Dimension groups and dynamical systems

Fabien Durand, Dominique Perrin

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Chapter 1

Introduction

In this monograph, we introduce the reader to the connection between topological dynamical systems and dimension groups. In this way, we will be able to distinguish topological dynamical systems, which can appear in many different forms by comparing their dimension groups, which are easier to handle.

Dimension groups are ordered Abelian groups associated with a family of associative algebras called approximately finite, or AF-algebras.

These algebras are themselves a class of $C^*$-algebras which are direct limits of finite dimensional algebras and were introduced by Bratteli (1972). The algebra is built from a special kind of graph called a Bratteli diagram. Dimension groups were introduced by Elliott (1976) tool for classifying AF-algebras and he proved that the dimension group (together with an additional information called the scale) provides a complete algebraic invariant for these algebras.

The connection of these ideas with dynamical systems was done by Vershik (1982) who used a lexicographic order on paths of the Bratteli diagrams to define a topological dynamical system on the set of infinite paths of the graph. Later, Herman, Putnam and Skau showed that every minimal system on a Cantor space is isomorphic to such system. As a consequence, a dimension group is attached to any minimal Cantor system and subsequent work by Giordano, Putnam and Skau (Giordano et al., 1995) showed that this group is related to the orbit structure of the system.

In this expository presentation, written after the unpublished notes by Bernard Host (Host, 1995) (see also (Host, 2000)), we present the basic elements of this theory, insisting on the computational and algorithmic aspects allowing one to effectively compute the dimension groups. The computation applies in particular to the case of substitution shifts, explicitly presented previously in Durand et al. (1999).

In the first chapter (Chapter 2) we present the basic notions of topological dynamical systems. We restrict our attention to Cantor systems on which acts the group $\mathbb{Z}$ or the semigroup $\mathbb{N}$. We define recurrent systems and minimal dynamical systems (Section 2.1). Next, we introduce in Section 2.2 subshift
dynamical systems, which are the basic systems we are interested in. We define return words and higher block shifts. In Section 2.4 we introduce substitution shifts. We define the notion of recognizable substitution and we state the Theorem of Mossé (Theorem 2.4.28) asserting that any aperiodic primitive substitution is recognizable.

In the second chapter (Chapter 3), we shift to an algebraic and combinatorial environment. We first introduce, in Section 3.1 ordered groups (considering only Abelian groups). We define several notions, as that of order unit and order ideal. We also define a simple ordered group as one with no nontrivial ideals. In Section 3.3 we define direct limits of ordered groups and we give examples of the computation of these ordered groups. In the last part of this section (Section 3.4), we finally define dimension groups. These groups are defined as direct limits of groups \( \mathbb{Z}^n \) with the usual ordering. We prove the abstract characterization by Effros, Handelman and Shen (Effros et al. 1980) using the property of Riesz interpolation.

In Chapter 4, we come to notions of cohomology defined in a Cantor system. We first introduce the notion of coboundary (Section 4.1) and prove in Section 4.2 the Gottshalk Hedlund Theorem (Proposition 4.2.3) characterizing the continuous functions on a Cantor set which are coboundaries. We next define the ordered cohomology group \( K^0(X, T) \) of a recurrent system \((X, T)\) as the quotient of the group of integer valued continuous functions on \(X\) by the subgroup formed by coboundaries. In the next two sections (Sections 4.6 and 4.7), we consider the effect on the ordered cohomology group of applying a factor map or taking the system induced on a clopen set. In a second part of this chapter, beginning with Section 4.8, we define invariant probability measures on a Cantor system and recall that a substitutive shift defined by a primitive substitution has a unique invariant probability measure. We indicate a method to compute this measure. We show in Section 4.9 that there is a close connection between the cohomology group and the cone of invariant measures (Proposition 4.9.3). We use this connection to give a description of the dimension groups of Sturmian shifts (Theorem 4.9.4).

In Chapter 5, we introduce the fundamental tool of partitions in towers, or Kakutani-Rokhlin partitions. We prove the theorem of Herman, Putnam and Skau which shows that any minimal Cantor system can be represented as the limit of a sequence of partitions in towers (Theorem 5.1.7). In Chapter 6 we come back to partition in towers. We first show how to associate an ordered group to a partition in towers (Section 5.2). Next, in Section 5.3 we use a sequence of partitions in towers to prove that the group \( K^0(X, T) \) is, for any minimal dynamical system \((X, T)\), a simple dimension group (Theorem 5.3.4). In the next sections, we present explicit methods to compute the dimension group of a minimal shift space. In Section 5.4 we use return words and in Section 5.5 we use Rauzy graphs. Finally, in Section 5.6 we show how to compute the dimension group of a substitutive shift, as exposed in Durand et al. (1999).

We introduce Bratteli diagrams in Chapter 6. We define the telescoping of a diagram. We define the dimension group of a Bratteli diagram and prove that it is a complete invariant for telescoping equivalence (Theorem 6.1.5). We
next introduce ordered Bratteli diagrams and show that one may associate a
dynamical system to every properly ordered Bratteli diagram. We prove the
Bratteli-Versik model Theorem (Theorem 6.3.3) showing the completeness of
the model for minimal Cantor systems. We next prove the Strong Orbit Equi-
valance Theorem (Theorem 6.5.1) showing that dimension groups are a complete
invariant for strong orbit equivalence and the related Orbit Equivalence Theo-
rem (Theorem 6.5.3).

In Chapter 7, we focus on substitution shifts and their representations. We
begin by considering odometers, which have BV-representations close to sub-
stitution shifts. We characterise, as a main result, the family of BV-systems
associated with stationary Bratteli diagrams as the disjoint union of station-
ary odometers and substitution minimal systems (Theorem 7.2.1). We develop
next the description of linearly recurrent shifts, which are characterized by their
BV-representation (Theorem 7.3.6). We introduce in Section 7.4 the notion of
an $S$-adic representation. The main result is an explicit description of the
dimension group of a unimodular $S$-adic shift (Theorem 7.5.4). In the last section
(Section 7.6), we consider the family of substitutive shifts, a natural generakiza-
tion of substitution shifts. The main result is a characterization by a finiteness
condition of substitutive sequences (Theorem 7.6.1).

Chapter 8 describes the class of dendric shifts, defined by a restrtive condi-
tion on the possible extensions of a word. This class is a simultaneous gener-
alisations of several other classes of interest, such as Sturmian shifts or interval
exchange shifts (introduced in the next chapter). The main result is the Return
Theorem (Theorem 8.1.14) which states that the set of return words in a mini-
mal dendric shift is a basis of the free goup on the alphabet. We use this result to
describe the $S$-adic representation of dendric shifts and show that it can be de-
dined using elementary automorphisms of the free group (Theorem 8.1.40). We
illustrate these results by considering the class of Sturmian shifts (Section 8.2).
The last part of the chapter is devoted to specular shifts, a class of eventually
dendric shifts which plays a role in the next chapter, when we introduce linear
involutions. The main result is a description of the dimension group of a
specular shift (Theorem 8.3.40).

In Chapter 9, we introduce the notion of interval exchange transformation. We
prove Keane’s Theorem characterizing minimal interval exchanges (Theo-
rem 9.1.2). We develop the notion of Rauzy induction and characterize the
subintervals reached by iterating the transformation (Theorem 9.1.22). We
generalize Rauzy induction to a two-sided version and characterize the intervals
reached by this more general transformation (Theorem 9.2.3). We link
these transformations with automorphisms of the free group (Theorem 9.2.10).
We also relate these results with the theorem of Boshernizan and Carrol giving
a finiteness condition on the systems induced by an interval exchange when the
lengths of the intervals belong to a quadratic field (Theorem 9.3.2). In the last
section (Section 9.4) we define linear involutions and show that the natural cod-
ing of a linear involution without connexions is a specular shift (Theorem 9.4.9).

In the last chapter (Chapter 10) we give a brief introduction to the link
between Bratteli diagrams and the the vast subject of $C^*$-algebras. We define
approximately finite algebras and show their relation Bratteli diagrams. We
relate simple Bratteli diagrams and simple AF algebras (Theorem 10.3.9). We
prove Elliott’s Theorem showing that AF algebras are characterized by their
dimension groups (Theorem 10.3.17).

The book ends with three appendices, to be used as a reference for notions
used in this book in several domains of mathematics.

Acknowledgements These notes follow from a seminar held in Marne-la-
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Dolce, Pavel Heller, Revekka Kyriakoglou, Julien Leroy, Dominique Perrin and
Giuseppina Rindone. The text follows, at least in an initial version, closely
the notes of Bernard Host [Host] (1995), trying to develop more explicitly some
arguments. We have also chosen to follow a slightly different presentation, not
assuming systematically that the dynamical systems are minimal. We wish to
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chapters.
Chapter 2

Topological dynamical systems

We present in this chapter some definitions concerning topological dynamical systems and symbolic dynamical systems. We begin, in Section 2.1, with some general definitions concerning topological dynamical systems. The adjective topological is used to distinguish these systems which are based on a topological space from dynamical systems based on a measurable space. We define the notion of recurrent and of minimal system.

In Section 2.2, we consider shift spaces and their language. In Sections 2.3, 2.4, 2.5 and 2.6, we present several particular types of symbolic systems, namely shifts of finite type, Sturmian shifts, substitution shifts and finally Toeplitz shifts.

2.1 Recurrent and minimal dynamical systems

A topological dynamical system is a pair $(X, T)$ where $X$ is a compact metric space and $T : X \to X$ a continuous map.

Example 2.1.1 As a simple example, consider $X = [0, 1]$, which is metric and compact as a closed interval of the real line $\mathbb{R}$ and the transformation $T : x \mapsto (x + \alpha) \mod 1$ for some $\alpha \in \mathbb{R}$, which is is a continuous map from $X$ into $X$.

Thus, in such a system, to each point $x$ in the space $X$ is associated a sequence $(x, T(x), T^2(x), \ldots)$ of points. It is convenient to imagine the action of $T$ as the sequence of positions of the point $x$ in the space $X$ at discrete intervals of time $0, 1, 2, \ldots$. The effect of the hypothesis that $X$ is compact is to guarantee that the sequence of these points will remain at bounded distance of $x$.

When $T$ is a homeomorphism, we say that the system $(X, T)$ is invertible. Although we will meet most of the time invertible dynamical systems, we do not
make this hypothesis systematically, mentioning each time when it is necessary. Note that, since $X$ is assumed to be compact, if $T$ is invertible, its inverse is continuous and thus $T$ is a homeomorphism (Exercise 2.1).

**Example 2.1.2** The system of Example 2.1.1 is not invertible since 1 is not in the image of $T$. If we consider, instead of $[0, 1]$, the torus $T = \mathbb{R}/\mathbb{Z}$ in which 0 and 1 are identified, the transformation $T$ is simply $x \mapsto x + \alpha$ and becomes a homeomorphism.

For $x \in X$, we often denote $Tx$ instead of $T(x)$. We also denote $T^0$ for the identity on $X$ and for $n \geq 0$, $T^{-n}(x) = \{y \in X \mid T^n(y) = x\}$.

An important example is when $X$ is a Cantor space, that is, a totally disconnected compact metric space without isolated points. We say then that $(X, T)$ is a Cantor dynamical system or Cantor system (we shall come back shortly to Cantor spaces).

In particular, let $A$ be a finite set called an alphabet. The set $A^\mathbb{Z}$ of all bi-infinite sequences endowed with the product topology is a compact space. Let $d$ be the distance on $A^\mathbb{Z}$ defined for $x \neq y$ by

$$d(x, y) = 1/ \max\{n \geq 0 \mid x_i = y_i, |i| \leq n\}$$

The topology defined by this distance is the same as the product topology and thus $A^\mathbb{Z}$ is a compact metric space. It is actually a Cantor space (see below).

The shift transformation $S : A^\mathbb{Z} \to A^\mathbb{Z}$ is defined for $x = (x_n)_{n \in \mathbb{Z}}$ by $y = Sx$ where

$$y_n = x_{n+1}, \quad (2.1.1)$$

for all $n \in \mathbb{Z}$. The shift is obviously continuous and thus $(A^\mathbb{Z}, S)$ is a topological dynamical system.

As a variant, the set $A^\mathbb{N}$ of one-sided infinite words is also a topological space for the product topology and this topology is defined by the metric analogous to the one above. It is also a Cantor space. The one-sided shift transformation is defined by Equation (2.1.1) for $n \in \mathbb{N}$. It is not invertible as soon as $\text{Card}(A) \geq 2$. Thus $(A^\mathbb{N}, S)$ is an example of a non invertible dynamical system on a Cantor space.

### 2.1.1 Recurrent dynamical systems

A point $x \in X$ in a topological dynamical system $(X, T)$ is recurrent if for every open set $U$ containing $x$ there is an $n \geq 1$ such that $T^n(x) \in U$ (and, in fact, then an infinity of such $n$, see Exercise 2.3).

A system $(X, T)$ is recurrent (or topologically transitive) if for every pair of nonempty open sets $U, V$ in $X$, there is an integer $n > 1$ such that $U \cap T^{-n}V \neq \emptyset$ (or equivalently $T^nU \cap V \neq \emptyset$, see Exercise 2.3).

**Proposition 2.1.3** The following conditions are equivalent for a topological dynamical system.
2.1. RECURRENT AND MINIMAL DYNAMICAL SYSTEMS

(i) \((X, T)\) is recurrent.

(ii) The set of recurrent points is dense.

(iii) There is a recurrent point \(x \in X\).

The proof is left as Exercise 2.4.

A morphism of dynamical systems from \((X, T)\) to \((X', T')\) is a continuous map \(\phi : X \to X'\) such that \(\phi \circ T = T' \circ \phi\) (see the diagram below).

\[
\begin{array}{ccc}
X & \xrightarrow{T} & X \\
\downarrow^{\phi} & & \downarrow^{\phi} \\
X' & \xrightarrow{T'} & X'
\end{array}
\]

If \(\phi\) is onto, it is called a factor map and \((X', T')\) is called a factor system of \((X, T)\).

A factor of a topological dynamical system inherits its dynamical properties. For instance, a factor of a recurrent system is recurrent.

When it is bijective, the morphism \(\phi\) is called an isomorphism of dynamical systems, or also a topological conjugacy (or simply a conjugacy). Since \(X\) is compact, the inverse is also continuous (Exercise 2.1) and thus \(\phi\) is a homeomorphism.

Conjugate systems are indistinguishable concerning their dynamical properties. It can be very difficult to exhibit a conjugacy between dynamical systems and much of what will follow in this book addresses this problem. In particular, we will be looking for invariants, that is properties shared by conjugate systems and easier to determine than the conjugacy itself. Even more interesting are complete invariants which characterize the conjugacy class.

2.1.2 Minimal dynamical system

Given a topological dynamical system \((X, T)\), a subset \(Y\) of \(X\) is stable if \(TY \subseteq Y\). The empty set and \(X\) are always (trivial) stable sets. As a stronger condition, the set \(Y \subseteq X\) is invariant if \(T^{-1}Y = Y\).

A topological dynamical system is minimal if \(X\) is the only closed and stable nonempty set. A factor of a minimal dynamical system is minimal (Exercise 2.3).

The positive orbit of a point \(x \in X\) in a dynamical system \((X, T)\) is the set \(O_+(x) = \{T^n x \mid n \geq 0\}\). Its orbit is the set \(O(x) = \cup_{n \geq 0} T^{-n} (O_+(x))\). Thus, we have also \(O(x) = \{y \in X \mid T^m x = T^n y \text{ for some } m,n \geq 0\}\) and when \(T\) is invertible, \(O(x) = \{T^n x \mid n \in \mathbb{Z}\}\).

The following characterization of minimal systems is sometimes used for definition.

**Proposition 2.1.4** A topological dynamical system is minimal if and only if the positive orbit of every point is dense in \(X\).
CHAPTER 2. TOPOLOGICAL DYNAMICAL SYSTEMS

Proof. Assume first that $(X,T)$ is minimal. For every $x \in X$, the closure of the positive orbit of $x$ is a closed stable nonempty set and thus it is equal to $X$. Conversely, let $Y \subset X$ be a closed stable nonempty set. For any $y \in Y$, the closure of the positive orbit of $y$ is contained in $Y$. Since it is equal to $X$, we conclude that $Y = X$.

It follows directly from Proposition 2.1.4 that if $(X,T)$ is minimal, for every nonempty open sets $U,V$ and every $x \in U$ there exists an $n \geq 1$ such that $T^n x \in V$. In particular, $T^n U \cap V \neq \emptyset$ and thus $(X,T)$ is recurrent.

The simplest example of a minimal dynamical system is a finite system, formed of a finite set $X = \{1,2,\ldots,n\}$ on which acts a circular permutation $T$.

A dynamical system is called aperiodic if it does not contain any finite nonempty system. A minimal system is aperiodic if and only if it is infinite.

As a second example, we find the rotations of the circle.

Example 2.1.5 consider the unit circle $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ and fix some $\lambda \in S^1$. Let $R_\lambda$ be the map defined by $R_\lambda(z) = \lambda z$. Then $(S^1, R_\lambda)$ is a dynamical system. It is minimal if and only if $\lambda$ is not a root of unity (Exercise 2.6). Otherwise it is a disjoint union of periodic systems.

We have introduced above (Example 2.1.2) the torus $T$ as the topological space $\mathbb{R}/\mathbb{Z}$. For $\alpha \in \mathbb{R}$, let $T_\alpha : T \to T$ be the map $T_\alpha(x) = x + \alpha$ and set $\lambda = \exp(2i\pi \alpha)$. The map $\phi : x \to e^{2i\pi x}$ is a homeomorphism from $T$ onto $S^1$ and $R_\lambda \circ \phi = \phi \circ T_\alpha$ because $R_\lambda \circ \phi(x) = R_\lambda e^{2i\pi x} = e^{2i\pi x} e^{2i\pi \alpha} = \phi \circ T_\alpha(x)$.

Consequently the topological dynamical systems $(S^1, R_\lambda)$ and $(T, T_\alpha)$ are isomorphic. The transformation $R_\alpha$ is called a rotation of angle $\alpha$. The isomorphism is the map $\phi : x \mapsto \exp(2i\pi x)$ (see the diagram below).

$$
\begin{array}{ccc}
T & \xrightarrow{T_\alpha} & T \\
\downarrow \phi & & \downarrow \phi \\
S^1 & \xrightarrow{R_\lambda} & S^1
\end{array}
$$

2.1.3 Induced systems

Let $(X,T)$ be a minimal topological dynamical system and let $U$ be a nonempty clopen subset of $X$. Since $X$ is minimal, for every $x \in X$, there is an $n > 0$ such that $T^n x \in U$ and we can define the integer

$$n(x) = \inf \{ n > 0 \mid T^n x \in U \}$$

called the entrance time of $x$ in $U$.

Since $U$ is clopen, the function $x \mapsto n(x)$ is continuous. Indeed, for each $n \geq 1$, the set of $x \in X$ such that $n(x) = n$ is $T^{-n}(U) \setminus \cup_{i=1}^{n-1} T^{-i}(U)$ which is open. Thus, the map $T_U : U \to U$ defined by

$$T_U(x) = T^n(x)$$
is continuous as a composition of continuous functions. It is called the \textit{induced transformation} on \( U \) and \( (U, T_U) \) is called an \textit{induced system} of \((X, T)\) or also a \textit{derivative} of \((X, T)\). This system is minimal. Indeed, the orbit under \( T \) of every point \( x \) of \( U \) is dense in \( X \) and thus its orbit under \( T_U \) is dense in \( U \).

For \( x \in U \), the integer \( n(x) \) is called the \textit{return time} to \( U \). For \( x \in X \), the function

\[
m(x) = \begin{cases} 
n(x) & \text{if } x \notin U \\
0 & \text{otherwise}
\end{cases}
\]

is called the \textit{waiting time} to access \( U \).

Note that the induced system can be defined even if \((X, T)\) is not minimal, provided the clopen set \( U \) is such that the return time \( n(x) \) is bounded on \( U \).

The inverse operation can be described as follows. Let \((X, T)\) be a topological dynamical system and let \( f : X \to \mathbb{N} \) be a continuous function. Set

\[
\hat{X} = \{(x, i) \mid x \in X, 0 \leq i < f(x)\}
\]

and define a map \( \hat{T} : \hat{X} \to \hat{X} \) by

\[
\hat{T}(x, i) = \begin{cases} 
(x, i + 1) & \text{if } i + 1 < f(x) \\
(Tx, 0) & \text{otherwise}
\end{cases}
\]

Then \((\hat{X}, \hat{T})\) is a topological dynamical system, which is minimal if \((X, T)\) is minimal (see Exercise 2.7). The map \( x \mapsto (x, 0) \) identifies \( X \) to the system induced by \( \hat{X} \) on \( X \times \{0\} \). The space \( \hat{X} \) defined above is called the \textit{tower} over \( X \) relative to \( f \) and the system \((\hat{X}, \hat{T})\) is called a \textit{primitive} of \( X \).

\textbf{Example 2.1.6} Consider the system \((X, T)\) with \( X = [0, 1] \) and \( Tx = x + \alpha \mod 1 \) of Example 2.1.1 for \( 1/2 < \alpha < 1 \). The transformation \( T_U \) induced by \( T \) on \( U = [0, \alpha] \) is given by

\[
T_U x = \begin{cases} 
x + 2\alpha - 1 & \text{if } x < 1 - \alpha \\
x + \alpha - 1 & \text{otherwise}
\end{cases}
\]

Thus, \((U, T_U)\) is isomorphic to the system \((X, T_{2 - (1/\alpha)})\) via the conjugacy \( x \mapsto 1/\alpha \).

\subsection*{2.1.4 Dynamical systems on Cantor spaces}

We have defined a Cantor space as a totally disconnected compact metric space without isolated points. Recall that a topological space is called totally disconnected if every connected component is reduced to a point (see Appendix A).

Let us give classical examples of Cantor spaces. There is first the original one, as defined originally by Cantor. The \textit{Cantor set} is the subset \( C \) of \([0, 1]\) defined by \( C = \cap_n C_n \) where \( C_0 = [0, 1] \) and

\[
C_{n+1} = \frac{C_n}{3} \cup \left( \frac{2}{3} + \frac{C_n}{3} \right).
\]
It is a Cantor space for the induced topology.

There is the abstract topological one, that we used for the definition. It is a compact metric space which is totally disconnected without isolated point or, equivalently, a topological space with a countable basis of the topology consisting of clopen sets and without isolated points (see Appendix A).

There is the algebraic one, the ring $\mathbb{Z}_p$ of $p$-adic integers, where $p$ is a prime number, for the $p$-adic topology which is induced by the distance defined for $x \neq y$ by $d(x, y) = p^{-n}$ if $p^n$ is the higher power of $p$ dividing $x - y$. An important example of a minimal Cantor system is the odometer $(\mathbb{Z}_p, T)$ where $T$ is the transformation defined by $T(x) = x + 1$.

Finally, there is the symbolic one that we have already seen and consists in considering the set $A^\mathbb{Z}$ of two-sided infinite sequences on a finite alphabet $A$.

From a topological point of view all these topological spaces are the same as they are homeomorphic (see the notes section for a reference to a proof).

A Cantor system is a dynamical system $(X, T)$ where $X$ is a $T$-stable Cantor set. The odometers are Cantor systems. A symbolic system is a dynamical system $(X, T)$ where $X$ is a $T$-stable closed subset of $A^\mathbb{Z}$. The transformation $T$ need not be the shift (see the example of the odometer). Observe that symbolic systems are not necessarily Cantor systems since closed subsets could have isolated points. But infinite minimal symbolic systems are Cantor systems (Exercise 2.9).

The pair $(A^\mathbb{Z}, S)$ is called the full shift (on the alphabet $A$). If $X$ is a closed shift-invariant subset of $A^\mathbb{Z}$, then the topological dynamical system $(X, T)$, where $T$ is the restriction of $S$ to $X$, is called a subshift of the full shift on the alphabet $A$ or a shift space, or a two-sided shift space. Thus, a shift space is a symbolic system. For the full shifts on any finite alphabet and in general for shift spaces, the transformation will usually be denoted by $S$. Thus, we will often use the notation $X$ instead of $(X, S)$ for a shift space.

Similarly one can define the one-sided full shift as follows. We still denote by $S$ the one-sided transformation, called the one-sided shift which is defined for $x \in A^\mathbb{N}$ by $y = Sx$ if $y_n = x_{n+1}$, as in Equation (2.1.1). The pair $(A^\mathbb{N}, S)$ is called the one-sided full shift on the alphabet $A$. If $X$ is a closed $S$-invariant subset of $A^\mathbb{N}$, the pair $(X, S)$ is called a one-sided shift space or one-sided subshift.

To every shift space $(X, S)$, one may associate a one-sided shift space $(Y, S)$ called its associated one-sided shift space by considering the set $Y$ of $y_0y_1\cdots$ such that $y_i = x_i$ $(i \geq 0)$ for some $x = (x_n)_{n \in \mathbb{Z}}$. The map $\theta : X \to Y$ defined by $\theta(x) = y$ is a surjective morphism.

Conversely, for every one-sided shift space $(Y, S)$ there is a unique two-sided shift space $(X, S)$ such that $(Y, S)$ is associated to $(X, S)$. Indeed, the set $X$ of sequences $x \in A^\mathbb{Z}$ such that $x_nx_{n+1}\cdots \in Y$ for every $n \in \mathbb{Z}$ is closed and shift-invariant. It is clear that $(X, S)$ is the unique shift space such that $Y = \theta(X)$. We also say that $X$ is the two-sided shift space associated to $Y$.

This shows that one-sided and two-sided shift spaces are closely related objects. In general, in this book, we consider two-sided shift spaces rather than one-sided shift spaces because it is often convenient to have a transformation which is invertible. In general, by a shift space, we mean a two-sided shift space.
Example 2.1.7 The golden mean shift is the set $X$ of two-sided sequences on $A = \{a, b\}$ with no consecutive $b$. Thus $X$ is the set of labels of two-sided infinite paths in the graph of Figure 2.1.1. It is recurrent, as one may easily verify. It is not minimal since it contains the one-point set $\{a^2\}$ which is closed and shift invariant.

Figure 2.1.1: The golden mean shift.

Example 2.1.8 Let $\varphi : a \to ab, b \to a$ be the Fibonacci substitution. Since $\varphi(a)$ begins with $a$, any $\varphi^n(a)$ is a prefix of $\varphi^{n+1}(a)$. Let $x \in \{a, b\}^\mathbb{N}$ be the sequence $x$ having all $\varphi^n(a)$ as prefixes. Thus

$$x = abaababa \cdots$$

It is known as the Fibonacci word (we prefer to keep the name ‘Fibonacci sequence’ for the well-known sequence $F_{n+1} = |\varphi^n(b)|$). The subshift of $\{a, b\}^\mathbb{N}$ which is the closure of the orbit $x$ is the one-sided Fibonacci shift. We will see that it is minimal (Example 2.4.13).

2.2 More on shift spaces

In this section, we develop in more detail the notions related to shift spaces and their language. We will see how the notions of recurrence and minimality can be expressed adequately for shift spaces. We will also introduce important notions like return words or Rauzy graphs.

2.2.1 Some combinatorics on words

Let $A$ be a set called an alphabet. We will generally assume that the alphabet is finite. A word over $A$ is an element of the free monoid generated by $A$, denoted by $A^*$. If $u = u_0u_1 \cdots u_{n-1}$ (with $u_i \in A$, $0 \leq i \leq n-1$) is a word, its length is $n$ and is denoted by $|u|$. For $a \in A$, we denote by $|u|_a$ the number of occurrences of the letter $a$ in $u$.

