr>
<tr>
<td>13K-50</td>
<td>94.15 1463.57</td>
<td>-8.44 1405.47</td>
<td>0.53 374.15</td>
</tr>
<tr>
<td>13K-55</td>
<td>94.65 1521.64</td>
<td>-12.59 1370.72</td>
<td>0.62 204.50</td>
</tr>
<tr>
<td>13K-60</td>
<td>95.09 1459.83</td>
<td>-8.26 1208.48</td>
<td>0.81 207.58</td>
</tr>
<tr>
<td>13K-65</td>
<td>95.44 1910.87</td>
<td>-9.16 1279.62</td>
<td>1.19 106.20</td>
</tr>
</tbody>
</table>

To achieve the desired infeasibility tolerance within 1000 iterations. For the modified ADMM [11], the penalty parameter quickly reaches the upper bound $10^{24}$, which indicates their assumption for convergence is not satisfied, and such large penalty leads to solutions with high generation costs. Jiang et al. [22] realize that ADMM can be used to solve the relaxed problem (15) with $\lambda^k = 0$ and large enough constant $\beta$. When $\beta = O(1/\epsilon^2)$, their proposed ADMM variant is guaranteed to find an $\epsilon$-stationary solution of the original problem. Using parameters suggested in their analysis, we observe that the true infeasibility $\|Ax^e + Bz\|$ decreases very slowly, though the three-block residual $\|Ax^e + Bz + z^*\|$ converges to near 0 fast. This indicates the need for a more careful tuning of the penalty $\beta$, but there is no principled way to select all hyperparameters in their algorithm for OPF instances.

VI. CONCLUSION

In this paper we propose a two-level distributed ADMM framework for solving AC OPF. Instead of relying on parameter tuning to exhibit convergence behaviour, the proposed algorithm is proven to have guaranteed global convergence. Iteration complexity bound is also obtained, and can be further accelerated by suitable heuristics. Promising numerical results over some large-scale test cases show that the proposed algorithm provides a new, robust distributed, and convergence-guaranteed algorithmic framework for solving real-world sized AC OPF problems.

REFERENCES

Appendix

A. Proof of Theorem 1

Proof. The claims under condition (20) are proved in Theorem 1-2 of [23]. We shall prove the claims under condition (21).

1) The first claim follows from the compactness of \( X_i \)'s and \( \tilde{X} \) and the fact that \( \|A_k + B_k + z^k\| \to 0 \).

2) First we assume the sequence \( \{\beta_k\}_k \) stays finite. Then there exists \( K > 0 \) such that \( \eta_k \geq \|z^k\| \) for all \( k \geq K \), which follows \( \|z^k\| \to 0 \). Next assume \( \beta^k \to +\infty \), and assume without loss of generality that \( (x^k, \tilde{x}^k, z^k) \to (x^*, \tilde{x}^*, z^*) \). If the first case in (21) is executed infinitely many times, we have \( \|z^k\| \to 0 \). Otherwise we must have \( \|\lambda^k\| \) stays constant for all sufficiently large \( k \).

Define \( \lambda^k := \lambda^k + \beta_k z^k = -\eta^k \). If \( \{\lambda^k\} \) has a bounded subsequence, then we have \( \|z^k\| \to 0 \) as \( \beta^k \) converges to infinity. In all the previous cases, we have \( \|A_k + B_k + z^k\| \leq \|A_k + B_k + z_k\| + \|z^k\| \leq \epsilon_k + \|z^k\| \to 0 \), and thus \( \|A_k + B_k\| = 0 \). In the last case we have \( \beta^k \to \infty \), \( \|\lambda^k\| \) stays bounded, and \( \|\lambda^k\| \to \infty \). By the definition of \( \lambda^k \), we can see \( \eta^k / \beta^k \to -z^* = A \tilde{x}^* + B \tilde{x}^* \). At termination of ADMM, we have