The empty word is denoted by $\varepsilon$. It is the unique word of length 0. The set of nonempty words over $A$ is denoted by $A^+$.

Two words $u, v$ are conjugate if $u = rs$ and $v = sr$ for some words $r, s$ or, equivalently, if $v$ is obtained from $u$ by a circular permutation of its letters. Conjugacy is an equivalence relation on words.
A word \( w \) is primitive if it is not a power of another word. Formally, \( w \) is primitive if \( w = u^n \) implies \( n = 1 \). A primitive word of length \( n \) has \( n \) distinct conjugates (Exercise 2.10).

A factor (also called a subword or a block) of a word \( u \) is a finite word \( y \) such that there exist two words \( v \) and \( w \) satisfying \( u = vyw \). When \( v \) (resp. \( w \)) is the empty word, we say that \( y \) is a prefix (resp. suffix) of \( u \). If \( k, l \) are integers such that \( 0 \leq k \leq l < |u| \), we let \( u_{[k,l]} \) denote the subword \( u_k u_{k+1} \cdots u_l \) of \( u \). We define \( u_{[k,l+1]} \) to be \( u_{[k,l]} \). If \( l < k \), then \( u_{[k,l]} \) is the empty word. If \( y \) is a factor of \( u \), the occurrences of \( y \) in \( u \) are the integers \( i \) such that \( u_{[i,i+|y|-1]} = y \).

If \( y \) has an occurrence in \( u \), we also say that \( y \) occurs in \( u \).

The reversal of a word \( u = u_0 u_1 \cdots u_n \), with \( u_i \in A \), is the word \( \hat{u} = u_n \cdots u_1 u_0 \).

The elements of \( A^K \), where \( K \) is equal to \( \mathbb{N}, \mathbb{N}^* \) or \( \mathbb{Z} \), are called sequences or infinite words. When we need to precise which kind of sequences we are dealing with we sometimes say right infinite sequence, or one-sided sequence when \( K = \mathbb{N} \), left-infinite sequence when \( K = \mathbb{N}^* \) and two-sided sequence in the last case. For \( x = (x_n)_{n \in \mathbb{Z}} \in A^K \), we let \( x^+ \) and \( x^- \) respectively denote the sequences \( (x_n)_{n \geq 0} \) and \( (x_n)_{n < 0} \). For \( x \in A^{-N} \) and \( y \in A^N \), we denote \( z = x \cdot y \) the two sided sequence \( z \) such that \( z = z^- \) and \( y = z^+ \). The notion of factor is naturally extended to sequences, as well as the notion of prefix when \( K = \mathbb{N} \). The set of subwords of length \( n \) of \( x \) is written \( \mathcal{L}_n(x) \) and the set of subwords of \( x \), or the language of \( x \), is denoted by \( \mathcal{L}(x) \). We also denote by \( \mathcal{L}_{\leq n}(x) \) the set of words of length at most \( n \) in \( \mathcal{L}(x) \).

The reversal of a right-infinite sequence \( x = x_0 x_1 \cdots \) is the left-infinite sequence \( \hat{x} = \cdots x_1 x_0 \).

For a finite word \( u \in A^+ \), we denote by \( u^\omega \) the right infinite sequence \( uu u \cdots \) and by \( u^\infty \) the two-sided infinite sequence \( \cdots uu u u u u \cdots \), where the dot is placed to the left of \( x_0 \). In this way, we have \( x^+ = u^\omega \).

The sequence \((p_n(x))_{n \geq 0}\) defined by \( p_n(x) = \text{Card}(\mathcal{L}_n(x)) \) is the factor complexity (or word complexity or simply the complexity) of \( x \). Note that \( p_0(x) = 1 \), that \( p_n(n) \leq p_{n+1}(x) \) and that \( p_{n+m}(x) \leq p_n(x) p_m(x) \) for all \( n, m \geq 0 \).

**Example 2.2.1** Let \( x \) be the Fibonacci word (Example 2.1.8). We have \( p_1(x) = 2 \) since every letter \( a, b \) appears in \( x \). Next \( p_2(x) = 3 \) since \( \mathcal{L}(x) = \{aa, ab, ba\} \) as one may verify. We will see that actually, one has \( p_n(x) = n + 1 \) for all \( n \geq 1 \) (see Section 2.3).

The sequence \( x \in A^\mathbb{N} \) is eventually periodic if there exist a word \( u \) and a nonempty word \( v \) such that \( x = u^\omega v^\omega \), where \( v^\omega = vvv \cdots \). A sequence that is not eventually periodic is called aperiodic. It is periodic if \( u \) is the empty word.

The following result is classical.

**Theorem 2.2.2** (Morse, Hedlund) Let \( x \) be a two-sided sequence. The following conditions are equivalent.

(i) For some \( n \geq 1 \), one has \( p_n(x) \leq n \).
2.2. MORE ON SHIFT SPACES

(ii) For some \( n \geq 1 \), one has \( p_n(x) = p_{n+1}(x) \).

(iii) \( x \) is periodic.

Moreover, in this case, the least period of \( x \) is \( \max p_n(x) \).

Proof. (i) \( \Rightarrow \) (ii). Since \( p_n(x) \leq p_{n+1}(x) \) for all \( n \geq 0 \), the hypothesis implies that \( p_n(x) = p_{n+1}(x) \) for some \( n \geq 0 \).

(ii) \( \Rightarrow \) (iii). For every \( w \in \mathcal{L}_n(x) \), there is a unique letter \( a \in A \) such that \( wa \in \mathcal{L}_{n+1}(x) \). This implies that two consecutive occurrences of a word \( u \) of length \( n \) in \( x \) are separated by a fixed word depending only on \( u \) and thus that \( x \) is periodic.

(iii) \( \Rightarrow \) (i) is obvious.

Let \( n \) be the least period of \( x \). Since a primitive word of length \( n \) has \( n \) distinct conjugates, we have \( p_n(x) = n \) and \( p_m(x) = n \) for all \( m \geq n \).

Thus, by Proposition 2.2.2 either \( p_n(x) \geq n + 1 \) for all \( n \geq 1 \) or \( p_n(x) \) is eventually constant. The case \( p_n(x) = n + 1 \) for all \( n \geq 1 \) corresponds to the Sturmian sequences (see below).

Proposition 2.2.3 The following conditions are equivalent for \( x \in A^\mathbb{Z} \).

(i) Every \( u \in \mathcal{L}(x) \) has at least two occurrences in \( x \).

(ii) Every \( u \in \mathcal{L}(x) \) has an infinite number of occurrences in \( x \).

(iii) For every \( u, w \in \mathcal{L}(x) \) there is \( v \in \mathcal{L}(x) \) such that \( uvw \in \mathcal{L}(x) \).

Proof. (i) \( \Rightarrow \) (ii). Assume that \( u \) has a finite number of occurrences in \( x \). Let \( v \) be the shortest prefix of \( x \) containing all these occurrences. Since \( v \) has a second occurrence in \( x \), we have a contradiction.

(ii) \( \Rightarrow \) (iii). Assume that \( u = x_{[i,j)} \). Since \( w \) has an infinite number of occurrences in \( x \), there is an index \( k \) larger than \( j \) such that \( w = x_{k,\ell} \). Set \( v = x_{[j,k)} \). Then \( uvw = x_{[i,\ell)} \).

(iii) \( \Rightarrow \) (i) is clear, considering \( u = w \).

A word \( u \) is recurrent in \( x \in A^\mathbb{Z} \) if condition (ii) above is satisfied. The sequence \( x \) itself is called recurrent if one of the conditions is satisfied. Thus a sequence \( x \in A^\mathbb{Z} \) is recurrent if and only if \( x \) is a recurrent point of the full shift. We could of course use Proposition 2.1.3 to prove Proposition 2.2.3. We also say that the language \( \mathcal{L}(x) \) is recurrent or irreducible if condition (iii) is satisfied.

Example 2.2.4 Let \( A = \{a, b\} \) and let \( x = abaaabbabb \cdots \) be the sequence formed of all words on \( A \) in radix order (that is, ordered first by length, then lexicographically). It is a recurrent sequence in which all words on \( A \) appear, that is, such that \( \mathcal{L}(x) = A^* \). As a variant of this example, the Champernowne sequence is the sequence \( x = 01234567891011121314151617181920 \cdots \) formed of the decimal representation of all numbers in increasing order.
Proposition 2.2.5 The following conditions are equivalent for a sequence $x \in A^\mathbb{N}$.

(i) Every $u \in \mathcal{L}(x)$ occurs infinitely often in $x$ and the greatest difference of two successive occurrences of $u$ is bounded.

(ii) For every $u \in \mathcal{L}(x)$, there is an $n \geq 1$ such that $u$ occurs in every word of $\mathcal{L}_n(x)$.

Proof. (i) $\Rightarrow$ (ii). Let $k$ be the maximum of the differences between successive occurrences of $u$. Then $u$ appears in every word of $\mathcal{L}_n(x)$ for $n = |u| + k$.

(ii) $\Rightarrow$ (i) is clear. $\blacksquare$

A sequence $x \in A^\mathbb{N}$ is uniformly recurrent if one of these conditions hold. We also say in this case that $\mathcal{L}(x)$ is uniformly recurrent.

Example 2.2.6 The Fibonacci word $x$ (Example 2.2.1) is uniformly recurrent. Indeed, let $u \in \mathcal{L}(x)$ and let $n$ be the minimal integer such that $u$ is a factor of $\varphi^n(a)$. Then $\varphi^{n+2}(a) = \varphi^n(a)\varphi^n(b)\varphi^n(a)$ Then $u$ has a second occurrence in $\varphi^{n+2}(a)$, and thus at bounded distance of the first one.

A sequence $x \in A^\mathbb{N}$ is linearly recurrent with constant $K$ if it is recurrent and the greatest difference between successive occurrences of $u$ is bounded by $K|u|$.

Most of this terminology extends naturally to a two-sided infinite sequence. In particular, $x \in A^\mathbb{Z}$ is periodic if $x = v^\infty$ for some $v \in A^+$. In this case, $x^+$ is periodic. Similarly, $x \in A^\mathbb{Z}$ is recurrent (resp. uniformly recurrent) if $\mathcal{L}(x)$ is recurrent (resp. uniformly recurrent). The same extension holds for linearly recurrent sequences.

Proposition 2.2.7 Let $x$ be a two-sided sequence which is linearly recurrent with constant $K$. Then

1. Every word of $\mathcal{L}_n(x)$ appears in every word of $\mathcal{L}_{(K+1)n-1}(x)$.

2. The factor complexity of $x$ is at most $Kn$.

Moreover, if $x$ is not periodic, it is $(K+1)$-power free, that is for every nonempty word $u \in \mathcal{L}(x)$, $u^n \in \mathcal{L}(x)$ implies $n \leq K$.

Proof. 1. Let $u \in \mathcal{L}_n(x)$. Since two successive occurrences of $u$ differ by at most $Kn$, every word of $\mathcal{L}_{(K+1)n-1}$ contains $u$ as a factor.

2. Set $p_n(x) = \text{Card}(\mathcal{L}_n(x))$. A word of length $(K+1)n - 1$ has at most $Kn$ factors of length $n$. Thus, by Assertion 1, $p_n(x) \leq Kn$.

Assume now that $u^{K+1} \in \mathcal{L}(x)$. Set $n = |u|$. The word $u^{K+1}$ has length $(K+1)n$ and at most $n$ factors of length $n$. By Assertion 1, this implies $p_n(x) \leq n$. By Theorem 2.2.2, this implies that $x$ is periodic. $\blacksquare$

Clearly, a linearly recurrent sequence is uniformly recurrent but the converse is not true (see Exercise 2.40).
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Example 2.2.8 The Fibonacci word is linearly recurrent (Exercise 2.13).

2.2.2 The language of a shift space

Let $X$ be a shift space. The language of $X$ is the set $\mathcal{L}(X)$ of subwords of elements belonging to $X$. The set $\mathcal{L}(X)$ is of course that union of the languages $\mathcal{L}(x)$ for $x \in X$. We also denote by $\mathcal{L}_n(X)$ the set of words of length $n$ in $\mathcal{L}(X)$ and by $\mathcal{L}_{\leq n}(X)$ the set of those of length at most $n$.

The same notation can be used for the language of a one-sided shift space. The languages of a two-sided shift space and of its associated one-sided shift space are actually the same.

A set $L$ of words on the alphabet $A$ is factorial if it contains the factors of its elements. It is said to be extendable if for every $u \in L$, there are letters $a, b \in A$ such that $au b \in L$. The language of a shift space is factorial and extendable and, conversely, for every factorial extendable set $L$, there is a unique shift space $X$ such that $\mathcal{L}(X) = L$ (Exercise 2.12).

For two words $u, v$ such that $uv \in \mathcal{L}(X)$, the set
$$[u \cdot v]_X = \{ x \in X \mid x_{|u|} = uv \}$$
is nonempty. It is called the cylinder with basis $(u, v)$. If $u$ is the empty word we set $[v]_X = [u \cdot v]_X$ or equivalently $[v]_X = \{ x \in X \mid x_{|v|} = v \}$. Any cylinder is open and every open set in a shift space is a union of cylinders. The clopen sets in $X$ are the finite unions of cylinders.

For any sequence $x \in A^\mathbb{Z}$ there is a smallest shift space containing $x$ called the subshift generated by $x$ and denoted $\Omega(x)$. It is the closure of the orbit of $x$.

The following property is sometimes taken for definition of shift spaces.

Proposition 2.2.9 A set $X \subset A^\mathbb{Z}$ is a shift space if and only if there is a set $F \subset A^*$ of finite words such that $X$ is the set $X_F$ of infinite words without factor in $F$.

Proof. Indeed, such a set is clearly closed and invariant by the shift. Conversely, let $X$ be a shift space. Since $X$ is closed, its complement $Y = A^\mathbb{Z} \setminus X$ is open. Thus for every $y \in Y$, there is a cylinder $[u_y, v_y]_{\mathbb{A}^z}$ containing $y$ and contained in $Y$. Set $F = \{ u_y v_y \mid y \in Y \}$. Then $X_F \subset X$ since $X_F \cap Y = \emptyset$. Conversely, if $x \in X$ has a factor $u_y v_y$ in $F$, then $T^n x$ is in $[u_y, v_y]_{\mathbb{A}^z}$ for some $n \in \mathbb{Z}$ and this is a contradiction since $X$ is shift invariant. Thus $X = X_F$.

Let $X$ and $Y$ be shift spaces on alphabets $A, B$ respectively. Given integers $m, n$ such that $-m \leq n$, let $f : \mathcal{L}_{m+n+1}(X) \to B$ be a map called a block map. We call $m$ the memory and $n$ the anticipation of the block map $f$.

The sliding block code induced by $f$ is the map $\varphi : X \to B^\mathbb{Z}$ defined by $y = \varphi(x)$ if (see Figure 2.2.1)
$$y_i = f(x_{i-m} \cdots x_{i+n}) \quad (i \in \mathbb{Z}).$$

When $\varphi(X) \subset Y$, we write $\varphi : X \to Y$. 
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Figure 2.2.1: The sliding block code.

\[
x_{i-m} \quad \cdots \quad x_{i+n}
\]
\[
f \quad y_i
\]

Theorem 2.2.10 (Curtis, Hedlund, Lyndon) Let \(X\) and \(Y\) be shift spaces. A map \(\varphi : X \to Y\) is a morphism of dynamical systems if and only if it is a sliding block code.

Proof. A sliding block code is clearly continuous and commutes with the shifts, that is \(\varphi \circ T = S \circ \varphi\). Thus it is a morphism of dynamical systems. Conversely, let \(\varphi : X \to Y\) be a morphism. Since the alphabet \(B\) of the shift \(Y\) is finite, the set \(\varphi^{-1}[b]\) is, for every \(b \in B\), a clopen set. Thus there is an integer \(n\) such that \(\varphi(x)\) depends only on \(x[{-n,n}]\). Set \(f(x_{-n} \cdots x_n) = \varphi(x)_0\). Then \(\varphi\) is the sliding block code associated to \(f\).

The factor complexity (or word complexity or simply the complexity) of the shift space \((X,S)\) is the sequence \(p_n(X) = \text{Card}(L_n(X))\).

Observe that \(p_0(X) = 1\) and that \(p_n(X) \leq p_{n+1}(X)\). Indeed, for every \(w \in L_n(X)\), there is a letter \(a \in A\) such that \(wa \in L(X)\).

Example 2.2.11 Let \(X\) be the golden mean shift (Example 2.1.7). The factor complexity of \(X\) is \(p_n(X) = F_{n+1}\) where \(F_n\) is the Fibonacci sequence defined by \(F_0 = 0, F_1 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for \(n \geq 1\). Indeed, the number of words in \(L_{n+1}(X)\) ending with \(a\) is equal to \(p_n(X)\) and the number of those ending with \(b\) is \(p_{n-1}(X)\) since the \(b\) has to be preceded by \(a\). Thus \(p_{n+1}(X) = p_n(X) + p_{n-1}(X)\).

The following result is the counterpart for shift spaces of Theorem 2.2.2.

Theorem 2.2.12 (Morse, Hedlund) Let \(X\) be a shift space. The following conditions are equivalent.

(i) For some \(n \geq 1\), one has \(p_n(X) \leq n\).

(ii) For some \(n \geq 1\), one has \(p_n(X) = p_{n+1}(X)\).

(iii) \(X\) is finite.

Proof. (i) \(\Rightarrow\) (ii). Since \(p_n(X) \leq p_{n+1}(X)\) for all \(n \geq 0\), the hypothesis implies that \(p_n(X) = p_{n+1}(X)\) for some \(n \geq 0\).

(ii) \(\Rightarrow\) (iii). By Proposition 2.2.2 every \(x \in X\) is periodic and its period is bounded by \(\max p_n(X)\).
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(iii) ⇒ (i) is obvious.

Thus, by Proposition 2.2.12 either \( p_n(X) \geq n + 1 \) for all \( n \geq 1 \) or \( p_n(X) \) is eventually constant. The case \( p_n(X) = n + 1 \) for all \( n \geq 1 \) corresponds to the Sturmian shifts (see below).

A shift space \( X \) is irreducible if the language \( \mathcal{L}(X) \) is irreducible, that is, if for every \( u, v \in \mathcal{L}(X) \) there is a \( w \) such that \( uuv \in \mathcal{L}(X) \).

**Proposition 2.2.13** A shift space is recurrent if and only if it is irreducible.

*Proof.* Assume first that \( X \) is recurrent and consider \( u, v \in \mathcal{L}(X) \). Let \( n > |u| \) be such that \( S^{-n}([v]_X) \cap [u]_X \neq \emptyset \). Then every \( x \in S^{-n}([v]_X) \cap [u]_X \) is in \( [uuv]_X \) for some \( w \). Thus \( uuv \in \mathcal{L}(X) \), showing that \( X \) is irreducible.

Conversely, assume that \( X \) is irreducible. Let \( U, V \) be open sets in \( X \). We can find cylinder sets \( [u]_X \subset U \) and \( [v]_X \subset V \). Since \( X \) irreducible there is some word \( w \) such that \( uuv \in \mathcal{L}(X) \). Then \( [uuv]_X \) is nonempty and in \( S^{-n}V \cap U \) for \( n = |uuv| \). Thus \( X \) is recurrent.

A shift space is of course irreducible if and only if it is recurrent as a topological dynamical system. Thus we could also have used Proposition 2.1.3 to prove Proposition 2.2.13.

Thus, for a shift space, the property of being recurrent can be expressed by a property of its language \( \mathcal{L}(X) \). Likewise, we can translate the property of being minimal. A shift space \( X \) is uniformly recurrent if the language \( \mathcal{L}(X) \) is uniformly recurrent, that is, if for every \( u \in \mathcal{L}(X) \) there is an \( n \geq 1 \) such that \( u \) is a factor of every word in \( \mathcal{L}_n(X) \).

**Proposition 2.2.14** The following conditions are equivalent for a shift space \((X, S)\).

(i) The shift space \( X \) is minimal.

(ii) Every \( x \in X \) is uniformly recurrent and \( \mathcal{L}(x) = \mathcal{L}(X) \).

(iii) The shift space \( X \) is uniformly recurrent.

*Proof.* (i) ⇒ (ii). Let \( u \in \mathcal{L}(X) \). Since \( [u]_X \) is clopen and \( X \) is minimal, for every \( x \in X \), the entrance time

\[
n(x) = \min\{n > 0 \mid T^n x \in [u]_X\}
\]

exists and is bounded. This shows that every \( x \in X \) is uniformly recurrent and that \( \mathcal{L}(x) = \mathcal{L}(X) \).

(ii) ⇒ (iii) Let \( n \) be the maximum of the integers \( n(x) \) defined previously. Then \( u \) is a factor of every word of \( \mathcal{L}(X) \) of length \( n + |u| \).

(iii) ⇒ (i) is clear since the orbit of every \( x \in X \) is dense.

Again, we could have used Proposition 2.1.3 to prove Proposition 2.2.14.
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2.2.3 Special words

For \( w \in \mathcal{L}(X) \), there is at least one letter \( a \in A \) such that \( wa \in \mathcal{L}(X) \) and symmetrically, at least one letter \( a \in A \) such that \( aw \in \mathcal{L}(X) \). The word \( w \) is called right-special if there is at least two letters \( a \in A \) such that \( wa \in \mathcal{L}(X) \). Symmetrically, \( w \) is left-special if there is at least two letters \( a \in A \) such that \( aw \in \mathcal{L}(X) \). It is bispecial if it is both left and right-special.

Special words are closely linked with the factor complexity of a shift space. Let indeed \( p_n(X) = \text{Card}(\mathcal{L}_n(X)) \) be the factor complexity of the shift \((X,S)\) on the alphabet \( A \). For all \( n \geq 0 \),

\[
\begin{align*}
    s_n(X) &= p_{n+1}(X) - p_n(X), \\
    b_n(X) &= s_{n+1}(X) - s_n(X).
\end{align*}
\]

For a word \( w \in \mathcal{L}(X) \), let

\[
\begin{align*}
    \ell_X(w) &= \text{Card}\{a \in A \mid aw \in \mathcal{L}(X)\} \\
    r_X(w) &= \text{Card}\{b \in A \mid wb \in \mathcal{L}(X)\} \\
    e_X(w) &= \text{Card}\{(a,b) \in A \times A \mid awb \in \mathcal{L}(X)\}
\end{align*}
\]

Thus \( \ell_X(w) > 1 \) (resp. \( r_X(w) > 1 \)) if and only if \( w \) is left-special (resp. right-special). Define also the multiplicity of \( w \in \mathcal{L}(X) \) as

\[
m_X(w) = e_X(w) - \ell_X(w) - r_X(w) + 1.
\]

The word \( w \) is called neutral if \( m_X(w) = 0 \).

**Proposition 2.2.15** We have for all \( n \geq 0 \),

\[
\begin{align*}
    s_n(X) &= \sum_{w \in \mathcal{L}_n(X)} (\ell_X(w) - 1) = \sum_{w \in \mathcal{L}_n(X)} (r_X(w) - 1) \quad (2.2.1) \\
    b_n(X) &= \sum_{w \in \mathcal{L}_n(X)} m_X(w). \quad (2.2.2)
\end{align*}
\]

In particular, the number of left-special (resp. right-special) words of length \( n \) is bounded by \( s_n(X) \).

**Proof.** We have

\[
\begin{align*}
    \sum_{w \in \mathcal{L}_n(X)} (\ell_X(w) - 1) &= \sum_{w \in \mathcal{L}_n(X)} \ell_X(w) - \text{Card}(\mathcal{L}_n(X)) \\
    &= \text{Card}(\mathcal{L}_{n+1}(X)) - \text{Card}(\mathcal{L}_n(X)) = p_{n+1}(X) - p_n(X) \\
    &= s_n(X)
\end{align*}
\]

with the same result for \( \sum_{w \in \mathcal{L}_n(X)} (r_X(w) - 1) \). Next,
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\[ \sum_{w \in \mathcal{L}_n(X)} m_X(w) = \sum_{w \in \mathcal{L}_n(X)} (e_X(w) - \ell_X(w) - r_X(w) + 1) = p_{n+2}(X) - 2p_{n+1}(X) + p_n(X) = s_{n+1}(X) - s_n(X) = b_n(X). \]

2.2.4 Return words

Let \( X \) be a shift space. For \( w \in \mathcal{L}(X) \) a right return word to \( w \) is a word \( u \) such that \( uw \) is in \( \mathcal{L}(X) \), has \( w \) as a proper suffix and has no factor \( w \) which is not a prefix or a suffix. Thus a right return word is a word \( u \) such that, reading from left to right and have already seen \( w \), reading \( u \), one sees again the word \( w \) for the first time.

![Figure 2.2.2: A right return word](image)

For example, in the golden mean shift, the word \( u = aab \) is a right return word to \( w = b \).

Symmetrically, a left return word to \( w \) is a word \( u \) such that \( uw \) is in \( \mathcal{L}(X) \), has \( w \) as a proper prefix and has no other factor \( w \).

We denote by \( R_X(w) \) (resp. \( R'_X(w) \)) the set of right (resp. left) return words to \( w \). The set \( R_X(w) \) is a prefix code, that is, no word in \( R_X(w) \) is a prefix of another one. The following property of return words is easy to verify.

**Proposition 2.2.16** Every word \( w \in \mathcal{L}(X) \) which begins and ends with \( u \) has a unique factorisation \( w = u w_1 w_2 \cdots w_n \) with \( w_i \in R_X(u) \) and \( n \geq 0 \).

For example, if \( X \) is the golden mean shift, we have

\[ R_X(b) = \{ab, aab, aaab, \ldots\} \] and \( R'_X(b) = \{ba, baa, \ldots\} \).

Clearly a recurrent shift space \((X, S)\) is minimal if and only if \( R_X(w) \) is finite for every \( w \in \mathcal{L}(X) \).

2.2.5 Rauzy graphs

Let \((X, S)\) be a shift space on the alphabet \( A \). The Rauzy graph of \( X \) of order \( n \), denoted \( \Gamma_n(X) \), is the following labeled graph. The set of vertices of \( \Gamma_n(X) \) is the set \( \mathcal{L}_{n-1}(X) \) and the set of edges is \( \mathcal{L}_n(X) \). The origin and end of the edge \( w \) are the words \( u, v \) such that \( w = ua = bv \) with \( a, b \in A \). The label of the edge \( w \) is \( a \).
Example 2.2.17 Let $X$ be the Fibonacci shift. The Rauzy graphs of order $n = 1, 2, 3$ are represented in Figure 2.2.3 (with the edge from $w = ua$ labeled $a$).

Every infinite path $\cdots \rightarrow p_{i-1} \rightarrow p_i \rightarrow p_{i+1} \rightarrow \cdots$ in $\Gamma_n(X)$ has a label, which is the sequence $(a_i)_{i \in \mathbb{Z}}$. The set of these labels is a shift space $X_n$. We have $X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots \supset X$ and $X = \cap_{n \geq 0} X_n$. Thus the sequence $X_n$ approximates $X$ from above. A graph is strongly connected if for every pair of vertices $v, w$ there is a path from $v$ to $w$.

Proposition 2.2.18 A shift space $X$ is recurrent if and only if all graphs $\Gamma_n(X)$ are strongly connected.

Proof. Assume that $X$ is recurrent (or, equivalently, irreducible). If $u, v \in L_{n-1}(X)$, there is some $w \in L(X)$ such that $uvw \in L(X)$. But then there is a path labeled $uv$ from $u$ to $v$ in $\Gamma_n(X)$. Thus $\Gamma_n(X)$ is strongly connected.

Conversely, assume that all $\Gamma_n(X)$ are strongly connected. Consider $u, v \in L(X)$. Let $p, s$ be such that $pu, vs$ have the same length $n$. Since $\Gamma_{n+1}(X)$ is strongly connected, there is a path from $pu$ to $vs$ in $\Gamma_{n+1}(X)$, which can be chosen of length at least $n$. Let $w = rvs$ be the label of this path. Then $puw = purvs$ is in $L(X)$, which implies that $urv \in L(X)$. Thus $X$ is irreducible.

2.2.6 Higher block shifts

Let $X$ be a shift space on the alphabet $A$ and let $k \geq 1$ be an integer. Let $f : L_k(X) \rightarrow A_k$ be a bijection from the set $L_k(X)$ of blocks of length $k$ of $X$ onto an alphabet $A_k$. The map $\gamma_k : X \rightarrow A_k^\mathbb{Z}$ defined for $x \in X$ by $y = \gamma_k(x)$ if for every $n \in \mathbb{Z}$

$$y_n = f(x_n \cdots x_{n+k-1})$$

is the $k$-th higher block code on $X$. The shift space $(X^{(k)}, S)$ where $X^{(k)} = \gamma_k(X)$ is called the $k$-th higher block presentation of $X$. The higher block code is an
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\[
\begin{array}{cccc}
  x_n & x_{n+1} & \cdots & x_{n+k-1} \\
  f & & & y_n \\
\end{array}
\]

Figure 2.2.4: The \(k\)-th higher block code.

isomorphism of dynamical systems and the inverse of \(\gamma_k\) is the map \(y \mapsto x\) such that \(x_n\) is the first letter of \(f^{-1}(y_n)\) for all \(n\).

We sometimes, when no confusion arises, identify \(A_k\) and \(L_k(X)\) and write simply \(y_0y_1\cdots = (x_0x_1\cdots x_{k-1})(x_1x_2\cdots x_k)\cdots\).

**Example 2.2.19** Consider again the golden mean shift \((X,S)\) (Example 2.1.7). We have \(L_3(X) = \{aaa, aab, aba, baa, bab\}\). Set \(f : aaa \to x, aab \to y, aba \to z, baa \to t, bab \to u\). The third higher block shift \(X^{(3)}\) of \(X\) is the set of two-sided infinite paths in the graph of Figure 2.2.5 on the left (this graph is, up to the labeling, the Rauzy graph \(\Gamma_3(X)\), see below).

Note that for \(n \geq k\), the Rauzy graph \(\Gamma_n(X^{(k)})\) is the same, up to the labels, as the graph \(\Gamma_{n+k-1}(X)\). As an example, the graph of Figure 2.2.5 on the left can be identified either with \(\Gamma_2(X^{(2)})\) or with \(\Gamma_3(X)\) (see Figure 2.2.5 on the right).

### 2.3 Shifts of finite type

A shift space \(X\) is of finite type if \(\mathcal{L}(X) = A^* \setminus A^*IA^*\) where \(I \subset A^*\) is finite set. In other words, a two-sided infinite words \(x\) is in \(X\) if and only if it has no factor in the finite set \(I\). The elements of \(I\) are called the **forbidden blocks** of \(X\).

The class of shifts of finite type is closed under conjugacy (Exercise 2.1.15).

A well-know example is the **golden mean shift** \((X,S)\) where \(X\) is the set of two-sided sequences on \(A = \{a,b\}\) with no consecutive \(b\). Thus the set of forbidden blocks is \(I = \{bb\}\) and \(X\) is the set of labels of two-sided infinite paths in the graph of Figure 2.1.1.

As a more generic example, for every shift space \(X\) and \(n \geq 1\), the set \(X_n\) of labels of infinite paths in the Rauzy graph \(\Gamma_n(X)\) is a shift of finite type.
Indeed, it is defined by the finite set of forbidden blocks which is the set of words of length \( n \) which are not in \( \mathcal{L}_n(X) \).

Given a finite graph \( G = (V, E) \), the *edge shift* on \( G \) is the shift space \((X, S)\) where \( X \subset \mathbb{Z}^\mathbb{Z} \) is the set of biinfinite paths in \( G \) and \( S \) is the shift on \( \mathbb{Z}^\mathbb{Z} \). An edge shift is a shift of finite type, since it is defined by forbidden blocks of length 2. Moreover, if the graph \( G \) is strongly connected, the edge shift on \( G \) is irreducible (and thus recurrent).

**Proposition 2.3.1** Any shift of finite type is conjugate to an edge shift on some graph. The shift is recurrent if and only if the graph can be chosen strongly connected.

The proof is left as an exercise (Exercise 2.16).

**Example 2.3.2** The edge shift on the graph \( G \) represented in Figure 2.3.1 is conjugate to the golden mean shift by the 1-block map \( e \mapsto a, f \mapsto b, g \mapsto a \).

![Figure 2.3.1: An edge shift](image)

2.4 Substitution shifts

Let \( A \) and \( B \) be finite alphabets. By a **morphism** from \( A^* \) to \( B^* \) we mean a morphism of free monoids, that is a map \( \sigma : A^* \to B^* \) such that \( \sigma(\varepsilon) = \varepsilon \) and \( \sigma(uv) = \sigma(u)\sigma(v) \) for all \( u, v \in A^* \). When \( \sigma(A) = B \), we say \( \sigma \) is a **letter-to-letter morphism**. Thus, letter-to-letter morphisms are onto.

We set \( |\sigma| = \max_{a \in A} |\sigma(a)| \) and \( \langle \sigma \rangle = \min_{a \in A} |\sigma(a)| \). The morphism \( \sigma \) is of **constant length** if \( \langle \sigma \rangle = |\sigma| \). It is is **growing** if \( \lim_{n \to \infty} \langle \sigma^n \rangle = +\infty \) or, equivalently, if \( |\sigma^n(a)| \to \infty \) for every \( a \in A \).

We say that a morphism \( \sigma \) is **erasing** if there exists \( b \in A \) such that \( \sigma(b) \) is the empty word and **non-erasing** otherwise. A growing morphism is non-erasing. If \( \sigma \) is non-erasing, it induces, by infinite concatenation, a map from \( A^\mathbb{N} \) to \( B^\mathbb{N} \) defined for \( x \in A^\mathbb{N} \) by

\[
\sigma(x) = \sigma(x_0)\sigma(x_1) \cdots
\]

and a map from \( A^\mathbb{Z} \) to \( B^\mathbb{Z} \) defined for \( x \in A^\mathbb{Z} \) by

\[
\sigma(x) = \cdots \sigma(x_{-1}) \cdot \sigma(x_0)\sigma(x_1) \cdots
\]

These maps are also denoted by \( \sigma \).
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The language of the morphism \( \sigma : A^* \to A^* \) is the set \( \mathcal{L}(\sigma) \) of words occurring in some \( \sigma^n(a) \), \( a \in A \), \( n \in \mathbb{N} \) and \( X(\sigma) \) will denote the set of sequences \( y \in A^\mathbb{Z} \) whose subwords belong to \( \mathcal{L}(\sigma) \). The set \( X(\sigma) \) is clearly a shift space since it is closed and invariant. It is called the shift generated by \( \sigma \).

By definition \( \mathcal{L}(\sigma) \subset \mathcal{L}(\sigma^n) \) and \( X(\sigma) \subset X(\sigma^n) \) for all \( n \geq 1 \) (the equality does not always hold, see Exercise 2.20). Observe that \( \mathcal{L}(\sigma) \) is finite if and only if \( X(\sigma) \) is empty. Observe also that \( \mathcal{L}(X(\sigma)) \subset \mathcal{L}(\sigma) \) but that the equality might not hold in general. For example, if \( \sigma : a \to a, b \to ba \), then \( X(\sigma) \) is reduced to \( a^\mathbb{Z} \) and thus \( b \) is not a factor of \( X(\sigma) \). We will soon reduce to a class of morphisms (the substitutions) where this kind of phenomenon cannot happen.

Example 2.4.1 The Fibonacci morphism (Example 2.1.8) is defined by \( \varphi : a \mapsto ab, b \mapsto a \). The shift space associated to \( \varphi \) is called the Fibonacci shift. An example of a two-sided infinite sequence in \( X(\varphi) \) is \( z = \tilde{x} \cdot x \) where \( x \) is the Fibonacci word. Indeed, \( \varphi^2(a) = aba \) ends with \( a \) and thus there is a unique left-infinite word \( y \) having all \( \varphi^{2n}(a) \) as suffixes. Since \( \varphi^2(a) \) is a palindrome, we have \( y = \tilde{x} \). Then all factors of \( z \) are factors of some \( \varphi^{2n}(aa) \) and thus \( z \in X(\varphi) \).

Example 2.4.2 The Thue-Morse morphism is the morphism defined by \( \tau : a \mapsto ab, b \mapsto ba \). The associated shift space is the Thue-Morse shift.

A morphism \( \sigma : A^* \to A^* \) is called periodic if its associated shift space is periodic and aperiodic otherwise. When \( \sigma \) is periodic, its period can be bounded a priori and thus this property is decidable for a morphism (Exercise 2.27).

Example 2.4.3 The shift space associated to the morphism \( \varphi : a \mapsto aba, b \mapsto b \) is periodic. Indeed, \( \varphi(ab) = (ab)^2 \) and thus the set \( X(\varphi) \) is formed of two points.

2.4.1 Fixed points

Let \( \sigma : A^* \to A^* \) be a morphism. If there exist a letter \( a \in A \) such that \( \sigma(a) \) begins with \( a \) and if, moreover, \( \lim_{n \to +\infty} |\sigma^n(a)| = +\infty \), then \( \sigma \) is said to be right-prolongable on \( a \).

Suppose that \( \sigma \) is right-prolongable on \( a \in A \). Since for all \( n \in \mathbb{N} \), \( \sigma^n(a) \) is a prefix of \( \sigma^{n+1}(a) \) and because \( |\sigma^n(a)| \) tends to infinity with \( n \), there is a unique right infinite word denoted \( \sigma^\omega(a) \) which has all \( \sigma^n(a) \) as prefixes. Indeed, for every \( i \geq 0 \), the \( i \)-th letter of \( x \) is the common \( i \)-th letter of all words \( \sigma^n(a) \) longer than \( i \). Then \( x = \sigma^\omega(a) \) is a fixed point of \( \sigma \) since \( \sigma(x) = x \).

By an admissible one-sided fixed point of \( \sigma \), we mean a one-sided infinite sequence \( x = \sigma^\omega(a) \) where \( \sigma \) is right-prolongable on \( a \).

Observe that a morphism can have other fixed points, either finite or infinite. In fact, if \( \sigma \) is the morphism \( a \to a, b \to ba \), then \( a \) is a finite fixed point since \( \sigma(a) = a \) and \( a^\omega \) is also a fixed point since \( \sigma(a^\omega) = a^\omega \).
Proposition 2.4.4 Every growing morphism has a power which has an admissible one-sided fixed point.

Proof. Let $\sigma : A^* \to A^*$ be a growing morphism. Let $a \in A$. Since $\sigma$ is growing, it is non-erasing and thus all $\sigma^n(a)$ are nonempty. Since $A$ is finite, there are $n,p \geq 1$ such that $\sigma^n(a)$ and $\sigma^{n+p}(a)$ begin with the same letter, say $b$. Since $\sigma$ is growing, $\lim_{k \to \infty} \sigma^{kp}(b) = \infty$. Thus $\sigma^p$ is right-prolongable on $b$ and has an admissible one-sided fixed point.

We will use term substitution for a morphism $\sigma : A^* \to A^*$ which is right-prolongable on some $a \in A$ and moreover such that every letter of $A$ appears in $x = \sigma^\omega(a)$. The second condition is harmless, since it can always be satisfied by removing the useless letters.

A sequence $x$ which is an admissible one-sided fixed point of a substitution $\sigma$ is said to be purely substitutive (with respect to $\sigma$). The subshift generated by $x$ is then called a substitution shift.

In this way, the substitution shift generated by $x$ coincides with $X(\sigma)$. Moreover we will always have $L(X(\sigma)) = L(\sigma)$. Thus the shift generated by $\sigma$ and the subshift generated by $x$ are fortunately the same thing.

Example 2.4.5 The Thue-Morse substitution $\tau : a \to ab, b \to ba$ is prolongable on $a$ and $b$. The fixed point $x = \lim \tau^n(a)$ is called the Thue-Morse sequence. One has

$$x = abbabaab \cdots$$

One may show that $x_n = a$ if and only if the number of 1 in the binary expansion of $n$ is even (Exercise 2.21).

If $x \in A^\mathbb{N}$ is purely substitutive (with respect to $\sigma$) and $\phi : A^* \to B^*$ is a letter-to-letter morphism, then $y = \phi(x)$ is said to be substitutive (with respect to $(\sigma, \phi)$) and the shift space it generates, denoted $X(\sigma, \phi)$, is called a substitutive shift.

Two-sided fixed points of $\sigma$ can be similarly defined. Assume that $\sigma$ is right-prolongable on $a \in A$ and left-prolongable on $b \in A$, that is, $\sigma(b)$ ends with $b$ and $\lim_{n \to -\infty} |\sigma^n(b)| = +\infty$. Let $x = \sigma^\omega(a)$ and let $y$ be the left infinite sequence having all $\sigma^n(b)$ as suffixes. Let $z \in A^\mathbb{Z}$ be the two-sided sequence such that $x = z^+$ and $y = z^-$. Then $z$ is a fixed point of $\sigma$ denoted $\sigma^\omega(b \cdot a)$. It can happen that $\sigma^\omega(b \cdot a)$ does not belong to $X(\sigma)$. In fact, $\sigma^\omega(b \cdot a)$ belongs to $X(\sigma)$ if and only if $ba$ belongs to $L(\sigma)$. In this case, we say that $\sigma^\omega(b \cdot a)$ is an admissible fixed point of $\sigma$. Equivalently, $z \in A^\mathbb{Z}$ is an admissible fixed point of $\sigma$ if, and only if $z^+, z^-$ are admissible one-sided fixed points and $z$ belongs to $X(\sigma)$.

When $z \in A^\mathbb{Z}$ is a two-sided admissible fixed-point of $\sigma$, we say, as in the one-sided case, that $z$ is a pure substitutive two-sided sequence and if $\phi$ is a letter-to-letter morphism, we say that $x = \phi(z)$ is a substitutive two-sided sequence. Likewise, the shift generated by $x$, denoted $X(\sigma, \phi)$ is called a substitutive shift.
Proposition 2.4.6 Every growing substitution has a power with an admissible two-sided fixed point.

Proof. Let \( \sigma : A^* \to A^* \) be a growing substitution. Let \( a, b \in A \) be such that \( ab \in \mathcal{L}(\sigma) \). There are integers \( n, p \) such that simultaneously \( \sigma^n(a), \sigma^{n+p}(a) \) end with the same letter \( c \) and \( \sigma^n(b), \sigma^{n+p}(b) \) begin with the same letter \( d \). This will be also true, with the same \( p \) but possibly different letters \( c, d \), for \( n+1, n+2, \ldots \) and thus we may also assume that \( p \) divides \( n \). Thus \( \tau^\omega(c.d) \) is a two-sided fixed point of \( \tau = \sigma^p \). But \( cd \) is a factor of \( \sigma^{n+p}(ab) \) and thus \( cd \in \mathcal{L}(\tau) \), showing that \( \tau^\omega(c.d) \) is an admissible two-sided fixed point of \( \tau \).

Example 2.4.7 Let \( \varphi : a \mapsto ab, b \mapsto a \) be the Fibonacci substitution. Then \( \varphi^2 : a \mapsto aba, b \mapsto ab \) is right prolongable on \( a \) and left prolongable on \( a \) and \( b \). Since moreover \( ba \in \mathcal{L}(\varphi) \), the two sided infinite sequences \( \varphi^\omega(a \cdot a) \) and \( \varphi^\omega(b \cdot a) \) are admissible fixed points of \( \varphi^2 \).

Observe that a morphism \( \sigma \) could have non admissible fixed points.

Example 2.4.8 Let \( \sigma : A^* \to A^* \), with \( A = \{a, b, c\} \) be a substitution defined by \( a \mapsto ab, b \mapsto ac, c \mapsto aa \). It can be checked that \( \sigma \) has a unique fixed point in \( A^3 \) but no admissible fixed point in \( A^2 \) whereas \( \sigma^3 \) has three admissible fixed points: \( \sigma^\omega(a.a), \sigma^\omega(b.a) \) and \( \sigma^\omega(c.a) \).

2.4.2 Primitive morphisms

A morphism \( \varphi : A^* \to A^* \) is said to be primitive if there is an \( n \geq 1 \) such that for every \( a, b \in A \), the letter \( b \) appears in the word \( \varphi^n(a) \).

Proposition 2.4.9 If \( \sigma : A^* \to A^* \) is a primitive morphism and \( A \) has at least two letters, then \( \sigma \) is growing.

Proof. Let \( n \geq 1 \) be such that every letter \( b \in A \) occurs in \( \sigma(a) \) for every \( a \in A \). Then \( |\sigma^n(a)| \geq \text{Card}(A) \) and thus \( |\sigma^{n+m}(a)| \geq \text{Card}(A)^m \) for all \( m \geq 1 \).

In particular, every primitive morphism on at least two letters has a power which is a substitution. The converse is not true since, for example, \( a \mapsto aba, b \mapsto bb \) is a growing substitution which is not primitive.

The shift \( X(\sigma) \) generated by a primitive morphism \( \sigma \) is a substitution shift. Indeed, if \( \sigma \) is primitive, some \( \sigma^n \) is a substitution and \( X(\sigma) = X(\sigma^n) \).

The following result is well known.

Proposition 2.4.10 A primitive substitution shift is minimal.

Proof. Let \( \sigma : A^* \to A^* \) be a primitive morphism. Let \( n \geq 1 \) be such that every \( b \in A \) occurs in every \( \sigma^n(a) \) for \( a \in A \). By Proposition 2.2.14 it is enough to prove that \( \mathcal{L}(\sigma) \) is uniformly recurrent. Let \( u \in \mathcal{L}(\sigma) \). Let \( a \in A \) and \( m \geq 1 \) be such that \( u \) is a factor of \( \sigma^m(a) \). Then \( u \) is a factor of every \( \sigma^{n+m}(b) \) and thus
a factor of every word of $L(\sigma)$ of length $2|\sigma|^{n+m}$. Therefore $L(\sigma)$ is uniformly recurrent.

The converse of Proposition 2.4.10 is not true. A well-known example is the binary Chacon substitution $\sigma : 0 \rightarrow 0010, 1 \rightarrow 1$. The substitution is not primitive but the corresponding shift space is minimal (see Exercise 2.23). The converse is true however for a substitution which is growing (Exercise 2.24).

When $\sigma$ is primitive, we easily check that, for all $k \geq 1$, $L(\sigma^k) = L(\sigma)$ and that $L(\sigma) = L(\sigma^\omega(a))$ for any letter $a \in A$ on which $\sigma$ is right-prolongable. In particular, we also have that for all $x \in X(\sigma)$, $\Omega(x) = X(\sigma) = X(\sigma^k)$, for all $k \geq 1$. We say that $(X(\sigma), S)$ is a primitive substitution shift. A sequence $x$ is primitive substitutive if it is substitutive with respect to a primitive substitution. The subshift $(\Omega(x), S)$ that it generates is then primitive substitutive. Since a letter-to-letter morphism is a morphism of dynamical systems, we have the following corollary of Proposition 2.4.10.

**Corollary 2.4.11** A primitive substitutive shift is minimal.

The composition matrix of a morphism $\sigma : A^* \rightarrow B^*$ is the $B \times A$-matrix $M(\sigma)$ defined for every $a \in A$ and $b \in B$ by

$$M(\sigma)_{b,a} = |\sigma(a)|_b$$

where $|\sigma(a)|_b$ denotes the number of occurrences of the letter $b$ in the word $\sigma(a)$. The transposed matrix of $M(\sigma)$ is called the incidence matrix of $\sigma$. Thus, on the composition matrix, the columns correspond to the words $\sigma(a)$ (in the sense that the column of index $a$ is the vector $(|\sigma(a)|_b)_{b \in B}$) while, in the incidence matrix, this role is played by the rows.

Such a matrix gives, as we shall see, important information on a morphism. (although distinct morphisms may well have the same composition matrix). An important property is that if $\sigma : A^* \rightarrow B^*$ and $\tau : B^* \rightarrow C^*$ are morphisms, then

$$M(\tau \circ \sigma) = M(\tau)M(\sigma).$$

Note that, with incidence matrices, the relative order of $\sigma, \tau$ is reversed since $M(\tau \circ \sigma)^t = M(\sigma)^tM(\tau)^t$.

When $A = B$, the composition matrix is a square $A \times A$-matrix. It follows for (2.4.1) that for every $k \geq 0$, one has

$$M(\sigma^k) = M(\sigma)^k$$

For example, the composition matrix of the substitution $\sigma : 0 \rightarrow 01, 1 \rightarrow 00$ is

$$M(\sigma) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}.$$
Proposition 2.4.12 The morphism $\sigma$ is primitive if and only if the matrix $M(\sigma)$ is primitive.

Example 2.4.13 The Fibonacci substitution is primitive. Indeed, $\varphi^2(a) = aba$ and $\varphi^2(b) = ab$ both contain $a$ and $b$. Accordingly

$$M(\varphi) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(\varphi)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

Thus the Fibonacci shift is minimal.

Let $\varphi : A^* \to A^*$ be a primitive substitution. Then, the matrix $M$ is a primitive non-negative matrix. The following result is well-known.

Theorem 2.4.14 (Perron, Frobenius) Let $M$ be a nonnegative primitive real square matrix. Then

(i) $M$ has a positive simple eigenvalue $\lambda_M$ such that $|\mu| < \lambda_M$ for every other eigenvalue $\mu$ of $M$.

(ii) There corresponds to $\lambda$ a positive eigenvector $v$. Every nonnegative eigenvector of $M$ is colinear to $v$.

(iii) The sequence $(1/\lambda_M^n M^n)$ converges at geometric rate to the matrix $wv$ where $v, w$ are positive left and right eigenvectors such that $Mw = \lambda_Mw$, $vM = \lambda_Mv$ and $vw = 1$.

The theorem expresses in particular that if a matrix $M$ is primitive, its spectral radius $\rho(M) = \max\{ |\lambda| \mid \lambda \in \text{Spec}(M) \}$ is an eigenvalue of $M$ which is algebraically simple. Furthermore, any eigenvalue of $M$ different from $\rho(M)$ has modulus less than $\rho(M)$. We call $\rho(M)$ the dominant eigenvalue of $M$. By abuse of language, when $M$ is the composition matrix of a primitive endomorphism $\sigma$, we call $\rho(M)$ the dominant eigenvalue of $\sigma$.

The term geometric rate of convergence used in assertion (iii) means that there is constant $c > 0$ and a real number $r < 1$ such that for all $n \geq 0$

$$|| \frac{1}{\lambda_M^n} M^n - wv || \leq cr^n. \quad (2.4.3)$$

We can choose for $r$ the quotient $r = \mu/\rho(M)$ where $\mu$ is the maximum of the $|\lambda|$ for $\lambda$ an eigenvalue of $M$ other than $\rho(M)$.

We illustrate this theorem with the following example.

Example 2.4.15 Let $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of $M$ are $\lambda = (1 + \sqrt{5})/2$ and $\lambda = (1 - \sqrt{5})/2$. Since $\lambda < \lambda$, we have $\lambda_M = \lambda$. Since $\lambda^2 = \lambda + 1$, a left eigenvector corresponding to $\lambda$ is $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$. The sequence $(1/\lambda_M^n M^n)$ tends to the matrix

$$\frac{1}{1 + \lambda^2} \begin{bmatrix} \lambda^2 & \lambda \\ \lambda & 1 \end{bmatrix} = \frac{1}{1 + \lambda^2} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix}.$$
Example 2.4.16 Let \( \varphi \) be the Morse substitution. Then

\[
M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.
\]

The dominant eigenvalue is equal to 2 and the corresponding row eigenvector is \( v = [1/2, 1/2] \).

Thus, if \( M \) is the incidence matrix of a primitive morphism, we have for any \( a \in A \),

\[
\lim_{n \to \infty} |\varphi^{n+1}(a)|/|\varphi^n(a)| = \lambda_M.
\]  
(2.4.4)

since \( 1/\lambda_M^n |\varphi^n(a)| \) is the sum of coefficients of the row of index \( a \) of \( 1/\lambda_M^n M^n \).

We now prove the following important property of primitive morphisms. For a morphism \( \varphi \), we denote \( \langle \varphi \rangle = \min_{a \in A} |\varphi(a)| \) and \( |\varphi| = \max_{a \in A} |\varphi(a)| \).

Proposition 2.4.17 For every primitive morphism, the quotient \( |\varphi^n|/|\varphi^n| \) is bounded.

For this we need the following lemma (which will also be used again later).

Lemma 2.4.18 Let \( M \) be a primitive matrix with dominant eigenvalue \( \lambda_M \) and positive left and right eigenvectors \( x = (x_a) \), \( y = (y_b) \) such that \( \sum_{a \in A} x_a = 1 \) and \( \sum_{a \in A} x_a y_a = 1 \). There are \( c > 0 \) and \( \tau < \lambda_M \) such that for every \( a, b \in A \) and \( n \geq 1 \), we have

\[
\begin{align*}
|||\varphi^n(a)||_b - \lambda_M^n y_a x_b| & \leq c \tau^n \quad \text{(2.4.5)} \\
|||\varphi^n(a)| - \lambda_M^n y_a| & \leq c \text{Card}(A) \tau^n \quad \text{(2.4.6)} \\
|||\varphi^n(a)||_b - |\varphi^n(a)|x_b| & \leq c(1 + \text{Card}(A)) \tau^n \quad \text{(2.4.7)}
\end{align*}
\]

Proof. The first equation results directly from (2.4.3). For the second one, we write, using the triangular inequality and (2.4.6)

\[
|||\varphi^n(a)||_b - \lambda_M^n y_a x_b| = |\sum_{b \in A} |\varphi^n(a)||_b - \lambda_M^n \sum_{b \in A} y_a x_b| \\
\leq \sum_{b \in A} |||\varphi^n(a)||_b - \lambda_M^n y_a x_b| \\
\leq c \text{Card}(A) \tau^n
\]
which is (2.4.6). Finally, using (2.4.3) and (2.4.6), we obtain, since \( x_b \leq 1 \),

\[
|||\varphi^n(a)||_b - |\varphi^n(a)|x_b| = |||\varphi^n(a)||_b - \lambda_M^n y_a x_b + \lambda_M^n y_a x_b - |\varphi^n(a)|x_b| \\
\leq |||\varphi^n(a)||_b - \lambda_M^n y_a x_b| + x_b |||\varphi^n(a)| - \lambda_M^n y_a| \\
\leq c(1 + \text{Card}(A)) \tau^n
\]
which is (2.4.7). 

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Proof of Proposition 2.4.17 Let $M$ be the incidence matrix of $\varphi$. Equation (2.4.6) shows that for every $a \in A$, $|\varphi^n(a)|/\lambda^n_M$ converges to $y_a > 0$. This implies that for every $a, b \in A$, the quotient $|\varphi^n(a)|/|\varphi^n(b)|$ converges to $y_a/y_b$ and thus proves the statement.

We say that the shift space $(X, S)$ is linearly recurrent (LR) (with constant $K \geq 0$) if it is minimal and if for all $u$ in $L(X)$ and for all right return words $w$ to $u$ in $X$ we have $|w| \leq K|u|$.