\[
\begin{align*}
\hat{d}_1^k &\in \nabla c(x^k) + A^T y^k + N_X(x^k) \quad (24a) \\
\hat{d}_2^k &\in B^T y^k + N_{\tilde{X}}(\tilde{x}^k) \quad (24b) \\
\hat{d}_3^k &= A \tilde{x}^* + B \tilde{x}^* + z^k \quad (24c)
\end{align*}
\]

where \( \max\{\|\hat{d}_1^k\|, \|\hat{d}_2^k\|, \|\hat{d}_3^k\|\} \leq \epsilon_k \to 0 \). By the closeness of normal cone, dividing vectors in (24a), (24b) by \( \beta^k \), and then taking limit, we have \( 0 \in A^T (A \tilde{x}^* + B \tilde{x}^*) + N_X(x^*) \) and \( 0 \in B^T (A \tilde{x}^* + B \tilde{x}^*) + N_{\tilde{X}}(\tilde{x}^*) \), which imply \( (x^*, \tilde{x}^*) \) is stationary for (22).

3) Using the same case analysis as in Part (2), we can see \( \|z^k\| \to 0 \) (along the subsequence converging to \( z^* \)). Thus taking limit on (24) completes the proof.

B. Proof of Theorem 2

Proof. We use \( T_k \) to denote the number of inner iterations of the \( k \)-th ADMM, which produces a \( \epsilon \)-stationary solution point of (15). By Lemma 3 of [23], we can find \( (x^k, \tilde{x}^k, z^k, y^k) \) and corresponding \( (d_1^k, d_2^k, d_3^k) \) (see Definition 2) such that

\[
\max\{\|d_1^k\|, \|d_2^k\|, \|d_3^k\|\} \leq \rho^k \tau \left( \frac{2(\overline{T} - L)}{\beta^k T_k} \right)^{1/2}.
\]

Recall \( \rho^k = 2\beta^k = 2\beta^0 c^k \); it is sufficient to find a \( T_k \) with

\[
T_k \leq \frac{8\beta^0 (\overline{T} - L) \tau^2 c^k}{\epsilon^2} + 1
\]

in order to get \( \max\{\|d_1^k\|, \|d_2^k\|, \|d_3^k\|\} \leq \epsilon \). Let \( K \) denote the number of outer-level ALM iterations. Then the total number of inner iterations is bounded by

\[
\sum_{k=1}^K T_k \leq \left[ \left( \frac{8\beta^0 (\overline{T} - L) \tau^2 c^k}{\epsilon^2} \left( \frac{e^K - 1}{c - 1} \right) \right) + K \right].
\]

Now it remains to bound the number of outer-level iterations \( K \). Notice that the dual residuals are controlled by \( \epsilon \) at the end of each inner-level ADMM, so it is sufficient to ensure \( \|Ax^k + Bz^k\| \leq c \). By Theorem 3 of [23],

\[
\|Ax^k + Bz^k\|^2 \leq \frac{2(\overline{T} - L) \tau^2 c^k}{\beta^k}\epsilon^2
\]

and the claimed \( K_1 \) is sufficient to ensure \( \|Ax^k_1 + Bz^k_2\| \leq \epsilon \). Furthermore, if \( \|\lambda^k + \beta z^k\| \leq \Lambda \) for all outer index \( k \), then we have

\[
\|z^k\| = \|\lambda^k + \beta z^k\| \leq \frac{\lambda + M}{\beta^0 c^k}.
\]

Again by Lemma 3 of [23], we have

\[
\|Ax^k + Bz^k + z^k\| \leq \frac{1}{2} \left( \frac{2(\overline{T} - L)}{\beta^k T_k} \right)^{1/2} \leq \frac{\epsilon}{4\beta^0 c^k T_k}
\]

where the last inequality is due to the choice of \( T_k \). It is straightforward to verify that the claimed \( K_2 \) ensures

\[
\|Ax^k + Bz^k + z^k\| \leq \|Ax^k + Bz^k + z^k\| + \|z^k\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

This completes the proof.