Proposition 2.4.19 A primitive substitution shift is linearly recurrent

Proof. Let $\sigma : A^* \to A^*$ be a primitive substitution and let $X = X(\sigma)$ be the corresponding shift space. Set as usual $|\sigma| = \max_{a \in A} |\sigma(a)|$ and $\langle \sigma \rangle = \min_{a \in A} |\sigma(a)|$.

Since $\sigma$ is primitive, it follows from Proposition 2.4.17 that there is a constant $k$ such that

$$|\sigma^n| \leq k\langle \sigma^n \rangle$$

for all $n \geq 1$.

For every $w \in L(X)$, there is an integer $n$ such that $|w| \leq \langle \sigma^n \rangle$. Then, since $w$ has no factor in $\sigma^n(A)$, it is a factor of $\sigma^n(ab)$ for some $a, b \in A$. If $n$ is chosen minimal, we have $\langle \sigma^{n-1} \rangle < |w|$. Every return word to $w$ is a factor of a return word to some $\sigma^n(ab)$. Let $R$ be the maximal length of return words to words of length 2. Then, for every return word $u$ to $w$, we have

$$|u| \leq R|\sigma^n| \leq Rk\langle \sigma^n \rangle$$

$$< Rk|\sigma||w|.$$ 

This shows that $X$ is linearly recurrent with constant $K = kR|\sigma|$.

We have seen an illustration of this result for the Fibonacci shift (Exercise 2.13). We add an illustration on the Thue-Morse shift.

Example 2.4.20 Let $\sigma : a \mapsto ab, b \mapsto ba$ be the Thue-Morse substitution. According to the above, since $k = 1$ for a constant length morphism, and since the maximal length $R$ of return words of length 2 is 6, the Thue-Morse shift is LR with $K = 12$.

2.4.3 Circular codes

We say that a set $U \subset A^+$ of nonempty words is a code if every word $u \in A^*$ admits at most one decomposition in a concatenation of elements of $U$. Thus $U$ is a code if the submonoid $U^*$ generated by $U$ is isomorphic with the free monoid on $U$. A prefix code is obviously a code.

A coding morphism for a code $U \subset A^+$ is a morphism $\varphi : B^* \to A^*$ which is one-to-one from $B$ to $U$. 


The set \( U \subset A^+ \) is a circular code whenever it is a code and moreover if for \( p, q \in A^* \) one has
\[
pq, qp \in U^* \Rightarrow p, q \in U^*. \tag{2.4.9}
\]

One may visualize this definition as follows. Imagine the word \( pq \) written on a circle (or infinitely repeated). Then \( (pq)(pq) \) and \( p(qp)q \) are two decomposition in words of \( U \). Thus the circular codes are such that the decomposition is unique on a circle (or, equivalently, on the infinite repetition \( \cdots pqpq \cdots \)).

**Example 2.4.21** The set \( U = \{0, 01\} \) is a circular code but \( U' = \{010, 101\} \) is a code which is not circular.

**Proposition 2.4.22** Let \( (X, S) \) be a shift space. For every \( u \in \mathcal{L}(X) \), the set \( R_X(u) \) is a circular code.

**Proof.** The set \( U = R_X(u) \) is a prefix code and thus it is a code. Next, assume that \( pq, qp \in U^* \). Set \( p = u_1u_2 \cdots u_ns \) with \( u_i \in U \) and \( s \) a proper prefix of \( U \). Let \( k \geq 1 \) be such that \( (pq)^k \) is longer than \( u \). Since \( (pq)^k \) is in \( U^* \), the word \( u(pq)^k \) ends with \( u \) and thus \( (pq)^k \) ends with \( u \). Next, since \( (qp)^{k+1} = q(pq)^k p \) also ends with \( u \) for the same reason, we obtain that \( up \) ends with \( u \). This forces \( s = \varepsilon \) and thus \( p \in U^* \), which implies that \( q \) is also in \( U^* \) because \( U \) is a prefix code.

Circular codes have a property of unique decomposition of sequences.

**Proposition 2.4.23** Let \( U \subset A^+ \) be a finite circular code and let \( \varphi : B^* \to A^* \) be a coding morphism for \( U \). Then for every \( x \in A^2 \) there is at most one pair \((k, y)\) such that \( x = S^k \varphi(y) \) with \( 0 \leq k < |\varphi(y_0)| \). In particular, \( \varphi : B^2 \to A^2 \) is injective.

The proof is left as an exercise (Exercise 2.31). Note that in the case where \( U = R_X(u) \), the proof is immediate since the occurrences of \( u \) in a sequence determine the factorisation in words of \( U \).

### 2.4.4 Recognizable morphisms

Let \( \varphi : A^* \to B^* \) be a nonerasing morphism. Let \( X \) be a shift space on the alphabet \( A \) and let \( Y \) be the closure under the shift of \( \varphi(X) \). Then every \( y \in Y \) has a representation as \( y = S^k \varphi(x) \) with \( x \in X \) and \( 0 \leq k < |\varphi(x_0)| \). We say that \( \varphi \) is recognizable on \( X \) if every \( y \in Y \) has only one such representation.

As an equivalent definition, consider the tower over \( X \) associated to the function \( x \mapsto |\varphi(x_0)| \), already introduced in Section 2.3. It is the dynamical system \((X^\varphi, T)\) where
\[
X^\varphi = \{(x, i) | x \in X, \ 0 \leq i < |\varphi(x_0)|\}
\]
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and

\[ T(x, i) = \begin{cases} (x, i + 1) & \text{if } i + 1 < |\varphi(x_0)| \\ (Sx, 0) & \text{otherwise.} \end{cases} \]

The map \( \hat{\varphi} : (x, i) \to S^i \varphi(x) \) is a morphism of dynamical systems from \((X^\varphi, T)\) onto \((Y, S)\). The morphism \( \varphi \) is recognizable on \( X \) if \( \hat{\varphi}^{-1}(y) \) has only one element for every \( y \in Y \). Thus \( \varphi \) is recognizable on \( X \) if and only if \( \hat{\varphi} \) is injective and thus is a homeomorphism. Since \( X \) is isomorphic to the system induced by \( X^\varphi \) on \( X \times \{0\} \), we have proved the following useful statement.

**Proposition 2.4.24** If \( \varphi \) is recognizable on \( X \), then

1. The map \( \hat{\varphi} : X^\varphi \to Y \) is a homeomorphism.
2. \( \varphi(X) \) is a clopen subset of \( Y \).
3. \( X \) is isomorphic to the shift induced by \( Y \) on \( \varphi(X) \).

Note that we may consider \( X^\varphi \) as a shift space on the alphabet

\[ A^\varphi = \{(a, i) \mid a \in A, \ 0 \leq i < |\varphi(a)|\} \]

by identifying \((x, i)\) and the sequence \((a_n, j_n)_{n \in \mathbb{Z}}\) defined by \( T^nx, i = (y, j_n) \) with \( y_0 = a_n \).

The following is a reformulation of Proposition 2.4.23.

**Proposition 2.4.25** A nonerasing morphism \( \varphi : B^* \to A^* \) is recognizable on the full shift \( B^\omega \) if and only if \( \varphi \) is injective on \( B \) and \( \varphi(B) \) is a circular code.

As a positive example of application of Proposition 2.4.25, we have the case of the Fibonacci morphism.

**Example 2.4.26** The Fibonacci morphism \( \varphi : a \to ab, b \to a \) is recognizable on the full shift. Indeed, \( \{a, ab\} \) is a circular code.

The negative example of the Thue-Morse morphism shows the necessity of the hypothesis that \( \varphi(B) \) is circular in Proposition 2.4.23.

**Example 2.4.27** Let \( \tau : a \to ab, b \to ba \) be the Thue-Morse morphism. We have \( (ab)^\omega = \tau(ab)^\omega = S\tau(ba)^\omega \). Thus \( \tau \) is not recognizable on \( A^\omega \).

The following important theorem will be used several times. It will be used in particular in the representation of substitution shifts by sequences of partitions in towers in Chapter 5.

**Theorem 2.4.28** (Mossé) Every primitive nonperiodic morphism \( \sigma \) is recognizable on \( X(\sigma) \).
Example 2.4.29 The Thue-Morse morphism \( \tau : a \rightarrow ab, b \rightarrow ba \) is recognizable on the Thue-Morse shift \( X = X(\tau) \). Indeed, let \( x = S^k \tau(y) \) with \( 0 \leq k < 2 \). Every word in \( L(\tau) \) of length at least 5 contains \( aa \) or \( bb \). Thus \( x \) has an infinite number of occurrences of both \( aa \) and \( bb \). Consider an \( i > 1 \) such that \( x_i = x_{i+1} = a \). Then \( x_k \cdots x_i \) is in \( \tau(A^*) \) and thus has even length. Thus \( i \) and \( k \) have opposite parities. This shows that \( k \) is unique and thus also \( y \) since \( \tau \) has constant length.

The next example shows that the hypothesis that \( \sigma \) is nonperiodic is necessary.

Example 2.4.30 Let \( \sigma : a \rightarrow ab, b \rightarrow ca, c \rightarrow bc \). We have \( \sigma(abc) = (abc)^2 \) and thus \( X(\sigma) \) is formed of the three shifts of \( (abc)^\omega \). Therefore \( \sigma \) is primitive but periodic. Since

\[
(abc)^\omega = \varphi(abc)^\omega = S^2(bca)^\omega
\]

the morphism \( \sigma \) is not recognizable.

The proof of Theorem 2.4.28 uses the following concept. Let \( \varphi : A^* \rightarrow B^* \) be a nonerasing morphism, let \( X \) be a shift space on the alphabet \( A \), let \( Y \) be the closure of \( \varphi(X) \) under the shift. Let \( (p, q) \) be a pair of words on the alphabet \( B \) and \( (u, v) \) a pair of words on the alphabet \( A \). We say that \( (p, q) \) is \((u, v)\)-parsable if

(i) \( p \) is a suffix of \( \varphi(u) \) and

(ii) \( q \) is a prefix of \( \varphi(v) \).

We also say that \( (p, q) \) is parsable in \( L(X) \) if it is \((u, v)\)-parsable for \( (u, v) \) such that \( uv \in L(X) \).

A word \( z \) on \( A \) such that \( pq \) is a factor of \( \varphi(z) \) is synchronized with \( (p, q) \) if there is a factorization \( z = yt \) with \( p \) is a suffix of \( \varphi(y) \) and \( q \) is a prefix of \( \varphi(t) \) and the first letters of \( v, t \) are equal (see Figure 2.4.1). Finally, a parsable pair \( (p, q) \) is synchronizing if every \( z \in L(X) \) such that \( pq \) is a factor of \( \varphi(z) \) is synchronized with \( (p, q) \).

![Figure 2.4.1: A synchronizing pair](image)

Let \( \varphi : A^* \rightarrow B^* \) be a substitution, let \( X \) be a shift space on \( A \) and let \( Y \) be the closure of \( \varphi(X) \) under the shift.
2.4. SUBSTITUTION SHIFTS

**Proposition 2.4.31** The substitution \( \varphi \) is recognizable on \( X \) if and only if there is an integer \( L \) such that every pair of words of length \( L \) which is parsable in \( \mathcal{L}(X) \), is synchronizing.

**Proof.** Let us first show that the condition is sufficient.

We have to prove that for \( x, x' \in X \) such that \( S^j \varphi(x) \cap S^k \varphi(x') \neq \emptyset \) with \( 0 \leq j < |\varphi(x_0)| \) and \( 0 \leq k < |\varphi(x'_0)| \), one has \( j = k \) and \( x_0 = x'_0 \). We may suppose that \( j \leq k \).

By the hypothesis, for \( N \) large enough, the pair \((q, r)\) with \( q = \varphi(x[-N,-1]) \) and \( r = \varphi(x[0,N-1]) \) is synchronizing.

We can also assume that \( |\varphi(x[-N,-1])| \geq k - j \) and thus that \( q = q' q'' \) with \( |q''| = k - j \). Let \( M \geq 1 \) and let \( t = x'_{[-M,M-1]} \). For \( M \) large enough, we have \( \varphi(x'_{[-M,-1]}) = pq' \) and \( \varphi(x'_{[0,M-1]}) = q''rs \) for some word \( s \) (see Figure 2.4.2). Since \((q, r)\) is synchronizing, there is a prefix \( t \) of \( x'_{[0,M-1]} \) such that \( \varphi(t) = q'' \).

Since \( |q''| = k - j \leq k < |\varphi(x'_0)| \), this forces \( t = q'' = \varepsilon \), \( j = k \) and \( x_0 = x'_0 \).

![Figure 2.4.2: The intersection \( \varphi(y) \cap S^{k-j} \varphi(z) \).](image)

Conversely, assume that \( \varphi \) is recognizable on \( X \). Set \( \psi = \hat{\varphi}^{-1} \). Since \( \psi : Y \to X^\varphi \) is a conjugacy between shift spaces (we consider \( X^\varphi \) as a shift space on the alphabet \( A^\varphi \)), it is a sliding block code. Thus there is an integer \( L \geq 1 \) such that the symbol \((\psi y)_0\) depends only on \( y[-L,L-1] \).

![Figure 2.4.3: Proof that the condition is necessary.](image)

Let \((p, q)\) be a \((u, v)\)-parsable pair of words of length \( L \) with \( uv \in \mathcal{L}(X) \). Let \( y \in Y \) be such that \( \varphi(u) \) is a suffix of \( y^{-} \) and \( \varphi(v) \) is a prefix of \( y^{+} \). Let \( z \in \mathcal{L}(X) \) be such that \( \varphi(z) = rqps \). Let \( m, n, p', p'', q', q'' \) be words and \( a \in A \)}
be a letter such that \( z = man \) with (see Figure 2.4.3)

\[
\varphi(m) = rp', \quad \varphi(a) = p''q', \quad \varphi(n) = q''s, \quad p = p'p'', \quad q = q'q''
\]

and \( q' \) not empty.

Let \( y' \in Y \) be such that \( y' \) ends with \( m \) and \( y' \) begins with \( an \) (and consequently \( y'_0 = a \)). Since \( y[-L,L-1] = y'_0 = pq \), it follows from the definition of \( L \) that \( (\psi y)_0 = (\psi y')_0 \). But \( (\psi y)_0 = (b, 0) \) where \( b \) is the first letter of \( v \) while \( (\psi y')_0 = (a, |p''|) \). We conclude that \( q'' \) is empty and \( a = b \), which shows that the pair \((p, q)\) is synchronizing.

**Proof of Theorem 2.4.28.** Assume, by contradiction, that \( \varphi \) is not recognizable on \( X = X(\varphi) \). By Proposition 2.4.31, this implies that for every \( \ell \), there is a pair \((p, q)\) of words of length \( \ell \) which is parsable in \( L(X) \) but not synchronizing.

Fix an integer \( k \geq 1 \) which will be chosen later. By the hypothesis, there is for every \( n \geq 1 \) a pair \((u_n, v_n)\) of words of length \( k \) with \( u_n, v_n \in L(X) \) such that the pair \((p_n, q_n)\) with \( p_n = \varphi^n(u_n) \) and \( q_n = \varphi^n(v_n) \) is not synchronizing. Thus, there is for every \( n \geq 1 \) a word \( z_n \in L(X) \) such that (see Figure 2.4.4)

(i) \( \varphi^{n-1}(z_n) \) is not synchronized with \((p_n, q_n)\).

(ii) \( z_n = a_nb_n \) with \( a_n, b_n \in A \) and \( p_nq_n = s_n\varphi^n(w_n)t_n \).

![Figure 2.4.4: Proof of Mosse’s Theorem.](image)

Since \( \varphi^n(u_nv_n) = p_nq_n \) and \(|u_nv_n| = 2k\), we have

\[
|p_nq_n| \leq 2k|\varphi^n|
\]

And since \( p_nq_n = s_n\varphi^n(w_n)t_n \) we have

\[
|p_nq_n| \geq |w_n|/|\varphi^n|.
\]

Thus, we obtain

\[
|w_n| \leq k|\varphi^n|/|\varphi^n|.
\]

But, since \( \varphi \) is primitive, by Proposition 2.4.17, the right-hand side of (2.4.10) is bounded independently of \( n \). Thus, there is an infinity of \( n \) such that \((u_nv_n) = \)
(u, v), z_n = z, a_n = a, b_n = b and w_n = w for every n ∈ E. Thus we have the equalities
\[ z = awb, \quad \varphi^n(uv) = s_n \varphi^n(w)t_n \] (2.4.11)
(2.4.12)

Consider n, m ∈ E with n < m. We have
\[ s_n \varphi^m(w)t_n = \varphi^m(uv) = \varphi^{m-n}(\varphi^n(uv)) = \varphi^{m-n}(s_n) \varphi^m(w) \varphi^{m-n}(t_n). \]

Suppose that \( s_m \neq \varphi^{m-n}(s_n) \). We may assume that \( s_m > \varphi^{m-n}(s_n) \). Then, the word \( \varphi^m(w) \) is periodic of period \(|s_m| - \varphi^{m-n}(s_n) \leq |s_m| \). This implies that \( \mathcal{L}(X) \) contains powers of exponent larger than
\[ |\varphi^m(uv)|/|s_m| \geq 2k(|\varphi^m|)/|\varphi^m| \]
which tends to infinity with k. This contradicts Proposition 22.7 since, by Proposition 2.4.19 the shift X is linearly recurrent.

Thus \( s_m = \varphi^{m-n}(s_n) \). Now we have \( s_n \varphi^m(w)t_n = \varphi^n(uv) \) and thus, applying \( \varphi^{m-n-1} \) to both sides,
\[ \varphi^{m-n-1}(s_n) \varphi^{m-1}(w) \varphi^{m-n-1}(t_n) = \varphi^{m-1}(uv) \]

Let \( y, t \) be such that \( \varphi^{m-1}(w) = yt \) with
\[ \varphi^{m-1}(u) = \varphi^{m-n-1}(s_n)y, \quad \varphi^{m-1}(v) = t\varphi^{m-n-1}(t_n). \] (2.4.13)

Applying \( \varphi \) on both sides of (2.4.13), we obtain
\[ p_m = s_m \varphi(y), \quad q_m = \varphi(t)t_m \]
showing that \( \varphi^{m-1}(z) = \varphi^{m-1}(a)y\varphi^{m-1}(b) \) is synchronized with \((p_m, q_m)\), a contradiction.

### 2.4.5 Block presentations

We will need later to associate to a primitive substitution \( \varphi \) and an integer \( k \geq 1 \) the \( k \)-block presentation of \( \varphi \) denoted \( \varphi_k \).

Let \( \varphi : A^* \rightarrow A^* \) be a primitive substitution and let \((X, S)\) be its associated shift space. For \( k \geq 1 \), consider an alphabet \( A_k \) in one-to-one correspondence by \( f : \mathcal{L}_k(X) \rightarrow A_k \) with the set \( \mathcal{L}_k(X) \) of factors (or blocks) of length \( k \) of \( X \). The map \( f \) extends naturally to a map, still denoted \( f \), from \( \mathcal{L}_{k+n}(X) \) to \( \mathcal{L}_{k+n+1}(X(k)) \) defined for \( n \geq 0 \) by
\[ f(a_1a_2 \cdots a_{k+n}) = f(a_1 \cdots a_k)f(a_2 \cdots a_{k+1}) \cdots f(a_{n+1} \cdots a_{k+n}). \] (2.4.14)

We define a morphism \( \varphi_k : A^*_k \rightarrow A^*_k \) as follows. Let \( b \in A_k \) and let \( a \) be the first letter of \( f^{-1}(b) \) (see Figure 2.4.10). Set \( s = |\varphi(a)| \). To compute \( \varphi_k(b) \), we...
first compute the word \( \varphi(f^{-1}(b)) = c_1 c_2 \cdots c_\ell \). Note that \( \ell \geq |\varphi(a)| + k - 1 = s + k - 1 \). We set

\[
\varphi_k(b) = b_1 b_2 \cdots b_s
\]

where

\[
b_1 = f(c_1 c_2 \cdots c_k), \quad b_2 = f(c_2 c_3 \cdots c_{k+1}), \ldots, \quad b_s = f(c_s \cdots c_{s+k-1}).
\]

In other terms, \( \varphi_k(b) \) is the prefix of length \( s = |\varphi(a)| \) of \( f \circ \varphi \circ f^{-1}(b) \) where \( f \) is the map defined by (2.4.14) (see Figure 2.4.5).

**Example 2.4.32** Let \( \varphi : a \to ab, b \to a \) be the Fibonacci morphism. We have \( L_2(X) = \{aa, ab, ba\} \). Set \( A_2 = \{x, y, z\} \) and let \( f : aa \to x, ab \to y, ba \to z \).

Since \( f \circ \varphi \circ f^{-1}(x) = f(\varphi(aa)) = f(abab) = yzy \), keeping the prefix of length \( |\varphi(a)| = 2 \), we have \( \varphi_2(x) = yz \). Similarly, we have \( \varphi_2(y) = yz \) and \( \varphi_2(z) = x \).

Let \( \pi : A_k^+ \to A^+ \) be the morphism defined by \( \pi(b) = a \) where \( a \) is the first letter of \( f^{-1}(b) \).

Then we have for each \( n \geq 1 \) the following commutative diagram which expresses the fact that \( \varphi_k^n \) is the counterpart of \( \varphi^n \) for \( k \)-blocks.

\[
\begin{array}{ccc}
A_k^+ & \xrightarrow{\pi^n} & A_k^+ \\
\downarrow \pi & & \downarrow \pi \\
A^+ & \xrightarrow{\varphi_k^n} & A^+
\end{array}
\]  

(2.4.15)

Indeed, we have \( \varphi \pi(b) = \pi \varphi_k(b) \) for every \( b \in A_k \) by definition of \( \varphi_k \). Since \( \varphi \pi \) and \( \pi \varphi_k \) are morphisms, this implies \( \varphi \circ \pi = \pi \circ \varphi_k \) and thus \( \varphi^n \circ \pi = \pi \circ \varphi_k^n \) for all \( n \geq 1 \). This proves (2.4.16). In particular, since \( \pi \) is length preserving, we have

\[
|\varphi_k^n(b)| = |\varphi^n(a)|
\]  

(2.4.16)

for \( n \geq 1 \) and \( a = \pi(b) \).

We denote \( u \leq v \) to express that the word \( u \) is a prefix of \( v \).
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Proposition 2.4.33 We have for every $u \in \mathcal{L}(X)$ of length at least $k$.

$$\varphi_k(f(u)) \leq f(\varphi(u)).$$ \hspace{1cm} (2.4.17)

Proof. For a word $w$ of length at least $n$, we denote by $\text{Pref}_n(w)$ its prefix of length $n$ and we set $\rho(u) = \text{Pref}_{|u|-k+1}(u)$ for $u$ of length at least $k$. For $u \in \mathcal{L}(X)$, set $\ell(u) = |\varphi(\rho(u))|$. We prove the following result, showing that the definition of $\varphi_k$ extends to words in $f(\mathcal{L}_{\geq k}(X))$.

$$\varphi_k(f(u)) = \text{Pref}_{\ell(u)} f(\varphi(u)).$$ \hspace{1cm} (2.4.18)

The property is true for $u \in \mathcal{L}_k(X)$ since $\varphi_k(f(u))$ is the prefix of length $|\varphi(a)|$ of $f(\varphi(u))$ where $a$ is the first letter of $u$.

Arguing by induction on the length of $u$, consider now $a \in A$ and $u \in \mathcal{L}_{\geq k}(X)$. Set $f(au) = bv$ with $b \in A_k$ (see Figure 2.4.6). Then, since $\rho(au) = a\rho(u)$, since $f(\varphi(au)) = \varphi(a)f(\varphi(u))$ and since $f^{-1}(b) = \text{Pref}_k(au)$,

$$\text{Pref}_{\ell(au)} f(\varphi(au)) = \text{Pref}_{|\varphi(a)|+\ell(u)} f(\varphi(au))$$
$$= \text{Pref}_{|\varphi(a)|} f(\varphi(au)) \text{Pref}_{\ell(u)} f(\varphi(u))$$
$$= \text{Pref}_{|\varphi(a)|} f(\varphi(f^{-1}(b))) \text{Pref}_{\ell(u)} f(\varphi(u))$$
$$= \varphi_k(b)\varphi_k(v) = \varphi_k(bv)$$

proving (2.4.18) and thus (2.4.17).

![Figure 2.4.6: Comparing $\varphi_k(f(au))$ and $\varphi(au)$.](image)

Proposition 2.4.34 When $\varphi$ is primitive, then for $k \geq 1$,

1. the substitution $\varphi_k$ is primitive,

2. for every $b \in A_k$, $u = f^{(-1)}(b)$ and $n \geq 1$, $\varphi_k^n(b) \leq f(\varphi^n(u))$,

3. the shift associated to $\varphi_k$ is the $k$-th higher block presentation of $X$. 
Proof. 1. Since \( \varphi \) is primitive, there is an integer \( n \) such that for every \( a \in A \), the word \( \varphi^n(a) \) contains all the words of \( \mathcal{L}_k(\varphi) \) as factors. Then for every \( b \in A_k \), the word \( \varphi^n_k(b) \) contains all letters of \( A_k \). Thus \( \varphi_k \) is primitive.

2. Since \( \varphi_k(f(u)) \leq f(\varphi(u)) \) for every \( u \in \mathcal{L}_{\geq k}(X) \), we have also \( \varphi^n_k(f(u)) \leq f(\varphi^n(u)) \) for every \( n \geq 1 \).

3. This shows that for every \( b \in A_k \) and \( n \geq 1 \), the word \( \varphi^n_k(b) \) is in \( \mathcal{L}(X^{(k)}) \), which implies the conclusion since \( X^{(k)} \) is minimal.

We denote by \( M_k \) the incidence matrix of the morphism \( \varphi_k \). Thus, we have \( M_k = M(\varphi_k)^t \).

Proposition 2.4.35 All matrices \( M_k \) for \( k \geq 1 \) have the same dominant eigenvalue.

Proof. Since \( |\varphi^n_k(b)| = |\varphi^n(a)| \) for all \( n \geq 1 \) and \( b \in A_k \) with \( a = \pi(b) \) by (2.4.16), this follows from Equation (2.4.4).

Actually, one can prove that all matrices \( M_k \) for \( k \geq 2 \) have the same nonzero eigenvalues (Exercise 4.22).

Example 2.4.36 Let \( \varphi : a \mapsto ab, b \mapsto a \) be the Fibonacci substitution as in Example 2.4.1. Set \( A_2 = \{x, y, z\} \) and \( f : x \mapsto aa, y \mapsto ab, z \mapsto ba \). Then \( \varphi_2 : x \mapsto yz, y \mapsto yz, z \mapsto x \) as we have already seen. Thus

\[
M_2 = \begin{bmatrix}
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

2.5 Sturmian shifts

A shift space \((X, S)\) on a binary alphabet is called Sturmian if its word complexity is \( n + 1 \). Thus, by Proposition 2.2.12, Sturmian shifts have the minimal nonconstant word complexity. Since these shifts are defined by a property of their language, the definition applies to two-sided shifts as well as to one-sided shifts. An element of a Sturmian shift is a Sturmian sequence.

We shall see shortly an example of a Sturmian shift (Example 2.5.1).

The definition implies that Sturmian shifts are such that for every \( n \geq 1 \) there is exactly one right special word (and one left special word). Since a prefix of a left special word is left special, this implies that the left special words in a Sturmian shift are the prefixes of one right infinite word.

Example 2.5.1 The Fibonacci shift (see Example 2.4.1) is Sturmian (see Exercise 2.32).

As one might expect in view of its minimal word complexity, any Sturmian shift is minimal, moreover it is uniquely ergodic.
2.5. STURMIAN SHIFTS

As well known, Sturmian shifts correspond to the coding of rotations on the circle. Actually, for every irrational real number $\alpha$ with $0 \leq \alpha \leq 1$, let $s = (s_n)_{n \in \mathbb{Z}}$ be the biinfinite word defined by

$$s_n = \lfloor (n + 1)\alpha \rfloor - \lfloor n\alpha \rfloor$$  \hspace{1cm} (2.5.1)

It can be shown that the closure of the orbit of $s$ is a Sturmian shift and that every Sturmian shift is of this form (we actually prove this statement below in Proposition 2.5.5).

For example, if $\alpha = (3 - \sqrt{5})/2$, then $s_0 s_1 \cdots = 001001 \cdots$. The closure of the orbit of $s$ is the Fibonacci shift.

2.5.1 Episturmian shifts

Sturmian shifts can be generalized to arbitrary alphabets as follows. A shift space $X$ on an alphabet $A$ is called episturmian if $L(X)$ is closed under reversal and for every $n \geq 1$, there is at most one right-special word of length $n$. It is called strict episturmian (or also Arnoux-Rauzy) if for each $n \geq 1$, there is a unique right-special word $w$ of length $n$ which is moreover is such that $wa \in L(X)$ for every $a \in A$. Again, the definition applies to two-sided or one-sided shifts.

A Sturmian shift is strict episturmian. Indeed, if $X$ is Sturmian, it can be shown that $L(X)$ is closed under reversal.

A one-sided infinite word $x$ is called episturmian (resp. strict episturmian) if the subshift generated by $x$ is episturmian. It is called standard if its left special factors are prefixes of $x$. For every strict episturmian one-sided shift $X$, there is a unique standard infinite word $x$ in $X$. Accordingly, when $X$ is a strict episturmian two-sided shift, there is a unique standard infinite word $y$ which belongs to the one-sided shift associated to $X$, that is, such that $y = x^+$ for some $x \in X$.

Example 2.5.2 The morphism $\varphi : a \mapsto ab, b \mapsto ac, c \mapsto a$ is called the Tribonacci morphism. The fixed point $x = \varphi^\omega(a)$ is called the Tribonacci sequence. It is a standard episturmian word (Exercise 2.33). There are three two sided infinite words $z$ such that $z^+ = x$, namely $\varphi^3(a \cdot a)$, $\varphi^3(b \cdot a)$ and $\varphi^3(c \cdot a)$.

For $a \in A$, denote by $L_a$ the elementary automorphism defined for every $b \in A$ by

$$L_a(b) = \begin{cases} ab & \text{if } b \neq a \\ a & \text{otherwise} \end{cases}$$

We define $L_u$ for $u \in A^*$ by extending the map $a \mapsto L_a$ to a morphism $u \mapsto L_u$.

A palindrome is a word equal to its reversal. For a word $w$, the palindromic closure of $w$, denoted $w^{(+)}$, is the shortest palindrome which has $w$ as a prefix. The iterated palindromic closure of a word $w$, denoted $\text{Pal}(w)$, is defined by $\text{Pal}(\varepsilon) = \varepsilon$ and $\text{Pal}(ua) = (\text{Pal}(u)a)^{(+) for u \in A^* and a \in A.
Since \( \text{Pal}(u) \) is a prefix of \( \text{Pal}(uv) \), there is for every \( x \in A^\mathbb{N} \), a unique right infinite word \( y = \text{Pal}(x) \) such that \( \text{Pal}(u) \) is a prefix of \( y \) for every prefix \( u \) of \( x \). It can be shown that for every \( u, v \in A^* \), one has by Justin Formula

\[
\text{Pal}(uv) = L_u(\text{Pal}(v)) \text{Pal}(u) \tag{2.5.2}
\]

As a consequence, for every \( u \in A^* \) and \( v \in A^\mathbb{N} \), one has

\[
\text{Pal}(uv) = L_u(\text{Pal}(v)). \tag{2.5.3}
\]

Moreover, \( x \mapsto \text{Pal}(x) \) is the unique function \( f: A^\mathbb{N} \rightarrow A^\mathbb{N} \) such that \( f(uv) = L_u(f(v)) \) for every \( u \in A^* \) and \( v \in A^\mathbb{N} \).

We will use the following result in Chapter 5.

**Theorem 2.5.3** An infinite word \( s \) is a standard episturmian word if and only if there exists an infinite word \( x \) such that \( s = \text{Pal}(x) \). Moreover, \( s \) is episturmian strict if and only if every letter of \( A \) occurs infinitely often in \( x \).

The one-sided infinite word \( x \) is called the directive word of the standard word \( s \).

If \( x = x_0 x_1 \cdots \), the words \( u_n = \text{Pal}(x_0 \cdots x_{n-1}) \) are the palindrome prefixes of \( s \). It can be shown (Exercise 2.36) that moreover the set of left return words to \( u_n \) is

\[
\mathcal{R}_X(u_n) = \{ L_{x_0 \cdots x_{n-1}}(a) \mid a \in A \} \tag{2.5.4}
\]

This formula shows the remarkable fact that in a strict episturmian shift \( X \), the set of return words to the words \( u_n \) (and, as a consequence to every word in \( L(X) \)) has a constant cardinality. We shall have more to say about this later, when we introduce dendric shifts (Chapter 8).

**Example 2.5.4** The directive word of the Tribonacci word \( s \) (see Example 2.5.2) is \((abc)^\omega\). Indeed, we have by Justin’s Formula \( \text{Pal}(abc)^\omega = L_{abc}(\text{Pal}(abc)^\omega) \) whence the result since \( L_{abc} = \varphi^3 \). The palindrome prefixes of \( s \) are

\[
a, aba, abacaba, \ldots.
\]

We have for example

\[
\mathcal{R}_X(abacaba) = \{ L_{abc}(a), L_{abc}(b), L_{abc}(c) \} = \{ abacaba, abacab, abac \}.
\]

### 2.5.2 Sturmian words, rotations and continued fractions

Let us come back to binary Sturmian words. We will use the traditional alphabet \( A = \{0, 1\} \). To every standard Sturmian word \( s \in \{0, 1\}^\mathbb{N} \), we associate a real number \( \alpha(s) \) called its *slope* and defined as follows.

Let

\[
x = 0^{d_1}1^{d_2}0^{d_3} \ldots
\]
be the directive word of $s$. The slope of $s$ is the real number
\[ \alpha(s) = [0, 1 + d_1, d_2, d_3, \ldots]. \]
where $[a_0, a_1, a_2, \ldots]$ denotes the continued fraction with coefficients $a_0, a_1, \ldots$ (see Appendix C).

Recall that we denote $T = \mathbb{R}/\mathbb{Z}$ the one-dimensional torus. For $\alpha > 0$, let $R_\alpha : T \rightarrow T$ be the transformation defined by $R_\alpha(z) = z + \alpha$. The pair $(T, R_\alpha)$ is a topological dynamical system called the rotation of angle $\alpha$.

**Proposition 2.5.5** Let $s$ be a Sturmian word and let $\alpha$ be the slope of $s$. The map $\gamma_\alpha : T \rightarrow X$ defined for $z \in T$, by $y = \gamma_\alpha(z)$ if
\[
y_n = \begin{cases} 0 & \text{if } R_\alpha^n z \in [0, 1 - \alpha) \\ 1 & \text{otherwise} \end{cases}
\]
is such that $s = \gamma_\alpha(\alpha)$ and is an injective map from $T$ into the subshift $X$ generated by $s$ such that $\gamma_\alpha \circ R_\alpha = S \circ \gamma_\alpha$ (see Figure 2.5.1).

![Figure 2.5.1: The map $\gamma_\alpha$.](image)

The map $\gamma_\alpha$ is called the natural coding of $(T, R_\alpha)$. Note that $\gamma_\alpha(0)$ is the sequence defined by Equation 2.5.1.

Note also that $\gamma_\alpha$ is not a conjugacy since it is neither continuous nor surjective.

Indeed, it is not continuous since $\gamma_\alpha(0)_0 = 0$ while $\gamma_\alpha(z)_0 = 1$ for all $z \in (\alpha, 1)$ and thus for values of $z$ arbitrary close to 0.

Next, the sequence $s = \lim_{n \rightarrow 1-} \gamma_\alpha(z)$ is not in $\gamma_\alpha(T)$. Indeed, let $c_\alpha = \gamma_\alpha(\alpha)^+$. We have $s_+ = 1c_\alpha$ and thus $s = \gamma_\alpha(y)$ implies $y = 0$ although $\gamma_\alpha(0)_0 = 0$, a contradiction. Note that the left-special words in $L(X)$ are the prefixes of $c_\alpha$. Thus $c_\alpha$ is a standard sequence.

We first prove the following lemma.

**Lemma 2.5.6** For every irrational $\alpha$ with $0 < \alpha < 1$, the sequence $c_\alpha = \gamma_\alpha(\alpha)^+$ satisfies
\[
c_\alpha = \begin{cases} L_0(c_{\alpha/(1-\alpha)}) & \text{if } \alpha < 1/2 \\ L_1(c_{(1-\alpha)/\alpha}) & \text{otherwise} \end{cases}
\]
Proof. Assume that $d_1 > 0$ (or equivalently $\alpha < 1/2$) and set $x = 0y$. We have to prove that
\[ c_\alpha = L_0(c_{\alpha/(1-\alpha)}). \] (2.5.6)
We consider the transformation $R_\alpha$ as a map defined on $[0, 1)$ translating the semi intervals $[0, 1-\alpha)$ and $[1-\alpha, 1)$ as indicated in Figure 2.5.2 on the left. Consider the transformation $R'$ induced by $R = R_\alpha$ on the semi-interval $[0, 1-\alpha)$. It is defined by
\[ R'(z) = \begin{cases} R(z) & \text{if } z \in [0, 1-2\alpha) \\ R^2(z) & \text{otherwise} \end{cases} \]
The transformation $R'$ is obtained by cutting the interval $[0, 1)$ after the point $1-\alpha$ and grouping the two translations first from $[1-2\alpha, 1-\alpha)$ to $[1-\alpha, 1)$ and then to $[0, \alpha)$ into a single translation. In this way, we obtain a new rotation of angle $\alpha/(1-\alpha)$ and for every $z \in [0, 1-\alpha)$, $\gamma_\alpha(z) = L_0(R'(z))$. Since $R'(z) = R_{\alpha/(1-\alpha)}(z/(1-\alpha))$, this proves Equation (2.5.6).

The sequence $c_\alpha$ is called the characteristic sequence of slope $\alpha$.

![Figure 2.5.2: The action of $R_\alpha$.](image)

Proof of Proposition 2.5.5. For $y \in \{0,1\}^*$, let us denote by $\pi(y)$ the slope of the standard sequence $\text{Pal}(y)$. We have to prove that for all $u \in \{0,1\}^*$ and $y \in \{0,1\}^N$, we have
\[ c_{\pi(uy)} = L_u(c_{\pi(y)}). \] (2.5.7)
Indeed, by Justin Formula (2.5.3), this implies that $c_{\pi(x)} = \text{Pal}(x)$ and thus that $c_\alpha = s$.

Equation (2.5.7) is true for $u = \varepsilon$. Note that if $\alpha = \pi(x)$ and $x = 0x'$, then $\pi(x') = \alpha/(\alpha - 1)$. Arguing by induction on $|u|$, consider $u = 0v$. Then, by (2.5.6) and using the induction hypothesis, we have
\[ c_{\pi(uy)} = L_0(c_{\pi(vy)}) = L_0(L_u(c_{\pi(y)})) = L_u(c_{\pi(y)}). \]
The case $u = 1v$ is similar.

The transformation used in the proof is called a Rauzy induction (we shall meet again this notion in Chapter 7 when we consider interval exchange transformations).
Example 2.5.7 The slope of the Fibonacci word is \( \alpha = \frac{1 - \sqrt{5}}{2} \). Indeed, its directive word is 0101\ldots and thus \( \alpha = [0, 2, 1, 1, \ldots] \). The Fibonacci word is actually an approximation of the line of equation \( y = \alpha x + \alpha \) (see Figure 2.5.3).

Figure 2.5.3: The Fibonacci word as an approximation of the line \( y = \alpha (x + 1) \).

2.6 Toeplitz shifts

A sequence \( x = (x_n)_{n \in \mathbb{K}} \), with \( \mathbb{K} = \mathbb{Z} \) or \( \mathbb{N} \), on the alphabet \( A \) satisfying

\[
\forall n \in \mathbb{K}, \exists p \in \mathbb{N}, \forall k \in \mathbb{N}, x_n = x_{n+kp}
\]

is called a Toeplitz sequence.

For such a sequence \( x \) and \( p \geq 1 \), we let

\[
\text{Per}_p(x) = \{ n \in \mathbb{Z}; x_n = x_{n+kp} \text{ for all } k \in \mathbb{K} \}.
\]

It is clear that \( x \) is a Toeplitz sequence if there exists a sequence \( (p_n)_{n \geq 1} \) in \( \mathbb{N} \setminus \{0\} \) such that \( \mathbb{Z} = \bigcup_{n \geq 1} \text{Per}_{p_n}(x) \). Equivalently, \( x \) is a Toeplitz sequence if all finite blocks in \( x \) appear periodically. Hence, Toeplitz sequences are uniformly recurrent. We say that \( p_n \) is an essential period if for any \( 1 \leq p < p_n \) the sets \( \text{Per}_p(x) \) and \( \text{Per}_{p_n}(x) \) do not coincide. If the sequence \( (p_n)_{n \geq 1} \) is formed by essential periods and \( p_n \) divides \( p_{n+1} \), we call it a periodic structure of \( x \). Clearly, if \( (p_n)_{n \geq 1} \) is a periodic structure, then \( (p_{i_n})_{n \geq 1} \) is also a periodic structure for any strictly increasing sequence of positive integers \( (i_n)_{n \geq 1} \).

A shift space \((X, S)\) is a Toeplitz shift if \( X \) is generated by a Toeplitz sequence \( x \in X \).

Example 2.6.1 Let \( \sigma \) be the substitution \( 0 \rightarrow 01, 1 \rightarrow 00 \). The substitution shift \( X(\sigma) \) is a Toeplitz shift. Indeed, let us show that the admissible fixed point \( x = \sigma^{2n}(1 \cdot 0) \) is a Toeplitz sequence (sometimes known as the period-doubling sequence). First all symbols \( x_{2n} \) of even index are equal to 0. This implies that for \( k \in \mathbb{Z} \) the blocks \( x_{[2k2^N, 2k2^N+2^N]} \) of length \( 2^N \) are all equal. Thus, for any \( n \geq 1 \), let \( N \) be such that \( n < 2^N \). Then \( x_n = x_{n+kp} \) for all \( k \in \mathbb{Z} \) with \( p = 2^{N+1} \). The period structure is \( (2^n)_{n \geq 1} \).
More generally, a substitution $\sigma$ of constant length $n$ is said to have a coincidence at index $k$ for $1 \leq k \leq n$ if the $k$-th letter of every $\sigma(a)$ is the same.

**Proposition 2.6.2** A fixed point of a constant length substitution having a coincidence is a Toeplitz sequence.

The proof is similar to the one above.

There is a constructive way to obtain all Toeplitz sequences. Let $A$ be a finite alphabet and $?$ a letter not in $A$ (usually the symbol $?$ is referred as a “hole”). Let $x \in (A \cup \{?\})^\mathbb{Z}$. Given $x, y \in (A \cup \{?\})^\mathbb{Z}$, define $F_x(y)$ as the sequence obtained from $x$ replacing consecutively all the $?$ by the symbols of $y$, where $y_0$ is placed in the first $?$. In particular, if $x$ has no holes, $F_x(y) = x$ for every $y \in (A \cup \{?\})^\mathbb{Z}$. In addition, observe that:

$$\text{if } z = F_x(y) \text{ then } F_z = F_x \circ F_y.$$ (2.6.1)

Now, consider a sequence of finite words $(w(n))_{n \in \mathbb{N}}$ in $A \cup \{?\}$. For each $n$, let $y(n)$ be the periodic sequence $w(n) = \cdots w(n)w(n)w(n)w(n)w(n)w(n) \cdots \in (A \cup \{?\})^\mathbb{Z}$, where the central dot indicates the position to the left of coordinate 0.

We define the sequence $(z(n))_{n \geq 1}$ by: $z(1) = y(1)$ and, for every $n \geq 1$,

$$z(n + 1) = F_{z(n)}(y(n + 1)).$$ (2.6.2)

It is not complicated to see that $z(n) = u_n^\infty$ for some word $u_n$ of length $|w(1)| + |w(2)| + \cdots + |w(n)|$ and that the limit $z = \lim_{n \to \infty} z(n)$ is well defined as a sequence in $(A \cup \{?\})^\mathbb{Z}$. Moreover, if the $w(n)$ does not start or finish with a hole for infinitely many $n$, then the limit sequence belongs to $A^\mathbb{Z}$, i.e., $z$ has no holes. It is clear that $z$ is a Toeplitz sequence.

On the other way round, let $(p_n)_{n \geq 1}$ be a periodic structure of the Toeplitz sequence $x$. Then for every $n \geq 1$ we can define the skeleton of $x$ at scale $p_n$ by $z(n)_m$ equal to $y_m$ if $m \in \text{Per}_{p_n}(y)$ and to $?$ otherwise. Since $z(n)$ has period $p_n$ and $p_n$ divides $p_{n+1}$, there exists a periodic sequence $y(n)$ such that (2.6.2) holds.

### 2.7 Exercises

**Section 2.1**

2.1 Let $\phi : X \to Y$ be a continuous map between compact metric spaces $X, Y$. Show that if $\phi$ is bijective, its inverse is continuous.

2.2 Let $(X, T)$ be a topological dynamical system. Show that for two nonempty sets $U, V \subset X$ and $n \geq 0$, one has $U \cap T^{-n}V \neq \emptyset$ if and only if $T^nU \cap V \neq \emptyset$.

2.3 Let $(X, T)$ be a topological dynamical system. A point $x \in X$ is recurrent if for every open set $U$ containing $x$, there is an $n > 0$ such that $T^n(x) \in U$. 
2.7. EXERCISES

Show that if \( x \) is recurrent, for every open set \( U \) containing \( x \), there is an infinity of \( n > 0 \) such that \( T^n x \in U \).

2.4 Prove Proposition 2.1.3

2.5 Show that a factor of a minimal system is minimal.

2.6 Let \( R_\alpha \) be the transformation \( x \mapsto x + \alpha \) on the torus \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). Show that the system \((X, T)\) is minimal if and only if \( \alpha > 0 \) is irrational.

2.7 Let \((X, T)\) be a topological dynamical system and let \( h : X \to \mathbb{N} \) be a continuous function. The set

\[ X^h = \{(x, i) \mid 0 \leq i < h(x)\} \]

is a compact metric space as a closed subset of the product of finitely many copies of \( X \) (indeed, since \( h \) is continuous on the compact space \( X \), it is bounded). Moreover, the map

\[ T^h(x) = \begin{cases} (x, i + 1) & \text{if } i + 1 < h(x) \\ (Tx, 0) & \text{otherwise} \end{cases} \]

is continuous. Thus \((X^h, T^h)\) is a topological dynamical system called the tower over \( X \) defined by \( h \). Show that \((X^h, T^h)\) is invertible if \((X, T)\) is invertible, minimal if \((X, T)\) is minimal and that the system induced on \( X \times \{0\} \) is isomorphic with \((X, T)\).

2.8 Let \( \phi : [0, 1] \to \{0, 1\}^\mathbb{N} \) be the map defined by \( y = \phi(x) \) if \( y_n = 2^n x \bmod 1 \) or equivalently

\[ x = \sum_{n \geq 0} y_n 2^{-n}. \]

Show that \( \phi \) is a morphism from the dynamical system \((X, T)\) with \( Tx = 2x \bmod 1 \) to the Cantor space \( \{0, 1\}^\mathbb{N} \).

2.9 Show that a minimal infinite symbolic system is a Cantor system.

Section 2.2

2.10 Show that a primitive word of length \( n \) has \( n \) distinct conjugates.

2.11 A word \( w = a_1 a_2 \cdots a_n \) with \( a_i \in A \) has period \( p \) if \( w_i = w_{i+p} \) for \( 1 \leq i \leq n-p \). Prove the following property of periods of words, known as Fine-Wilf Theorem. Let \( p, q \geq 1 \) be integers and \( d = \gcd(p, q) \). If a word \( w \) has period \( p \) and \( q \) with \( |w| \geq p + q - d \), it has period \( d \).

2.12 Show that for every factorial extendable set \( L \), there is a unique shift space \( X \) such that \( \mathcal{L}(X) = L \).

2.13 Show that the Fibonacci word is linearly recurrent.
Section 2.3

2.14 Show that a shift space $X$ is of finite type if and only if there is an $n \geq 1$ such that every $v \in L_n(X)$ satisfies

$$uv, vw \in L(X) \Rightarrow uvw \in L(X) \quad (2.7.1)$$

for every $u, w \in L(X)$.

2.15 Show that the class of shifts of finite type is closed under conjugacy.

2.16 Prove Proposition 2.3.1

2.17 Given a finite graph $G$ with edges labeled by letters of an alphabet $A$, we denote by $X_G$ the set of labels of infinite paths in $G$. A shift space $X$ on the alphabet $A$ is sofic if there is a finite labeled graph $G$, called a presentation of $X$, such that $X = X_G$. Show that sofic shifts are the factors of shifts of finite type.

2.18 A labeled graph $G$ is right-resolving if the edges going out of the same vertex have different labels. A right-resolving presentation of a sofic shift is a right-resolving graph $G$ such that $X = X_G$. Show that

1. every sofic shift has a right-resolving presentation
2. every irreducible sofic shift has a unique minimal strongly connected right-resolving presentation.

Hint: consider the follower sets $F(u) = \{v \in L(X) \mid uv \in L(X)\}$ for $u \in L(X)$. If $G = (Q, E)$ is a right-resolving presentation of $X$, denote $I(u) = \{q \in Q \mid$ there is a path ending at $q$ labeled $u\}$ for $u \in L(X)$. Show that $I(u) = I(v)$ implies $F(u) = F(v)$.

2.19 Two nonnegative integral square matrices $M, N$ are elementarly equivalent if there are nonnegative integral matrices $U, V$ such that

$$M = UV, \quad N = VU$$

The matrices are strong shift equivalent if there is a sequence $(M_1, M_2, \ldots, M_k)$ of matrices such that $M_1 = M, M_k = N$ and $M_i$ elementary equivalent to $M_{i+1}$ for $1 \leq i \leq k - 1$.

Let $M$ be a nonnegative $n \times n$-matrix. Denote by $X_M$ the edge shift on a graph with adjacency matrix $M$. Show that if $M, N$ are strong shift equivalent, then $X_M, X_N$ are conjugate.
2.7. EXERCISES

Section 2.4

2.20 Let \( \sigma : A^* \to A^* \) be a morphism. Show that if \( \sigma \) is primitive, then \( \mathcal{L}(\sigma^n) = \mathcal{L}(\sigma) \) for every \( n \geq 1 \). Give an example of a morphism such that \( \mathcal{L}(\sigma^2) \) is strictly contained in \( \mathcal{L}(\sigma) \).

2.21 Let \( x \) be the Thue-Morse sequence. Prove that \( x^n = a \) if and only if the number of 1 in the binary expansion of \( n \) is even.

2.22 Let \( \varepsilon_n \in \{-1, 1\} \) be the parity of the number of (possibly overlapping) factors 11 in the binary representation of \( n \). The sequence \( \varepsilon_0 \varepsilon_1 \varepsilon_2 \cdots \) is the Rudin-Shapiro sequence. Show that it is the image under the morphism \( \phi : a \to 1, b \to 1, c \to -1, d \to -1 \) of the fixed point \( x \) beginning with \( a \) of the substitution \( \sigma : a \to ab, b \to ac, c \to db, d \to dc \).

2.23 Show that the Chacon binary shift generated by the Chacon binary substitution \( \sigma : 0 \to 0010, 1 \to 1 \) is minimal.

2.24 Let \( \sigma : A^* \to A^* \) be a growing substitution. Show that if the shift generated by \( \sigma \) is minimal, then \( \sigma \) is primitive.

2.25 A morphism \( \varphi : A^* \to A^* \) is elementary (not to be confused with the elementary automorphisms of Section 2.5) if it cannot be written \( \varphi = \alpha \circ \beta \) with \( \beta : A^* \to B^*, \alpha : B^* \to A^* \) and \( \text{Card}(B) < \text{Card}(A) \). Prove that an elementary nonerasing morphism defines an injective map from \( A^\mathbb{N} \) to \( A^\mathbb{N} \).

2.26 Prove that if \( x \) is a periodic fixed point of a primitive elementary morphism, then every letter \( a \in A \) can be followed by at most one letter in \( x \) and thus that the period of \( x \) is at most \( \text{Card}(A) \).

2.27 Prove that if a fixed point \( x \) of a primitive morphism \( \varphi \) is periodic, then the period of \( x \) is at most \( |\varphi|^{\text{Card}(A)-1} \) where \( |\varphi| = \max\{|\varphi(a)| \mid a \in A\} \) (Hint: Use Exercise 2.26 and the fact that if \( \varphi = \alpha \circ \beta \) and if \( x \) is a periodic fixed point of \( \varphi \), then \( y = \beta(x) \) is a periodic fixed point of \( \psi = \beta \circ \alpha \) and that the period of \( x \) is at most the period of \( y \) times \( |\alpha| \leq |\varphi| \)).

2.28 Prove that a submonoid \( M \) of \( A^* \) is generated by a code if and only if it satisfies
\[ u, uv, vw, w \in M \implies v \in M. \] (2.7.2)
for all \( u, v, w \in A^* \).

2.29 Let \( U \subset A^+ \) be finite set of words. The flower automaton of \( U \) is the following labeled graph \( A(U) = (Q, E) \). The set \( Q \) of vertices of \( A(U) \) is the set of pairs \( (u, v) \) of nonempty words such that \( uv \in U \) plus the special vertex \( \omega \). For every \( u, a, v \) with \( a \in A \) and \( u, v \in A^* \) such that \( uav \in U \) there is an edge \( e \) labeled \( a \) with

\[ u, uv, vw, w \in M \implies v \in M. \] (2.7.2)
(i) $e : (u, av) \rightarrow (ua, v)$ if $u, v \neq \varepsilon$

(ii) $e : (\omega, av) \rightarrow (a, v)$ if $u = \varepsilon, v \neq \varepsilon$

(iii) $e : (u, a) \rightarrow (ua, \omega)$ if $u \neq \varepsilon, v = \varepsilon$

(iv) and finally $e : \omega \rightarrow \omega$ if $u = v = \varepsilon$.

Show that the number of paths labeled $w$ from $\omega$ to $\omega$ is equal to the number of factorizations of $w$ in words of $U$. Show that $U$ is

1. a code if and only if there is a unique path from $p$ to $q$ labeled $w$ for every $p, q \in Q$ and $w \in A^*$,

2. a circular code if for every nonempty word $w$ there is at most one $p \in Q$ such that there is a cycle labeled $w$ from $p$ to $p$.

2.30 Let $U$ be a code. A pair $(x, y)$ of words in $U^*$ is synchronizing if for every $u, v \in U^*$, one has

$uxyv \in U^* \Rightarrow ux, yv \in U^*$.

Note that this definition is coherent with the definition of synchronizing pair given in Section 2.4. Consider the flower automaton $A(U) = (Q, E)$. Let $\mu$ be the morphism from $A^*$ into the monoid of $Q \times Q$-matrices with integer elements defined by

$\mu(w)_{p,q} = \begin{cases} 1 & \text{if } p \xrightarrow{w} q \\ 0 & \text{otherwise} \end{cases}$

Show that the following conditions are equivalent for $x, y \in U^*$

(i) $(x, y)$ is synchronizing.

(ii) $\mu(xy)_{p,q} = \mu(x)_{p,\omega} \mu(y)_{\omega,q}$ for all $p, q \in Q$, and thus $\mu(xy)$ has rank one.

Moreover, if $\mu(x), \mu(y)$ have rank one, then $(x, y)$ is synchronizing.

2.31 A code $U \subset A^+$ is said to have finite synchronization delay $n$ (or to be uniformly synchronized) if there is an integer $n$ such that every pair $x, y$ of words in $U^*$ of length at least $n$ is synchronizing.

Show that the following conditions are equivalent for a finite code $U$ on the alphabet $A$.

(i) $U$ is a circular code.

(ii) $U$ has finite synchronization delay.

(iii) every sequence in $A^\mathbb{Z}$ has at most one factorization in words of $U$, that is for every $x \in A^\mathbb{Z}$ there is at most one pair $(k, y)$ with $y \in B^\mathbb{Z}$ and $0 \leq k < |\varphi(y_0)|$ such that $x = S^k \varphi(y)$.

Hint: for (i) \Rightarrow (ii), use Exercise 2.30.
Section 2.5

2.32 Show that the Fibonacci shift is Sturmian. Hint: show that the left special words are the prefixes of the \( \varphi^n(a) \).

2.33 Prove that the Tribonacci shift is standard episturmian.

2.34 Show that for every word \( u \), the palindromic closure of \( u \) is \( u^{(+)} = uv^{(-)}\overline{u} \) where \( v \) is the longest palindrome suffix of \( u \).

2.35 Prove that for every \( w \in A^* \) and \( a \in A \), one has

\[
\text{Pal}(wa) = \begin{cases} 
\text{Pal}(w)a\text{Pal}(w) & \text{if } |w|_a = 0 \\
\text{Pal}(w)\text{Pal}(w)^{-1}\text{Pal}(w) & \text{if } w = w_1aw_2 \text{ with } |w_2|_a = 0 
\end{cases}
\]

2.36 Show that if \( s = \text{Pal}(x) \) is a standard strict episturmian one-sided sequence with directive word \( x = x_0x_1\cdots \), one has

\[
\mathcal{R}_X(u_n) = \{L_{x_0\cdots x_{n-1}}(a) \mid a \in A\} 
\]

where \( u_n = \text{Pal}(x_0x_1\cdots x_{n-1}) \) are the palindrome prefixes of \( s \).

2.37 Let \( (d_1, d_2, \ldots) \) be a sequence of integers with \( d_1 \geq 0 \) and \( d_n > 0 \) for \( n \geq 2 \). The standard sequence with directive sequence \( (d_1, d_2, \ldots) \) is the sequence \( (s_n) \) of words defined by \( s_0 = 0, s_1 = 0d_11 \) and \( s_n = s_{n-1}^{d_n}s_{n-2} \) for \( n \geq 2 \). Show that each \( s_n \) is a primitive word which is prefix of the characteristic sequence of slope \( \alpha = [0, 1 + d_1, d_2, \ldots] \).

2.38 Let \( (s_n) \) be the standard sequence with directive sequence \( (d_1, d_2, \ldots) \) and let \( \alpha = [0, 1 + d_1, d_2, \ldots] \). Show that for \( n \geq 3 \), the word \( s_n^{1+d_n+1} \) is a prefix of the characteristic word \( c_\alpha \) but not the word \( s_{n-1}^{2+d_n+1} \).

2.39 A sequence \( x \) is said to be \( d \)-power free if for every nonempty word \( w, w^n \in \mathcal{L}(x) \) implies \( n < d \). Show that if a Sturmian word of slope \( \alpha = [a_0, a_1, a_2, \ldots] \) is \( d \)-power free for some \( d \), the \( a_i \) are bounded (note that the converse is also true, see the Notes). Hint: use Exercise 2.38.

2.40 Show that if a Sturmian word of slope \( \alpha = [a_0, a_1, \ldots] \) is linearly recurrent, the \( a_i \) are bounded.

Section 2.6

2.41 Let \( x \in \{0, 1\}^\mathbb{N} \) be the period-doubling sequence, which is the fixed point of the substitution \( \sigma : 0 \rightarrow 01, 1 \rightarrow 00 \). Show that

\[
x_n = \nu_2(n + 1) \mod 2
\]

where \( \nu_2(m) \) is the number of 0 ending the binary representation of \( m \).
2.8 Solutions

Section 2.1

2.1 Let \( y_n \) be a sequence in \( Y \) converging to \( y \). Set \( x_n = \phi^{-1}(y_n) \). Since \( X \) is compact, the sequence \( x_n \) has accumulation points. Since \( f \) is continuous they are equal to \( x = f^{-1}(y) \) and thus \( x_n \) converges to \( x \).

2.2 Assume that \( U \cap T^{-n}V \neq \emptyset \). Let \( x \in U \cap T^{-n}V \neq \emptyset \). Then \( T^n x \in T^nU \cap V \). Conversely, if \( y \in T^nU \cap V \), there is some \( x \in U \) such that \( T^n(x) = y \). Thus \( x \in U \cap T^{-n}V \).

2.3 If \( T^n x \in U \), then \( x \in U \cap T^{-n}U \). Since the latter is an open set, there is an \( m > 0 \) such that \( T^m x \in U \cup T^{-n}U \) and thus \( T^{n+m}x \in U \). The same argument can be repeated to obtain the conclusion.

2.4 (i)⇒(ii) Let \((U_n)_{n \geq 0}\) be a countable basis of open sets (this exists for any compact metric space). Since \((X,T)\) is recurrent, the set

\[
S = \cap_{n \geq 0} \cup_{m \geq 0} T^{-m}U_n.
\]

is dense in \( X \). Every \( x \in S \) is recurrent since for every open set \( U \) containing \( x \), there is an \( U_n \) contained in \( V \) and not containing \( x \). Then \( x \in T^{-m}U_n \) for some \( m > 0 \), showing that \( x \) is recurrent. (ii)⇒(iii) is obvious. (iii)⇒(i) Let \( x_0 \in X \) be a recurrent point. For every pair \( U,V \) of nonempty open sets, there are, by Exercise 2.3, arbitrary large integers \( n, m \) such that \( T^n x \in U, T^m x \in V \). Choosing \( n < m \), we obtain \( U \cap T^{-n}V \neq \emptyset \).

2.5 Let \( \phi : (X,T) \to (X',T') \) be a morphism from a minimal system \((X,T)\) to \((X',T')\). Let \( Y' \) be a closed stable nonempty subset of \( X' \). Then \( Y = f^{-1}(Y') \) is nonempty and closed. It is also stable because for \( y \in Y \), we have \( \phi(T(y)) = T'(\phi(y)) \in T'Y' \subset Y' \) and thus \( T(y) \in Y \). Thus \( Y = X \) which implies \( Y' = X' \).

2.6 Assume that \( \alpha \) is irrational. For every \( q \geq 1 \), there is a \( p \) such that \( \alpha \in (p/q, (p+1)/q) \). Then every \( x \in [0,1) \) is an interval \([np/q, (n+1)p/q)\) and thus \( |x - n\alpha| \leq 1/q \).

2.7 If \( T \) is invertible, the inverse of \( T^h \) is the map which sends \( (x,i) \) to \( (x,i-1) \) if \( i > 1 \) and to \( (T^{-1}x,1) \) otherwise. If \((X,T)\) is minimal, the orbit of every \((x,i) \in X^h \) is clearly dense in \( X^h \). Thus \((X^h,T^h)\) is minimal. The last assertion result of the fact that \( f(x) \) is the return time to \( U = X \times \{0\} \) and thus

\[
(T^h)_U(x,0) = (T^h)^{f(x)}(x,0) = (T x, 0).
\]

2.8 The map \( \phi \) is continuous and \( \phi \circ T = S \circ \phi \).
2.8. SOLUTIONS

2.9 Let \((X, T)\) be an infinite minimal symbolic system. Since \(X\) is a closed subset of \(A^Z\), it is compact and totally disconnected. For every \(x \in X\), the orbit of \(x\) is dense in \(X\) since \(X\) is minimal. Since \(X\) is infinite, it contains points arbitrary close to \(x\) but distinct of \(x\). Thus \(x\) is not isolated.

Section 2.2

2.10 Assume that \(w\) is a primitive word of length \(n\) and that \(w = uv = vu\). Then for every \(k \geq 1\) we have \(w^k = u^kv\), as shown easily by induction on \(k\). This implies that \(w^\omega = u^\omega\) and thus \(w = u\).

2.11 Assume \(p < q\). We use induction on \(p + q\). The result is trivial if \(p = d\). Otherwise, let \(u\) be the prefix of length \(q - d\) of \(w\). Then, for \(1 \leq i \leq p - d\), we have \(u_i = u_{i+p} = u_{i+p-q}\). This shows that \(u\) has period \(q - p\). By induction hypothesis, \(u\) has period \(d\). Since \(|u| \geq p\), the word \(w\) has also period \(d\).

2.12 Let \(X\) be the set of \(x \in A^Z\) with all its factors in \(L\). Clearly \(X\) is the largest shift space such that \(L(X) \subseteq L\). For \(u \in L\), there is a sequence \((a_n, b_n) \in A \times A\) such that \(a_n \cdots a_1 b_1 b_2 \cdots b_n \in L\) for every \(n \geq 0\). Then \(\cdots a_2 a_1 \cdot b_1 b_2 \cdots\) is in \(X\) and thus \(u \in L(X)\).

2.13 Let \(\varphi : a \to ab, b \to a\) be the Fibonacci morphism and let \(x\) be the Fibonacci word. Let \(F_n\) be the Fibonacci sequence defined by \(F_0 = 0, F_1 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for \(n \geq 1\). Note that \(F_n \leq F_{n+1}\) implies \(F_{n+1} \leq 2F_n\) and \(F_{n+2} \leq 3F_n\) for \(n \geq 1\).

For every \(n \geq 1\), we have \(\varphi^{n+1}(a) = \varphi^n(a)\varphi^n(b)\). Thus \(|\varphi^n(a)| = F_{n+2}\) and \(|\varphi^n(b)| = F_{n+1}\).

Let \(w \in L(x)\) and let \(n\) be the least integer such that \(|w| < F_{n+1}\). By the choice of \(n\), we have \(F_n \leq |w|\).

Since \(w\) has no factor in \(\varphi^n(A)\), it is a factor of \(\varphi^n(A^2)\). But the largest difference between the occurrences of a word of length \(2\) in the Fibonacci word is \(5\) (this bound is reached by \(aa\) in \(aababa\)). Thus two occurrences of \(w\) in \(x\) are separated by at most \(5F_{n+2} \leq 15F_n \leq 15|w|\). This shows that \(x\) is linearly recurrent.

Section 2.3

2.14 Assume first that \((X, S)\) is a shift of finite type defined by a finite set \(I\) of forbidden blocks. Let \(n\) be the maximal length of the words of \(I\). It is clear that every \(v \in L_n(X)\) satisfies \((\ref{eq:2.7.1})\).

Conversely, consider the Rauzy graph \(G = \Gamma_n(X)\). By condition \((\ref{eq:2.7.1})\), the label of every every infinite path in \(G\) is in \(X\). Thus the edge shift on \(G\) can be identified with \((X, S)\).
2.15 Let $\varphi : X \to Y$ be a sliding block code from $(X, S)$ to a shift of finite type $(Y, S)$ which is a conjugacy. We may suppose that $\varphi, \varphi^{-1}$ are defined by block maps $f, g$ with memory and anticipation $\ell$. Let $k$ be the integer such that every word in $L_k(Y)$ satisfies (2.7.1). Set $m = k + 4\ell$. Then, one may verify that every word in $L_m(X)$ satisfies (2.7.1). Thus $(X, S)$ is a shift of finite type.

2.16 Let $I \subset A^*$ be a finite set of forbidden blocks and let $(X, S)$ be the shift of finite type corresponding to $I$. Let $n$ be the maximal length of the words in $I$ and consider the Rauzy graph $\Gamma_n(X)$. Then the edge shift on $\Gamma_n(X)$ is the $n$-th higher block presentation of $(X, S)$. If $(X, S)$ is recurrent, then $\Gamma_n(X)$ is strongly connected.

2.17 Suppose first that $X$ is sofic and let $G = (V, E)$ be a finite labeled graph such that $X = X_G$. The one-block map assigning to an edge its label is a factor map from the edge shift on $G$ onto $X$. Thus $X$ is a factor of a shift of finite type.

Conversely, let $Y$ be a shift of finite type, which may be assumed to be an edge shift. Let $X$ be the image of $Y$ by a sliding block code $f : \mathcal{L}_{m+n-1}(X)$ with memory $m$ and anticipation $n$. Consider the $n + m - 1$th higher block presentation $Z$ of $Y$. Then $Z$ is conjugate to $Y$ and thus is a shift of finite type by Exercise 2.15. Let $G$ be the Rauzy graph $\Gamma_{m+n-1}(Y)$. Then $Y$ can be identified with the edge shift on $G$. Let $H$ be the graph which the same as $G$ but with the labeling defined by the block map $f$. Clearly $X = X_H$ and thus $X$ is a sofic shift.

2.18 Let $X = X_G$ be a sofic shift where $G = (V, E)$ is a finite graph with labels in $A$. Let $H = (W, F)$ be the following graph. The set $W$ is the set $\Psi(V)$ of subsets of $V$. For $P, Q \subset V$, there is an edge from $P$ to $Q$ labeled $a$ if

$$Q = \{ q \in V \mid \text{there is an edge } p \xrightarrow{a} q \text{ with } p \in P \}.$$

Then $H$ is right-resolving and it is easy to see that $X = X_H$.

Assume now that $X$ is an irreducible sofic shift. Any minimal right-resolving presentation of $X$ is clearly strongly connected.

Set $L = \mathcal{L}(X)$. Let $M$ be the following labeled graph. Its vertices are the follower sets

$$F(u) = \{ v \in L \mid uv \in L \}$$

for $u \in L$. There is an edge from $p$ to $q$ labeled $a$ if $p = F(u)$ and $q = F(ua)$. Let us show that any strongly connected right-resolving minimal presentation $G = (Q, E)$ of $X$ can be identified with $M$. For every $p \in Q$, let $F(p)$ be the set of all $u \in L$ such that there is a path with label $u$ starting at $p$. Define an equivalence on $Q$ by $p \sim q$ if $F(p) = F(q)$. The quotient $Q/\sim$ is clearly again a right-resolving presentation of $X$. Since $G$ is minimal, the equivalence $\sim$ is the equality.

Consider, for $u \in L$, the set $I(u)$ of all $q \in Q$ such that there is a path in $G$ with label $u$ ending at $q$. Clearly, if $I(u) = I(v)$, then $F(u) = F(v)$. Take $u \in L$
such that $I(u)$ is of minimal cardinality. For every $p, q \in I(u)$, we have $p \sim q$. Thus $I(u)$ has only one element. Since $G$ is strongly connected, every element of $Q$ appears in this way. This allows us to associate to every $p \in Q$ the follower set $F(u)$ where $u \in L$ is such that $I(u) = \{p\}$. This identifies $G$ with $M$.

2.19 It is enough to consider the case of two elementary equivalent matrices $M = UV$ and $N = VU$. Let $G_M = (V_M, E_M)$ and $G_N = (V_N, E_N)$ be graphs with matrices $M, N$ respectively. Let also $G_U$ be the graph on $E_M \cup E_N$ having $U_{xy}$ edges from $x \in E_M$ to $y \in E_N$ and similarly for $G_V$.

Since $M = UV$, there is a bijection $e \mapsto (u(e), v(e))$ from $E_M$ onto the paths of length 2 made of an edge of $G_U$ followed by an edge of $G_V$. We denote $e(u, v)$ the inverse map. Similarly, we have a bijection $f \mapsto (v(f), u(f))$ from $E_N$ onto the paths formed of an edge of $G_V$ followed by an edge of $G_U$ with an inverse map denoted $f(v, u)$.

We define a 2-block map $s$ (with memory 0) from $X_M$ to $X_N$ by $s(e_0 e_1) = f(v(e_0) u(e_1))$. Let $\sigma : X_M \rightarrow X_N$ be the sliding block code defined by $s$. Similarly, let $t$ be the 2 block map (with memory 0) from $X_N$ to $X_N$ defined by $t(f_0 f_1) = e(u(f_0) v(f_1))$. Let $\tau : X_N \rightarrow X_N$ be the corresponding block map. We have (see Figure 2.8.1)

$$\tau \circ \sigma = S_M$$

where $S_M$ is the shift transformation on $X_M$.

![Figure 2.8.1: The conjugacies $\sigma$ and $\tau$.](image)

Section 2.4

2.20 Let $\sigma : a \rightarrow b, b \rightarrow aa$. Then $\mathcal{L}(\sigma) = \{a, b, aa, a aa, \ldots\}$ while $\mathcal{L}(\sigma^2) = \{a, aa, a aa, \ldots\}$. Assume that $\sigma$ is primitive. Let $m \geq 1$ be such that $|\sigma^m(b)|_a > 0$ for every $a, b \in A$. Consider $w \in \mathcal{L}(\sigma)$. Let $p \geq 1$ and $a \in A$ be such that $w$ is a factor of $\sigma^m(a)$. Let $q \geq m$ be such that $p + q$ is a multiple of $n$. Then $w$ is a factor of $\sigma^{p+q}(a)$ and thus $w \in \mathcal{L}(\sigma^n)$.

2.21 Let $\tilde{a} = b$ and $\tilde{b} = a$. With this notation, the Thue-Morse morphism is defined by $\tau(x) = x\tilde{x}$ for every $x \in \{a, b\}$. The property is true for $n =$
0, 1. Next, since \( \tau(x) = x \), and \( \tau(x_0 \cdots x_{n-1}) = x_0 \cdots x_{2n-1} \), we have \( \tau(x_n) = x_{2n}x_{2n+1} \). Thus \( x_{2n} = x_n \) and \( x_{2n+1} = \bar{x}_n \). This proves the property by induction on \( n \).

2.22 Consider the labeled graph represented in Figure 2.8.2.

![Figure 2.8.2: The automaton of the Rudin-Shapiro sequence.](image)

Let \( q_n \) be the vertex reached from \( a \) following a path labeled by the binary representation \( b(n) \) of \( n \). It is easy to verify by induction on the length of \( b(n) \) that

1. One has \( x_n = q_n \).
2. One has

\[
\varepsilon_n = \begin{cases} 
1 & \text{if } q_n \in \{a, b\} \\
-1 & \text{otherwise}
\end{cases}
\]

Thus we conclude that \( \varepsilon_n = \phi(x_n) \).

2.23 Set \( v_n = \sigma^n(0) \). Then \( v_{n+1} = v_nv_nv_n \). Any \( u \in \mathcal{L}(X(\sigma)) \) is a factor of some \( v_n \). Thus for every \( n \geq 1 \) there exists \( N \) such that every word of \( \mathcal{L}_n(X) \) is a factor of \( v_N \) and thus an integer \( M \) such that every word in \( \mathcal{L}_n(X(\sigma)) \) is a factor of every word in \( \mathcal{L}_M(X(\sigma)) \).

2.24 Since \( \sigma \) is a substitution, it is prolongable on some \( a \in A \) and every letter appears in \( x = \sigma^\omega(a) \). Thus every \( b \in A \) appears in \( x \). Since \( X(\sigma) \) is minimal, \( x \) is uniformly recurrent. Since \( \sigma \) is growing, for every \( b \in A \) there is some \( n \geq 1 \) such that \( a \) appears in \( \sigma^n(b) \). Since \( \sigma \) is prolongable on \( a \), the letter \( a \) also appears in all \( \sigma^m(b) \) for \( m \geq n \). Thus, there is an \( N \) such that \( a \) appears in all \( \sigma^N(b) \) for \( b \in A \). By minimality again, there is a \( k \geq 1 \) such that every \( c \in A \) appears in \( \sigma^k(a) \). Then every \( c \) appears in every \( \sigma^{N+k}(b) \).

2.25 Let \( \varphi : A^* \to A^* \) be a morphism which is not injective as a map from \( A^N \) to itself. Set \( U = \varphi(A) \). Let \( Y \) be the basis of the intersection of all free submonoids containing \( U^* \) and let \( \beta : B \to Y \) be a bijection from an alphabet \( B \) with \( Y \). Then there is a morphism \( \alpha : A^* \to B^* \) such that \( \varphi = \alpha \circ \beta \). For a word \( x \in U \), let \( \lambda(x) \in Y \) be the first symbol in its decomposition in words of \( Y \), that is \( x \in \lambda(x)Y^* \). If some \( y \in Y \) does not appear as an initial symbol
2.8. SOLUTIONS

in the words of \( U \), set \( Z = (Y \setminus y)g^* \). Then \( Z^* \) is free and \( U^* \subset Z^* \subset Y^* \). Thus \( Y = Z \), a contradiction. Since \( \varphi \) is not injective on \( A^N \), the map \( \lambda \) is not injective and thus \( \text{Card}(Y) < \text{Card}(U) \), showing that \( \varphi \) is not elementary.

2.26 Let \( p(n) \) be the number of factors of length \( n \) of \( x \). Let us show that if \( p(n) < p(n + 1) \), then there is an \( m > n \) such that \( p(m) < p(m + 1) \). Indeed if \( p(n) < p(n + 1) \) there is a factor \( u \) of length \( n \) of \( x \) which is right-special, that is, there are two distinct letters \( a, b \) such that \( ua, ub \in F(x) \). Since \( \varphi \) is elementary, it is injective on \( A^N \) (Exercise 2.25). Thus there are some \( v, w \) such that \( \varphi(av) \neq \varphi(bw) \). If \( |\varphi(u)| > |u| \) or if \( \varphi(a), \varphi(b) \) begin by the same letter, the longest common prefix of \( \varphi(uav), \varphi(ubw) \) is a right-special word of length \( m > n \). Otherwise we may replace \( u \) by \( \varphi(u) \). Since \( \varphi \) is primitive, the second case can happen only a bounded number of times. If \( x \) is periodic, then \( p(n) \) is bounded and by the preceding argument, no letter can be right-special, which implies that the period of \( x \) is at most \( \text{Card}(A) \).

2.27 We use an induction on \( \text{Card}(A) \). If \( \text{Card}(A) = 1 \), the result is true since the period is 1. Otherwise, either \( \varphi \) is elementary and thus injective by Exercise 2.25. Then, by Exercise 2.26 the period of \( x \) is at most equal to \( \text{Card}(A) \). Finally, assume that \( \varphi = \alpha \circ \beta \) with \( \beta : A^* \to B^* \) and \( \alpha : B^* \to A^* \) and \( \text{Card}(B) < \text{Card}(A) \). Set \( y = \beta(x) \) and \( \psi = \beta \circ \alpha \circ \beta \). Then \( \psi(y) = \beta \circ \alpha \circ \beta(x) = \beta(x) = y \). Thus \( y \) is a fixed point of \( \psi \). Since \( \psi \) is primitive, we may apply the induction hypothesis. Since the period of \( y \) is at most \( |\psi|^{\text{Card}(B)-1} \), the period of \( x \) is at most \( |\alpha| |\psi|^{\text{Card}(B)-1} \leq |\varphi|^{\text{Card}(A)-1} \).

2.28 Assume first that \( M = U^* \) where \( U \) is a code. Then, the unique factorisation of \( u(vw) = (uv)w \) forces \( v \in M \). Conversely, let \( U \) be the the set of nonempty words in \( M \) which cannot be written as a product of strictly shorter words in \( M \). Take \( x_1x_2 \cdots x_n = y_1y_2 \cdots y_m \) with \( x_i, y_j \in U \), and \( n, m \geq 0 \) minimal. By definition of \( U \), we have \( n, m \geq 1 \). Assume \( |x_1| \leq |y_1| \) and set \( y_1 = x_1v \). Then, \( x_2 \cdots x_n = vy_1 \cdots y_m \). By (2.7.2), we have \( v \in M \), which forces \( v = \varepsilon \) and \( x_1 = y_1 \), a contradiction with the minimality of \( n, m \).

2.29 The first assertion is clear since, by construction of \( A(U) \), the nonempty simple paths from \( \omega \) to \( \omega \) are in bijection with \( U \).

1. Suppose first that \( U \) is a code. By definition there can be at most one path from \( \omega \) to \( \omega \) labeled \( w \) since otherwise \( w \) would have two factorisations in words of \( U \). Next, if there are two distinct paths labeled \( w \) from \( p \) to \( q \), there would be two distinct paths labeled \( uvv \) from \( \omega \) to \( \omega \) for words \( u, v \) labeling paths from \( \omega \) to \( p \) and \( q \) to \( \omega \) respectively, a contradiction. The converse is obvious.

2. The number of paths labeled \( w \) from \( (u, v) \) to itself is the number of factorisations \( w = vxu \) with \( x \in U^* \). Assume that \( U \) is circular. There cannot be cycles labeled \( w \) from \( \omega \) to \( \omega \) and also from \( (u, v) \) to \( (u, v) \) by definition of a circular code. Next, suppose that there are cycles labeled \( w \) around \( (u, v) \) and
around \((u',v')\). Set \(w = v xu = v'x' u'\). Assuming \(|u| < |v'|\), set \(v' = vz\) and \(t = x' u'\). Then \(x u = z t\) implies \(z t v = x u v\), which shows that \(z t v \in U^*\). But \(vzt,z t v \in U^*\) implies \(z,t,v \in U^*\) by (2.4.9). Thus \(w = v xu = vzt\) is in \(U^*\), a contradiction. The converse is obvious.

2.30 (i) \(\Rightarrow\) (ii) the first assertion results from the definition of a synchronizing pair. The second one is clear since \(\mu(x)\) is the product of the column of index \(\omega\) of \(\mu(x)\) by the row of index \(\omega\) of \(\mu(y)\).

If \(\mu(x), \mu(y)\) have rank one, consider \(u,v \in A^*\) such that \(uxyv \in U^*\). Since \(\mu(x)\) has rank one, it follows from \(uxyv \in U^*\) that \(ux,yv \in U^*\). Similarly, \(uxyv \in U^*\). This shows that \((x,y)\) is synchronizing.

2.31 (i) \(\Rightarrow\) (ii). Let \(A(U) = (Q,E)\) be the flower automaton of \(U\) and let \(\mu\) be as in Exercise 2.30. Let \(S\) be the finite semigroup \(\mu(A^+)\). For \(m \in S\) write \(p^m q\) for \(m_{p,q} = 1\). Since \(S\) is finite, there is for every \(m \in S\) an integer \(k \geq 1\) such that \(m^{2k} = m^k\). Let \(e\) be such an idempotent. For every \(p,q \in Q\) there is, since \(e = e^3\) some \(r,s \in Q\) such that \(p \overset{e}{\rightarrow} r \overset{e}{\rightarrow} s \overset{e}{\rightarrow} q\). By unambiguity, \(r = s\), that is \(r\) is a fixed point of \(m\). But an element of \(S\) cannot have more than one fixed point by Exercise 2.29 and thus \(e\) has rank one.

Let \(J\) be the set of elements of \(S\) of rank one. Since all idempotents of \(S\) are in \(J\), we have \(S^n \subset J\) for \(n = \text{Card}(S) + 1\). Indeed, if \(m \in S^n\) with \(n \geq \text{Card}(S) + 1\), there is a factorisation \(m = uvw\) with \(uv = u\). Then \(uv^k = u\) for all \(k \geq 1\). When \(v^k\) is idempotent, we have \(v^k \in J\) and thus \(m \in J\). This shows that for every word \(x\) of length \(n\), \(\mu(x)\) has rank one. We conclude using Exercise 2.30 that \(U\) has finite synchronization delay.

(ii) \(\Rightarrow\) (iii) is clear by definition of a synchronizing pair.

(iii) \(\Rightarrow\) (i). Suppose that \(p,q \in A^*\) are such that \(pq,qp \in U^*\). Then the sequence \(\cdots qp \cdot pqpq \cdot \cdots\) has two factorizations unless \(p,q \in U^*\). Thus \(U\) is circular.

Section 2.5

2.32 The words \(F_n = \varphi^n(a)\) are left-special. Indeed, this is true for \(n = 0\) since \(aa,ba \in L(\varphi)\) and

\[ aF_{n+1} = \varphi(bF_n), \quad abF_{n+1} = \varphi(aF_n) \]

shows the claim by induction on \(n\). It is easy to see (again by induction) that conversely every left-special word is a prefix of some \(F_n\). This implies that there is exactly one left-special word of each length which is moreover extendable by every letter and is additionally a prefix of \(x\) and thus the conclusion that \(x\) is standard episturmian.
2.33 The words \( T_n = \varphi^n(a) \) are left-special with 3 extensions. Indeed, this is true for \( n = 0 \) since \( aa, ba, ca \in \mathcal{L}(X) \). Next the equalities
\[
aT_{n+1} = \varphi(cT_n), \quad abT_{n+1} = \varphi(aT_n), \quad acT_{n+1} = \varphi(bT_n)
\]
prove the property by induction. Conversely, any left-special word is a prefix of some \( T_n \), whence the assertion.

2.34 Any word \( uv^{-1} \tilde{u} \) is palindrome and has \( u \) as a prefix. Let us show that the palindromic closure is of this form. Set \( u(+)=ur=\tilde{r} \tilde{u} \). We have \(|r|<|u|\) since we can choose the last letter of \( u \) as palindrome suffix. Set \( u=\tilde{r} \tilde{s} \). Then \( ur=\tilde{r} \tilde{u} \) implies \( s=\tilde{s} \). This proves that \( u(+) \) is of the form above and thus corresponds to the longest possible \( v \).

2.35 The formula is clear when \( |w|_a = 0 \). Otherwise, Since the longest palindromic suffix of \( \text{Pal}(w)a \) is \( \text{Pal}(w_{1}) \) by Exercise 2.34.

2.36 By Justin’s Formula (2.5.2), we have for every \( a \in A \)
\[
L_{x_0x_1\cdots x_{n-1}}(a)u_n = \text{Pal}(x_0x_1\cdots x_{n-1}a).
\]
Since \( \text{Pal}(x_0x_1\cdots x_{n-1}a) \) is the shortest palindrome with prefix \( x_0x_1\cdots x_{n-1}a \), it begins and ends with \( u_n \) with no other occurrence of \( u_n \). Moreover, it is a factor of \( s \). Indeed, since every letter appears infinitely often in \( x \), there is an \( m \geq 0 \) such that \( x_{n+m} = a \). Then \( u_{n+m+1} = (u_{n+m}a)^{(+)} \) has a factor \( (u_n a)^{(+)i} = \text{Pal}(x_0x_1\cdots x_{n-1}a) \). This shows that \( L_{x_0x_1\cdots x_{n-1}}(a) \) is in \( R'_{X}(u_n) \). The converse is clear.

2.37 We first prove that the words \( s_n \) are primitive. For two words \( x,y \) on the alphabet \( \{0,1\} \), denote
\[
M(x,y) = \begin{bmatrix}
|x|_0 & |x|_1 \\
|y|_0 & |y|_1
\end{bmatrix}
\]
We prove by induction on \( n \geq 1 \) that \( \det M(s_n, s_{n-1}) = 1 \). This implies that \( |s_n|_0, |s_n|_1 \) are relatively prime and thus every \( s_n \) is primitive. The equality is true for \( n = 1 \). Next, for \( i \geq 1 \), we have
\[
M(s_n, s_{n-1}) = \begin{bmatrix}
d_i & 1 \\
1 & 0
\end{bmatrix} M(s_{n-1}, s_{n-2})
\]
whence the conclusion.

Let us now show that that every \( s_n \) is a prefix of \( c_\alpha \). For this, let
\[
h_n = \begin{cases} 
L_{0^d_11^d_2\cdots 1^d_n} & \text{if } n \text{ is even} \\
L_{0^d_11^d_2\cdots 0^d_n} & \text{if } n \text{ is odd}.
\end{cases}
\]
One can easily verify by induction on $n$ that $\{h_n(0), h_n(1)\} = \{s_n, s_n s_{n-1}\}$. By Justin Formula, this implies that $s_n$ is a prefix of $c_\alpha$.

2.38 We show that for $n \geq 3$, one has

$$s_{n-1} s_n = s_n t_{n-1} \text{ with } t_n = s_{n-1}^{d_{n-1}} s_{n-2} s_{n-1}.$$  

Indeed,

$$s_{n-1} s_n = s_{n-1} s_{n-1}^{d_{n-1}} s_{n-2} = s_{n-1}^{d_{n-1}} s_{n-2} s_{n-3} s_{n-2} = s_{n-1}^{d_{n-1}} s_{n-2} s_{n-3} s_{n-2} = s_n t_{n-1}.$$  

Observe that $t_{n-1}$ is not a prefix of $s_n$ since otherwise $s_n = t_{n-1} u$ for some word $u$ and $s_{n-1} s_n u = s_n^2$, a contradiction since $s_n$ is primitive by Exercise 2.37.

Clearly $s_{n+1} s_n$ is a prefix of the characteristic word $c_\alpha$. Since

$$s_{n+1} s_n = s_{n+1}^{d_{n+1}} s_{n-1} s_n = s_{n+1}^{d_{n+1}} s_n t_{n-1} = s_{n+1}^{d_{n+1}+1} t_{n-1},$$

the word $s_{n+1}^{d_{n+1}+1}$ is a prefix of $c_\alpha$ and since $t_{n-1}$ is not a prefix of $s_n$, the word $s_{n+1}^{d_{n+1}+2}$ is not a prefix of $c_\alpha$.

2.39 It suffices to consider the characteristic Sturmian word $c_\alpha$ with $\alpha = [0, 1 + d_1, d_2, \ldots]$. If the sequence of $a_i$ is unbounded, then $s_n^{d_{n+1}}$ is a prefix of $c_\alpha$ and consequently $c_\alpha$ is not $d$-power free for any $d$.

2.40 Let $x$ be a two-sided Sturmian sequence of slope $\alpha = [a_0, a_1, \ldots]$. Assume that $x$ is linearly recurrent with constant $K$. Since $x$ is Sturmian it is not periodic. Thus, by Proposition 2.2.7, it is $(K+1)$-power free. By Exercise ??, the coefficients $a_i$ are bounded.

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2.41 The formula is true for $n = 0$. Next, $\nu_2(2n + 1) \equiv \nu_2(n + 1) + 1 \pmod{2}$ and $\nu_2(2n + 1) = 0$ for all $n \geq 0$.

2.9 Notes

Topological dynamical systems are usually presented together with measure-theoretic ones (in which there is a measure on the space and the transformation preserves the measure, see Chapter 4). For a more detailed introduction to topological dynamical systems, see for example Brown (1976) or Petersen (1983).
2.9. NOTES

2.9.1 Topological dynamical systems

Cantor spaces (Section 2.1.4) are a classical object in topology. See Willard (2004) for more details and, in particular, (Willard, 2004, Theorem 30.3) for a proof that all Cantor spaces are homeomorphic.

The minimality of irrational rotations (Exercise 2.6) is known as the (one-dimensional) Kronecker Theorem (see Hardy and Wright (2008)).

2.9.2 Shift spaces

Shift spaces are the basic object of symbolic dynamics. The classical reference on symbolic dynamics is Lind and Marcus (1995). Many classes of shift spaces (such as shifts of finite type) are described there in much more detail than we do here.

The original reference to the Curtis-Hedlund-Lyndon Theorem is Hedlund (1969).

The classical reference for the Morse Hedlund Theorem (Theorem 2.2.12) is Morse and Hedlund (1938). The case of one-sided shifts is considered in Coven and Hedlund (1973). See also (Lothaire, 2002, Theorem 1.3.13) for example. Proposition 2.2.15 is from Cassaigne (1997) (see also (Berthé and Rigo, 2010, Theorem 4.5.4)).

The Champernowne sequence (Example 2.2.4) is from Champernowne (1933). It is usually referred to as the Champernowne constant, which is the real number 0.123456789101112... .

The bound \( K \leq 15 \) given in Exercise 2.13 for the linear recurrence of the Fibonacci word is far from optimal. It is shown in Du et al. (2014), using the software Walnut (Mousavi, 2016) based on decidable properties of substitutive sequences, that \( K \leq 3 \). Actually, one has more precisely \( K \leq (3 + \sqrt{5})/2 \), this bound being the best possible (Shallit, 2020).

Linearly recurrent sequences form an important class of sequences introduced in Durand et al. (1999). We shall study these sequences in more detail in Chapter 7.

2.9.3 Shifts of finite type


The notion of right-resolving presentation (Exercise 2.18) is close to the notion of deterministic automaton which is classical in automata theory (see Berstel et al. (2009)). The unique right-resolving minimal presentation of a sofic shift (Exercise 2.18) is due to Fischer (1975) and it is often called its Fischer cover. It is closely related with the notion of minimal automaton of a language (see Berstel et al. (2009) for example).

The notion of strong shift equivalence (Exercise 2.19) was introduced by Robert Williams in Williams (1973). The converse of the statement of Ex-
exercise 2.19 is also true, and the equivalence is Williams Classification Theorem. see (Lind and Marcus, 1995, Theorem 7.2.7). The proof of the converse uses the Decomposition Theorem which asserts that every conjugacy between shifts of finite type can be decomposed in elementary conjugacies called input splits and output splits and their inverses input merges and output merges (Lind and Marcus, 1995, Theorem 7.1.2). We will define output splits in Chapter 7. There is a close connection between strong shift equivalence and another notion called shift equivalence that we will present in Chapter 3 (see Exercise 3.12).

2.9.4 Substitution shifts

The substitution shifts of Section 2.4 form an important class of shifts and a good part of this book is focused on this class. For a more detailed treatment, see the classical reference (Queffélec, 2010) or (Berthé and Rigo, 2010), where, in particular, Theorem 2.4.10 appears.

The terminology concerning substitution shifts admits some variations. What we call purely substitutive sequence is also called a purely morphic sequence (see Allouche and Shallit (2003) for example) or a substitutive sequence (as in Queffélec (2010)). Similarly, what we call a substitutive sequence is called a morphic sequence in Queffélec (2010), Rigo (2014) and Fogg (2002).

We warn the reader that our definition of a substitution and of a substitution shift is slightly more general than that used in Queffélec (2010), where a substitution is always assumed to be growing and right-prolongable on some letter.

The Perron-Frobenius Theorem is a classical and very useful result. A proof can be found in Queffélec (2010) or in the classical Gantmacher (1959).

The property of the Thue-Morse sequence stated in Exercise 2.21 shows that the Thue-Morse sequence is an automatic sequence. Automatic sequences form a class of substitutive sequences introduced by Cobham (1972) under the name of uniform tag sequences. This theory is developed in Allouche and Shallit (2003). Another example of automatic sequence is the period-doubling sequence of Example 2.6.1 (Exercise 2.41). Still another example is the Rudin-Shapiro sequence, also called the Golay-Rudin-Shapiro sequence (Exercise 2.22). It is named after its independent invention by Golay, Rudin and Shapiro in connexion with extremal problems in analysis and problems in physics.

The definition of an elementary morphism and the property stated in Exercise 2.25 is due to Ehrenfeucht and Rozenberg (1978). The decidability of periodicity of fixed points of morphisms (Exercises 2.20 and 2.27) is due to Pansiot (1986) and Harju and Linna (1986) independently. (see also Kurka (2003)). It was extended by Durand (2013) to the decidability of a more general question, with periodicity replaced by eventual periodicity and purely substitutive sequences replaced by substitutive sequences.

The constant $K = 12$ for the linear recurrence of the Thue-Morse shift given in Example 2.4.20 is not optimal. It is shown in Schaeffer and Shallit (2012) that the optimal bound is a computable rational number for every constant length
substitution. Actually, the optimal bound for the Thue-Morse shift, computed using the software Walnut [Mousavi, 2016] is $K = 10$ (Shallit, 2020).

Codes and circular codes are described in detail in Berstel et al. (2009). A submonoid satisfying condition (2.4.2) is called stable and a submonoid satisfying (2.4.9) is called very pure. These notions were originally introduced by Schützenberger (Schützenberger, 1955). The flower automaton (Exercise 2.29) is a particular case of finite automaton. The uniqueness of paths with given origin, end and label defines the so-called unambiguous automata. The property of uniform synchronization of finite circular codes (Exercise 2.31) is due to Restivo (1975) (see also (Berstel et al., 2009, Theorem 10.2.7)).

The notion of recognizability for morphisms was introduced initially by Martin (1973) and its status remained uncertain during many years. The definition of recognizability given here is from Bezegh et al. (2009) who have proved that any (primitive or not) substitution $\sigma$ is recognizable on $X(\sigma)$ for aperiodic points. The basic result on the subject is Mossé’s Theorem (Theorem 2.4.28), which is from (Mossé, 1992; Mossé, 1996). Our presentation follows Kyriakoglou (2019) where Proposition 2.4.31 is from.

2.9.5 Sturmian shifts

The notion of Sturmian shifts (and many ideas of symbolic dynamics including the term ‘symbolic dynamics’ itself) was introduced by Morse and Hedlund (1938, 1940). An introduction can be found in Fogg (2002), Lothaire (2002) or Berthé and Rigo (2010). For a proof that Sturmian shifts are uniquely ergodic, see Berthé and Rigo (2010) for example.

Arnoux-Rauzy words are named after the paper (Arnoux and Rauzy, 1991) in which they are introduced as a generalization on more than two letters of Sturmian words. A reference for episturmian words is [Drubeau et al. (2001)] where Theorem 2.5.3 is proved. The function $\text{Pal}$ has been introduced by De Luca (1999) (see alsoRentenauer (2000)). Justin Formula (Equation 2.5.3) is from Justin and Vuillon (2000). The directive word of a standard episturmian word is called in Fogg (2002) the additive coding sequence.

Standard sequences (Exercise 2.37) are defined in Lothaire (2002) p. 75.

The statement of 2.39 (with a converse) is (Lothaire, 2002, Theorem 2.2.31) (see also Berstel (1999) where it is credited to Mignosi (1991). Powers in Sturmian sequences have been extensively studied (see Damanik and Lenz (2002, 2003)). We shall see in Chapter 8 a closely related statement concerning linearly recurrent sequences (Corollary 8.2.6).

2.9.6 Toeplitz shifts

We refer to Downarowicz (2005) for a survey on Toeplitz shifts (see also Williams (1984); Jacobs and Keane (1969)). The coincidences in substitutions of constant length have been introduced in Dekking (1977/78).
Chapter 3

Ordered groups

We now introduce notions concerning abelian groups: ordered abelian groups and direct limits of abelian groups. This will allow us to define edge dimension groups which are our main object of interest in the book.

In Section 3.1, we define abelian ordered groups and the notions of positive morphism or unit of an ordered group. In Section 3.3, we introduce direct limits of ordered groups, an essential notion for the following chapters. We come in Section 3.4 to the main focus of this book, that is, dimension groups, as direct limits of groups \( \mathbb{Z}^d \) with the usual order. We prove the important theorem of Effros, Handelman and Shen characterizing dimension groups among abelian ordered groups (Theorem 3.4.3). The use of the term ‘dimension’ in the name of dimension groups will be explained in the last chapter (Chapter 10) where the dimensions groups are related to the dimensions of some algebras.

3.1 Ordered abelian groups

By an ordered group we mean an abelian group \( G \) with a partial order \( \leq \) which is compatible with the group operations, that is, such that for all \( g, h \in G \) with \( g \leq h \), one has \( g + k \leq h + k \) for every \( k \in G \).

In an ordered group \( G \), the positive cone is the set \( G^+ = \{ g \in G \mid g \geq 0 \} \). It is a submonoid of \( G \), that is, it contains 0 and satisfies \( G^+ + G^+ \subset G^+ \). Moreover \( G^+ \cap (-G^+) = \{0\} \). Indeed let \( g \in G^+ \). If \( -g \in G^+ \), then \( g \geq 0 \) and \( 0 \geq g \) which implies \( g = 0 \) since \( \leq \) is an order relation.

The set \( G^+ \) is not a subgroup but, since \( G \) is abelian, the set \( G^+ - G^+ \) is a subgroup which is itself an ordered group with the same positive cone as \( G \).

**Proposition 3.1.1** For any pair \((G, G^+)\) formed of an abelian group \( G \) and and a subset \( G^+ \) of \( G \) such that

\[
G^+ + G^+ \subset G^+, \quad G^+ \cap (-G^+) = \{0\},
\]

the relation \( g \leq h \) if \( h - g \in G^+ \) is a partial order compatible with the group operation and such that \( G^+ \) is the positive cone.
Proof. The first condition on $G^+$ implies that $\leq$ is transitive and the second one that it is antisymmetric. For $g, h \in G$ such that $g \leq h$ and $k \in G$, one has $(g + k) - (h + k) = g - h$ and thus $g + k \leq h + k$. 

We will often denote an ordered group as a pair $(G, G^+)$ where $G^+$ is the positive cone of $G$.

**Example 3.1.2** The sets $\mathbb{R}^d$ and $\mathbb{Z}^d$ are, for $d \geq 1$, abelian groups for the componentwise addition

$$(x_1, x_2, \ldots, x_d) + (y_1, y_2, \ldots, y_d) = (x_1 + y_1, x_2 + y_2, \ldots, x_d + y_d)$$

with $0 = (0, \ldots, 0)$ as neutral element. The pairs $(\mathbb{R}^d, \mathbb{R}_+^d)$ and $(\mathbb{Z}^d, \mathbb{Z}_+^d)$ with $\mathbb{R}_+$ (resp. $\mathbb{Z}_+$) formed of the nonnegative reals (resp. integers), are ordered groups. The corresponding partial order on $\mathbb{R}^d$ is called the *natural order*. It is defined by

$$(x_1, x_2, \ldots, x_d) \leq (y_1, y_2, \ldots, y_d)$$

if $x_i \leq y_i$ for $1 \leq i \leq d$. This partial order is a *lattice order*, which means that every pair $x, y$ has a least upper bound, that is, an element $z$ such that $x, y \leq z$ and $z \leq z'$ for any $z'$ such that $x, y \leq z'$.

In the next example, the order is a total order.

**Example 3.1.3** Let $G = \mathbb{Z}^d$ ordered by the *lexicographic order* defined by $(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$ if there is an index $i$ with $1 \leq i \leq d$ such that $x_1 = y_1, \ldots, x_{i-1} = y_{i-1}$ and $x_i < y_i$. Then $G^+$ is the set

$$\{(x_1, \ldots, x_d) \mid x_1 = \ldots = x_{i-1} = 0 \text{ and } x_i > 0 \text{ for some } i \text{ with } 1 \leq i \leq d\} \cup \{0\}.$$

A *subgroup* of an ordered group $(G, G^+)$ is a pair $(H, H^+)$ where $H$ is a subgroup of $G$ and $H^+ = H \cap G^+$. Such a subgroup is itself an ordered group. Indeed, $H^+$ is clearly a submonoid and $H^+ \cap (-H^+) \subset G^+ \cap (-G^+) = \{0\}$. In this way, the order on $H$ is the restriction to $H \times H$ of the order on $G$.

An ordered group $G$ is *directed* if for every $x, y \in G$ there is some $z \in G$ such that $x, y \leq z$. In other terms, $G$ is directed if every pair of elements has a common upper bound.

**Example 3.1.4** Let $G = \mathbb{Z}^2$ with the positive cone $G^+ = \{(x_1, x_2) \mid x_1 > 0\} \cup \{(0, 0)\}$. It is a directed group.

In the next statement, we use the fact that if $S$ is a submonoid of an abelian group $G$, then the set $S - S = \{s - t \mid s, t \in S\}$ is the subgroup generated by $S$ (Exercise 6.1).

**Proposition 3.1.5** An ordered group $G$ is directed if and only if it is generated by the positive cone, that is, if $G = G^+ - G^+$.
3.1. ORDERED ABELIAN GROUPS

Proof. Assume first that \( G^+ \) generates \( G \). For \( x, y \in G \), there exist \( z, t, u, v \in G^+ \) such that \( x = z - t \) and \( y = u - v \). Then \( w = z + u \) is such that \( x, y \leq w \).

Conversely, for any \( x \in G \), considering the pair \( 0, x \), there is some \( y \in G \) such that \( 0, x \leq y \). Then \( x = y - (y - x) \in G^+ - G^+ \).

Note that a subgroup of a directed group need not be directed, as shown in the next example.

Example 3.1.6 Let \( G = \mathbb{Z}^2 \) with the positive cone \( G^+ = \{(x_1, x_2) \mid x_1 > 0\} \cup \{(0, 0)\} \) as in Example 3.1.4. Then \( H = \{0\} \times \mathbb{Z} \) is a subgroup of a directed group. But is not directed since \( H^+ = \{0\} \).

Let \((G, G^+)\) and \((H, H^+)\) be ordered groups. A morphism \( \varphi : G \rightarrow H \) is positive if \( \varphi(G^+) \subset H^+ \). Note that a morphism is positive if and only if it preserves the orders on \( G, H \), that is \( g \leq g' \) implies \( \varphi(g) \leq \varphi(g') \).

3.1.1 Ideals and simple ordered groups

An order ideal \( J \) of an ordered group \((G, G^+)\) is a subgroup \( J \) of \( G \) such that \( J = J^+ - J^+ \) (with \( J^+ = J \cap G^+ \)) and such that \( 0 \leq a \leq b \) with \( b \in J^+ \) implies \( a \in J \).

A face in \( G \) is a subset \( F \) of \( G^+ \) which is a submonoid and such that \( 0 \leq a \leq b \) with \( b \in F \) implies \( a \in F \).

Proposition 3.1.7 Let \( G = (G, G^+) \) be an ordered group.

1. For every \( g \in G^+ \), the set \( [g] = \{h \in G \mid 0 \leq h \leq ng \text{ for some } n \geq 0\} \)
   is a face.

2. For every face \( F \), the subgroup \( J = F - F \) satisfies \( J \cap G^+ = F \) and is the smallest order ideal of \( G \) containing \( F \).

Proof. 1. If \( h, k \in [g] \) then \( h \leq ng \) and \( k \leq mg \) for some \( n, m \geq 0 \). Thus \( h + k \leq (n + m)g \) showing that \( h, k \in [g] \). Thus \( [g] \) is a submonoid. If \( 0 \leq h \leq k \) with \( n \geq 0 \) such that \( k \leq ng \), then \( 0 \leq h \leq ng \) and thus \( h \in [g] \). This shows that \( [g] \) is a face.

2. The set \( J = F - F \) is clearly a subgroup. The set \( J^+ = J \cap G^+ \) is equal to \( F \). In fact \( F \subseteq J^+ \) by definition. Conversely if \( h \in J^+ \), set \( h = a - b \) with \( a, b \in F \). Then \( h \leq a \) implies \( h \in F \) since \( F \) is a face. This shows that \( J \) is an ideal. Finally, if \( K \) is an ideal containing \( F \), then \( J \subseteq K \) since \( K \) is a subgroup.

An ordered group is simple if it has no nonzero proper order ideals. Note that a simple group is directed since \( G^+ - G^+ \) is an ideal of \((G, G^+)\).
Proposition 3.1.8 A subgroup of a simple ordered group is simple.

Proof. Let \( (H, H^+) \) be a subgroup of the simple group \( (G, G^+) \). Let \( J \) be a nonzero order ideal of \( H \). Let \( k \) be a nonzero element of \( J^+ \). Then \( F = \{ h \in G \mid h \leq nk \text{ for some } n \geq 0 \} \) is a face by Proposition 3.1.7. Thus, by Proposition 3.1.7 again, \( K = F - F \) is an ideal and \( K \cap G^+ = F \). Since \( G \) is simple, we have \( K = G \) and also \( F = G^+ \). Thus every \( h \in H^+ \) is in \( F \) and consequently in \( J \), which shows that \( J = H \). Therefore \( H \) is simple. \( \blacksquare \)

Example 3.1.9 The ordered groups \( \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{R} \) (with the natural order) are simple. On the contrary, the ordered group \( (\mathbb{Z}^d, \mathbb{Z}_+^d) \) for \( d \geq 2 \) is not simple. Indeed, for \( d = 2 \), the set \( \mathbb{Z} \times \{0\} \) is an order ideal of \( \mathbb{Z}^2 \).

In contrast, for every irrational \( \alpha \), the group \( \mathbb{Z} + \alpha \mathbb{Z} \), with the order induced by \( \mathbb{R} \), is simple since it is a subgroup of \( \mathbb{R} \). We shall meet this simple ordered group several times. It is contained in the additive subgroup \( \mathbb{Z} [\alpha] \) of \( \mathbb{R} \) generated by the powers of \( \alpha \). When \( \alpha \) is an algebraic integer, that is, such that \( p(\alpha) = 0 \) for some polynomial \( p(x) = x^{k+1} + a_k x^k + \ldots + a_1 x + a_0 \) with \( a_i \in \mathbb{Z} \), the group \( \mathbb{Z} [\alpha] \) is generated by \( 1, \alpha, \ldots, \alpha^k \) and is thus finitely generated. For \( k = 2 \), the groups \( \mathbb{Z} + \alpha \mathbb{Z} \) and \( \mathbb{Z} [\alpha] \) coincide (see Appendix C).

An order unit of the ordered group \( G \) is a positive element \( u \) such that for every \( g \in G^+ \) there is an integer \( n > 0 \) such that \( g < nu \). Equivalently, \( u \) is an order unit if the set \( [u] \) defined in Proposition 3.1.7 is equal to \( G^+ \).

A unital ordered group is a triple \( (G, G^+, 1_G) \) formed of an ordered group \( (G, G^+) \) and an order unit \( 1_G \).

Proposition 3.1.10 A directed ordered group \( (G, G^+) \) is simple if and only if every nonzero element of \( G^+ \) is an order unit.

The proof is left as an exercise (Exercise 3.2).

Example 3.1.11 The triples \( (\mathbb{R}^d, \mathbb{R}_+^d, 1) \) and \( (\mathbb{Z}^d, \mathbb{Z}_+^d, 1) \) are unital ordered groups with order unit \( 1 = (1, 1, \ldots, 1) \).

A morphism of unital ordered groups from \( \mathcal{G} = (G, G^+, 1_G) \) to \( \mathcal{H} = (H, H^+, 1_H) \) is a group morphism \( \varphi : G \to H \) which is positive and such that \( \varphi(1_G) = 1_H \).

A subgroup of a unital ordered group \( \mathcal{G} = (G, G^+, u) \) is a unital ordered group \( \mathcal{H} = (H, H^+, u) \) such that \( H \) is a subgroup of \( G \) containing \( u \) and \( H^+ = H \cap G^+ \).

### 3.1.2 Unperforated ordered groups

An ordered group \( (G, G^+) \) is unperforated if for every \( g \in G \), and \( n > 0 \), if \( ng \in G^+ \) then \( g \in G^+ \). Otherwise, the group is perforated.

For example, \( \mathbb{Z}^d \) with the natural order is unperforated.

Example 3.1.12 The group \( G = \mathbb{Z} \) with positive cone the submonoid \( G^+ \) of \( \mathbb{Z} \) generated by the set \( \{2, 5\} \) is perforated since \( 3 + 3 \) belongs to \( G^+ \) whereas 3 does not.
A group $G$ is torsion-free if for every $g \in G$ and $n > 0$, $ng = 0$ implies $g = 0$.

**Proposition 3.1.13** An unperforated group is torsion-free.

**Proof.** Suppose that $ng = 0$ for $g \in G$ and $n > 0$. Then $g \in G^+$ since $G$ is unperforated. But $-g = (n - 1)g$ implies $-g \in G^+ \cap (-G^+)$ and thus $g = 0$. 

As an example, the group $G = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $G^+ = \{(\alpha, \beta) \in G \mid \alpha > 0\} \cup \{(0,0)\}$ is a perforated group. Indeed, $2(0,1) = (0,0)$ and thus $G$ has torsion.

### 3.2 States

Let $G = (G, G^+, 1)$ be a unital ordered group. A state on $G$ is a morphism of unital ordered groups from $G$ to the unital ordered group $(\mathbb{R}, \mathbb{R}^+, 1)$. Thus a group morphism $p : G \rightarrow \mathbb{R}$ is a state if $p(g) \geq 0$ for every $g \in G^+$ and $p(1) = 1$.

Let $S(G)$ denote the set of states of $G$. It is a convex set. Indeed, let $p, q \in S(G)$ and let $t \in [0, 1]$. Then $r = tp + (1 - t)q$ is a morphism since the set of morphisms from a group to $\mathbb{R}$ forms a vector space. It is positive because for every $g \in G^+$, we have $r(g) \geq \min\{p(g), q(g)\} \geq 0$. Moreover $r(1) = tp(1) + (1 - t)q(1) = 1$. Thus $tp + (1 - t)q \in S(G)$ which shows that $S(G)$ is convex.

**Example 3.2.1** Let $G$ be the group $\mathbb{Z}^d$ with the usual order and $u = (1, \ldots, 1)$. The set $S(G)$ is the $d - 1$ simplex formed of the maps $(\alpha_1, \ldots, \alpha_d) \mapsto p_1\alpha_1 + \ldots + p_d\alpha_d$ for $p_1, \ldots, p_d \geq 0$ of sum 1.

**Example 3.2.2** Let $\lambda$ be irrational. The unital ordered group $G = \mathbb{Z} + \lambda\mathbb{Z}$ (with the order induced by the reals and order unit 1) is simple, as we have already seen (Example 3.1.9). There is a unique state which is the identity. Indeed, let $p$ be a state on $G$ and let $\mu = p(\lambda)$. For every $x, y \in \mathbb{Z}$, if $x + \lambda y \geq 0$, then $x + \mu y \geq 0$. This implies $\mu = \lambda$ since $\mathbb{Q}$ is dense in $\mathbb{R}$.

**Proposition 3.2.3** Let $G$ be a directed unital ordered group. The set $S(G)$ is convex and compact for the product topology on $\mathbb{R}^G$.

**Proof.** For every $g \in G$, we have $g = g' - g''$ with $g', g'' \in G^+$ since $G$ is directed. Let $n \geq 1$ be such that $g', g'' \leq n1$. Then $|p(g)| \leq |p(g')| + |p(g'')| \leq 2n1$ for every $p \in S(G)$. This implies that $p(g) \in [-n, n]$ for every $p \in S(G)$ and thus that $S(G)$ is compact for the product topology.

We will now prove the following important result.

**Theorem 3.2.4** For every directed unital ordered group $G$, the set of states on $G$ is nonempty.
We first prove the following lemmas.

**Lemma 3.2.5** Let \( G = (G, G^+, u) \) be a unital ordered group and let \( H = (H, H^+, u) \) be a unital ordered subgroup of \( G \). Let \( p \) be a state on \( H \). For \( g \in G^+ \), let

\[
\alpha = \sup \{ p(x)/m \mid x \in H, m > 0, x \leq mg \}, \quad (3.2.1) \\
\beta = \inf \{ p(y)/n \mid y \in H, n > 0, ng \leq y \}. \quad (3.2.2)
\]

Then

1. \( 0 \leq \alpha \leq \beta < \infty \).
2. If \( q \) is a state on \( H + Zg \) which extends \( p \), then \( \alpha \leq q(g) \leq \beta \).
3. If \( \alpha \leq \gamma \leq \beta \), there is a state \( q \) on \( (H + Zg, H \cap (G^+ + Zg), u) \) such that \( q \) extends \( p \) and \( q(g) = \gamma \).

*Proof.*
1. Since \( 0 \leq g \), we have by (3.2.1) with \( x = 0 \) and \( m = 1 \), that \( \alpha \geq p(0)/1 = 0 \). Next, since \( u \) is an order unit, we have \( g \leq ku \) for some \( k > 0 \). Thus, using (3.2.2) with \( y = ku \) and \( n = 1 \), we obtain \( \beta \leq p(ku)/1 = k < \infty \).

Consider \( x, y \in H \) and \( m, n > 0 \) such that \( x \leq mg \) and \( ng \leq y \). Then \( nx \leq mg \) and \( ny \leq mg \), and consequently \( p(x)/m \leq p(y)/n \). Therefore \( \alpha \leq \beta \).
2. Let \( x \in H \) and \( m > 0 \) be such that \( x \leq mg \). Then \( p(x) = q(x) \leq mq(g) \) and thus \( \alpha \leq q(g) \). The proof that \( q(g) \leq \beta \) is similar.
3. We first claim that if \( z + kg \geq 0 \) for some \( z \in H \) and \( k \in Z \), then \( p(z) + k\gamma \geq 0 \). Indeed, if \( k = 0 \), then \( z \geq 0 \) and \( p(z) + k\gamma = p(z) \geq 0 \). If \( k > 0 \), then we have \( -z \leq kg \) with \( -z \in H \) whence, by (3.2.1), \( p(-z)/k \leq \alpha \leq \gamma \) and so \( p(-z) + k\gamma \geq 0 \). Finally, if \( k < 0 \), we have \( -kg \leq z \) with \( z \in H \) and \( -k > 0 \), hence \( \gamma \leq \beta \leq p(z)/(-k) \) and so \( p(z) + k\gamma \geq 0 \). This proves the claim.

As a consequence of the claim, we obtain that if \( z + kg = 0 \) for some \( z \in H \) and \( k \in Z \), then \( p(z) + k\gamma = 0 \). Indeed, we have both \( z + kg \geq 0 \) which implies \( z + k\gamma \geq 0 \) and \( -z + (-k)\gamma \geq 0 \) which implies \( -z + (-k)\gamma \geq 0 \). As a consequence, the map \( z + kg \rightarrow p(z) + k\gamma \) induces a morphism \( q \) from \( H + Zg \) to \( R \). Moreover, the claim shows that \( q \) is positive. Since \( q(g) = \gamma \) and \( q(u) = p(u) = 1 \), the proof is complete.

**Lemma 3.2.6** Let \( G = (G, G^+, u) \) be a directed unital ordered group and let \( H = (H, H^+, u) \) be a subgroup of \( G \). Every state on \( H \) extends to a state on \( G \).

*Proof.* Consider the family of pairs \((K, q)\) of a unital ordered group \( K = (K, K^+, u) \) such that \( K \) is a subgroup of \( G \) which contains \( H \) with \( K^+ = K \cap G^+ \) and a state \( q \) on \( K \) which extends \( p \). By Zorn’s Lemma, this family has a maximal element \((K, q) \in K \). Suppose that \( K \neq G \). Since \( G \) is directed, \( G^+ \) generates \( G \), which implies that there is some \( g \in G^+ \setminus K \). By Lemma 3.2.5 there is a state \( q \) on \( K + Zg \) which extends \( p \). But then \((K + Zg, q) \in K \), a contradiction. We conclude that \( K = G \) and that \( q \) is a state on \( G \) which extends \( p \).
We are now ready for the proof of Theorem 3.2.4.

Proof of Theorem 3.2.4. Set $\mathcal{G} = (G, G^+, u)$. The map $p : n u \mapsto n$ is a state on the unital ordered group $\mathcal{H} = (\mathbb{Z}u, \mathbb{Z}u^+, u)$ which is a unital ordered subgroup of $\mathcal{G}$. By Lemma 3.2.6, $p$ extends to a state of $G$.

Example 3.2.7 Let $\mathcal{G} = (G, G^+, u)$ with $G = \mathbb{Z}^2$, $G^+ = \{(x_1, x_2) \in G \mid x_1 > 0\} \cup \{(0,0)\}$ and $u = (1,0)$. There is a unique state which is the projection on the first component.

The following result shows that, for an unperforated simple ordered group, the order is determined by the set of states.

Proposition 3.2.8 If $\mathcal{G} = (G, G^+, u)$ is an unperforated simple unital group, then $G^+ = \{g \in G \mid p(g) > 0 \text{ for every } p \in S(\mathcal{G})\} \cup \{0\}$.

Proof. Since $G$ is simple, every nonzero element of $G^+$ is an order unit (Proposition 3.1.10). Thus if $g \in G^+ \setminus \{0\}$, there is $n \geq 1$ such that $ng \geq u$. Then, for any $p \in S(\mathcal{G})$ (which is nonempty by Theorem 3.2.4), we have $p(ng) \geq p(u) = 1$ and thus $p(g) > 0$ since $p(ng) = np(g)$. Conversely, if $p(g) > 0$ for every $p \in S(\mathcal{G})$, then since $S(\mathcal{G})$ is closed, there is $\varepsilon > 0$ such that $p(g) > \varepsilon$ for every $p \in S(\mathcal{G})$. Let $\alpha = r/s \in \mathbb{Q}$ with $0 < \alpha < \varepsilon$. Then $sg < ru$ implies $p(g) < r/s < \varepsilon$, a contradiction. Thus $sg \geq ru$, which implies $g \in G^+$ since $\mathcal{G}$ is unperforated.

Example 3.2.9 Let $G = \mathbb{Z}^2$ with $G^+ = \{(x_1, x_2) \mid x_1 > 0\} \cup \{(0,0)\}$ as in Example 3.2.7. There is a unique state $p : (x_1, x_2) \mapsto x_1$. Thus Proposition 3.2.8 is satisfied. In contrast, let $G = \mathbb{Z}^2$ with the lexicographic order, that is with $G^+ = \{(x_1, x_2) \mid x_1 > 0 \text{ or } x_1 = 0 \text{ and } x_2 \geq 0\}$. There is only one state which is the projection on the first component. Thus Proposition 3.2.8 does not hold. There is no contradiction since $G$ is not simple.

3.2.1 Infinitesimals

Let $(G, G^+, u)$ be a unital unperforated ordered group. We say that an element $g \in G$ is infinitesimal if $ng \leq u$ for every $n \in \mathbb{Z}$. It is easy to see that the definition does not depend on the choice of the order unit $u$ (Exercise 3.3). If $G = \mathbb{Z}^d$ with the usual order and unit $1 = (1, 1, \ldots, 1)$, there are no nonzero infinitesimals. On the contrary the following example exhibits nonzero infinitesimals.

Example 3.2.10 Let $G = \mathbb{Z}^2$ with $G^+ = \{(\alpha, \beta) \in G \mid \alpha > 0\} \cup \{(0,0)\}$ and $u = (1,0)$ as in Example 3.2.7. Any element $(0, \beta)$ with $\beta \in \mathbb{Z}$ is infinitesimal.

Let us introduce the following useful notation. For $\varepsilon \in \mathbb{Q}$, the inequality $g \leq \varepsilon u$ means that $pq \leq pu$ for some integers $p, q \geq 1$ such that $\varepsilon = p/q$.

Using this notation, one can give as an equivalent definition that $g$ is infinitesimal if $-\varepsilon u \leq g \leq \varepsilon u$ for all $0 < \varepsilon \in \mathbb{Q}^+$ (Exercise 3.5).

Another equivalent definition is the following.
**Proposition 3.2.11** Let \( G = (G, G^+, u) \) be an unperforated simple unital group. An element \( g \in G \) is infinitesimal if and only if \( p(g) = 0 \) for all \( p \in S(G) \).

**Proof.** By Theorem 3.2.4 the set \( S(G) \) of states is nonempty. Assume first that \( g \) is infinitesimal and let \( p \in S(G) \). Since \( -\varepsilon \leq p(g) \leq \varepsilon \) for every \( \varepsilon \in \mathbb{Q}_+ \), we conclude that \( p(g) = 0 \).

Conversely, suppose that \( |p(g)| \geq 1/n \) for some trace \( p \). Then \( u - ng \) and \( u + ng \) cannot be both in \( G^+ \). Thus either \( ng \leq u \) or \( -ng \leq u \) is false, a contradiction.

The collection of infinitesimal elements of \( G \) forms a subgroup, called the infinitesimal subgroup of \( G \), which we denote by \( \text{Inf}(G) \).

The quotient group \( G/\text{Inf}(G) \) of a simple ordered group \( G \) is also a simple ordered group for the induced order, and the infinitesimal subgroup of \( G/\text{Inf}(G) \) is trivial (see Exercise 3.6). Furthermore, an order unit for \( G \) maps to an order unit for \( G/\text{Inf}(G) \). Moreover the traces space of \( G \) and \( G/\text{Inf}(G) \) are isomorphic.

It may be interesting to summarize the properties of some of the various orders on \( \mathbb{Z}^2 \) that we have considered in the examples. The first row concerns the group \( \mathbb{Z}^2 \) ordered by the first component. It is not directed since \( G^+ \) generates \( \mathbb{Z} \times \{0\} \). The third row concerns the group of Example 3.1.3. The last one is the group of Example 3.1.6.

**3.2.2 Image subgroup**

Let \( G = (G, G^+, u) \) be a unital ordered group. The image subgroup associated to \( G \) is the subgroup of \( (\mathbb{R}, \mathbb{R}_+, 1) \) defined as

\[
I(G) = \bigcap_{p \in S(G)} p(G)
\]

When \( S(G) \) consists of a unique trace, the group \( I(G) \) is isomorphic to \( G/\text{Inf}(G) \).

**Example 3.2.12** Let \( G = \mathbb{Z}^2 \) with \( G^+ = \{ (\alpha, \beta) \in G \mid \alpha > 0 \} \cup \{ (0, 0) \} \) as in Example 3.2.10 Then the image subgroup is \( I(G) = (\mathbb{R}, \mathbb{R}_+, 1) \).
3.3 Direct limits

We now introduce the important notion of direct limit, which is central in this book. We will first formulate it for ordinary abelian groups and next for ordered ones.

3.3.1 Direct limits of abelian groups

Let $G_n$ be for each $n \geq 0$ an abelian group and let $i_{n+1,n} : G_n \to G_{n+1}$ be for every $n \geq 0$ a morphism. The sets

$$\Delta = \{(g_n)_{n \geq 0} | g_n \in G_n, g_{n+1} = i_{n+1,n}(g_n) \text{ for every } n \text{ large enough}\}$$

and

$$\Delta^0 = \{(g_n)_{n \geq 0} | g_n \in G_n, g_n = 0 \text{ for every } n \text{ large enough}\}$$

are subgroups of the direct product $\Pi_{n \geq 0} G_n$ and $\Delta^0 \subset \Delta$. Let $G$ be the quotient group $G = \Delta / \Delta^0$ and $\pi : \Delta \to G$ be the natural projection. The group $G$, denoted $G = \lim_{\to} G_n$, is called the direct limit (or inductive limit) of the sequence $(G_n)_{n \geq 0}$ with the maps $i_{n+1,n}$. The maps $i_{n+1,n}$ and more generally the maps $i_{m,n} = i_{m,m-1} \circ \cdots \circ i_{n+1,n}$ for $n < m$ are called the connecting morphisms.

See Exercise 3.5.3 for an alternative definition of the direct limit as a quotient of the union of the $G_n$.

Given $g \in G_n$, all the sequences $(g_k)_{k \geq 0} \in \Pi_{k \geq 0} G_k$ such that

$$g_n = g \text{ and } g_{m+1} = i_{m+1,m}(g_m) \text{ for all } m \geq n$$

belong to $\Delta$ and have the same projection, denoted $i_n(g)$ in $G$. The morphism $i_n$ is called the natural morphism from $G_n$ into $G$.

Note that $i_n : G_n \to G$ is such that $i_n = i_{n+1} \circ i_{n+1,n}$ for every $n \geq 0$.

The kernel of $i_n$ is

$$\ker(i_n) = \cup_{m \geq n} \ker(i_{m,m-1} \circ \cdots \circ i_{n+1,n}).$$

The group $G$ is the union of the ranges of the $i_n$. Thus, for every $g \in G$, there exist an integer $n \geq 0$ and an element $g_n \in G_n$ such that $g = i_n(g_n)$.

Example 3.3.1 Consider the case of the sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \ldots$ where each map is the multiplication by 2. The direct limit $G$ of this sequence can be identified with the group $\mathbb{Z}[1/2]$ of dyadic rationals, formed of all rational numbers $p/q$ with $q$ a power of 2. Indeed, $\Delta$ is formed of the sequences $(g_n)_{n \geq 0}$ such that for some $k \geq 1$, one has $g_{n+1} = 2g_n$ for every $n \geq k$. Consider the map $\pi : \Delta \to \mathbb{Z}[1/2]$ sending such a sequence on $2^{-k}g_k$. This map is a well defined group morphism and its kernel is $\Delta^0$. Thus it induces an isomorphism from $G$ onto $\mathbb{Z}[1/2]$. The natural morphism from $G_n = \mathbb{Z}$ to $G$ is $i_n(g) = 2^{-n}g$. 
Let \((G_n)\) be a sequence of abelian groups with connecting morphisms \(i_{n+1,n} : G_n \to G_{n+1}\). For every abelian group \(H\) and every sequence \((\alpha_n)\) of morphisms from \(G_n\) to \(H\) such that \(\alpha_n = \alpha_{n+1} \circ i_{n+1,n}\) (see Figure 3.3.1 on the right), there is a unique morphism \(\varphi\) from the direct limit \(G = \lim \alpha_n G_n\) to \(H\) such that \(\alpha_n = \varphi \circ i_n\) for all \(n \geq 0\) (see Figure 3.3.1 on the right). The verification is left as Exercise 3.9.

### 3.3.2 Direct limits of ordered groups

Let \(\mathcal{G}_n = (G_n, G_n^+, 1_n)\) be for each \(n \geq 0\) a directed unital ordered group and let \(i_{n+1,n} : G_n \to G_{n+1}\) be for every \(n \geq 0\) a morphism of unital ordered groups. Let \(G\) be the direct limit of the sequence \((G_n)_{n \geq 0}\), let \(G^+\) be the projection in \(G\) of the set

\[ \Delta^+ = \{(g_n)_{n \geq 0} \mid g_n \in G_n^+ \text{ for every large enough } n\} \]

and let \(1\) be the projection in \(G\) of the sequence \((1_n)_{n \geq 0} \in \Delta\).

**Proposition 3.3.2** The triple \(\mathcal{G} = (G, G^+, 1)\) is a directed unital ordered group and every \(i_n : \mathcal{G}_n \to \mathcal{G}\) is a morphism of unital ordered groups and \(G^+\) is the union of the \(i_n(G^+)^{n}\).

**Proof.** We first verify that \(G^+\) satisfies the two conditions defining an ordered group. First, if \(g, g'\) belong to \(\Delta^+\), then \(g_n, g'_n\) belong to \(G_n^+\) for every large enough \(n\) and thus \(g_n + g'_n\) also belong to \(G_n^+\) for every large enough \(n\). Thus \(g + g'\) belong to \(\Delta^+\). This shows that \(\Delta^+ + \Delta^+ \subset \Delta^+\) and it implies that \(G^+ + G^+ \subset G^+\). Similarly, the facts that \(G = G^+ - G^+\) and \(G^+ \cap (-G^+) = \{0\}\) follow from \(\Delta = \Delta^+ - \Delta^+\) and \(\Delta^+ \cap (-\Delta^+) = \Delta^0\). Thus \(\mathcal{G}\) is directed.

Finally, let us show that \(1\) is an order unit. Let \(g = (g_n)_{n \geq 0} \in \Delta\). If \(g_{n+1} = i_n(g_n)\) for some \(n \geq 0\), and if \(g_n \leq k1_n\) for some \(k \geq 1\), then, since \(i_{n+1,n}\) is a morphism of ordered groups with order unit, we have \(g_{n+1} \leq k1_{n+1}\). Thus, there is a \(k \geq 1\) such that \(g_n \leq k1_n\) for every \(n\) large enough. This implies that the projection of \(g\) in \(G\) is bounded by \(k1\).

The triple \(\mathcal{G}\) is called the direct limit (or inductive limit) of the sequence \((G_n)_{n \geq 0}\) (see Exercise 3.10 for an alternative definition).

![Figure 3.3.1: The universal property of the direct limit](image)
3.3. DIRECT LIMITS

Example 3.3.3 Consider again the case of the sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \ldots$ of Example 3.3.1 with $\mathbb{Z}$ ordered as usual. The direct limit of the corresponding sequence of ordered groups is the ordered group $(\mathbb{Z}[1/2], \mathbb{Z}_+[1/2], 1)$ where $\mathbb{Z}_+[1/2]$ is the set of non-negative dyadic rationals.

3.3.3 Ordered group of a matrix

We can generalize Example 3.3.1 by considering $G_n = \mathbb{Z}^d$ for some integer $d \geq 1$, an integer $d \times d$-matrix $M$ and the sequence of morphisms $i_n$ being the multiplication by $M$ on the elements of $\mathbb{Z}^d$ considered as column vectors.

Thus we address the description of the direct limit of a sequence $\mathbb{Z}^d \overset{M}{\rightarrow} \mathbb{Z}^d \overset{M}{\rightarrow} \mathbb{Z}^d \ldots$ of groups all equal to $\mathbb{Z}^d$ with the same connecting morphisms.

Define the eventual range of $M$ as $\mathcal{R}_M = \cap_{k \geq 1} M^k \mathbb{Z}^d$ and the eventual kernel of $M$ as $\mathcal{K}_M = \cup_{k \geq 1} \ker(M^k)$. Note that $\mathbb{Z}^d = \mathcal{R}_M \oplus \mathcal{K}_M$ (3.3.1)

and that the multiplication by $M$ defines an automorphism of $\mathcal{R}_M$. Indeed, since $\ldots \subset M^2 \mathbb{Z}^d \subset M \mathbb{Z}^d \subset \mathbb{Z}^d$ and $\ker M \subset \ker M^2 \subset \ldots$ there is some $h \geq 0$ such that $\mathcal{R}_M = M^h \mathbb{Z}^d$ and $\mathcal{K}_M = \ker M^h$. Then $M \mathcal{R}_M = M^{h+1} \mathbb{Z}^d = M^h \mathbb{Z}^d = \mathcal{R}_M$ and thus the multiplication by $M$ is an automorphism of $\mathcal{R}_M$. If $x \in \mathcal{R}_M \cap \mathcal{K}_M$, then $Mx = 0$ implies $x = 0$. Thus $\mathcal{R}_M \cap \mathcal{K}_M = \{0\}$.

Next, for every $x \in \mathbb{Z}^d$ there is since $M^h x \in \mathcal{R}_M$, some $y \in \mathcal{R}_M$ such that $M^h x = M^h y$. Then $x = y + (x - y) \in \mathcal{R}_M + \mathcal{K}_M$. This proves Equation (3.3.1)

Let also $\Delta_M = \{ v \in \mathcal{R}_M \mid M^k v \in \mathbb{Z}^d \text{ for some } k \geq 0 \}$ (3.3.2)

The following result describes direct limits with identical connecting morphisms $i_{n,n+1}$ for all $n \geq 0$.

Proposition 3.3.4 For every integer $d \times d$-matrix, the direct limit $G$ of the sequence $\mathbb{Z}^d \overset{M}{\rightarrow} \mathbb{Z}^d \overset{M}{\rightarrow} \mathbb{Z}^d \ldots$, where each map is the multiplication by $M$, is isomorphic to $\Delta_M$. If moreover the matrix $M$ is nonnegative, the triple $(\Delta_M, \Delta_M^+, 1_M)$, where $\Delta_M^+ = \{ v \in \mathcal{R}_M \mid M^k v \in \mathbb{Z}_+^d \text{ for every large enough } k \geq 0 \}$ and $1_M$ is the projection on $\mathcal{R}_M$ along $\mathcal{K}_M$ of the vector $[1 \ 1 \ldots 1]^t$, is a unital ordered group. If $M$ is primitive, the group $(\Delta_M, \Delta_M^+, 1_M)$ is simple.
Proof. Let

\[ \Delta = \{(x_n)_{n \geq 0} \mid x_{n+1} = Mx_n \text{ for every } n \text{ large enough}\} \]

and

\[ \Delta^0 = \{(x_n)_{n \geq 0} \mid x_n = 0 \text{ for every } n \text{ large enough}\}. \]

We have by definition \( G = \Delta / \Delta^0 \). Let \( x \in \Delta \). We may assume, by choosing \( k \) large enough, that \( x_k \in R^M \) and \( x_{n+1} = Mx_n \) for every \( n \geq k \). Since the multiplication by \( M \) is an automorphism of \( R^M \) there is a unique \( y \in R^M \) such that \( M^k y = x_k \). The map \( \pi : x \in \Delta \mapsto y \in \Delta^M \) is a well-defined group morphism and its kernel is \( \Delta^0 \). Thus \( \pi \) induces an isomorphism from \( G \) onto \( \Delta^M \). This proves the first statement.

Assume now that \( M \) is nonnegative. Let

\[ \Delta^+ = \{(x_n)_{n \geq 0} \mid x_n \in Z^d_+ \text{ for every } n \text{ large enough}\}. \]

Since \((G,G^+)\) is an ordered group by Proposition 3.3.2 and since \( \pi(\Delta^+) = \Delta^+_M \), the group \((\Delta_M,\Delta^+_M)\) is an ordered group. Next, let \( u = [1 \ 1 \ldots \ 1]^t \). For \( v \in \Delta^+_M \) let \( n \geq 1 \) be such that \( nu - v \in \mathbb{R}_d^+ \). Then \( M^k(nu - v) \in \mathbb{Z}_d^+ \) for large enough \( k \) and thus \( nu - v \in \Delta^+_M \), showing that \( u \) is an order unit. This proves the second statement.

Finally, if \( M \) is primitive, there is an integer \( k \) such that \( M^k \) is strictly positive. Thus for any nonzero element \( u \) of \( \Delta^+_M \), there is an integer \( k \) such that \( M^k u \) is strictly positive. For every \( v \in \Delta^+_M \) there is an integer \( n \geq 1 \) such that \( nM^k u - M^k v \in \mathbb{R}_d^+ \). Then \( nu - v \in \Delta^+_M \) and thus \( u \) is an order unit. This shows that \( u \) is an order unit and thus proves the last statement by Proposition 3.1.10.

The ordered group \((\Delta_M,\Delta^+_M,1_M)\) is called the ordered group of the matrix \( M \).

The following result is very useful.

**Proposition 3.3.5** Let \( M \) is primitive matrix. Then

\[ \Delta^+_M = \{v \in \Delta_M \mid z \cdot v > 0\} \cup \{0\} \]

where \( z \) is a positive left eigenvector of \( M \) for the dominant eigenvalue.

Proof. Let \( \lambda \) be the dominant eigenvalue of \( M \). Assume first that \( v \in \Delta^+_M \setminus \{0\} \). Then \( M^k v \in \mathbb{Z}_d^+ \) for some \( k \geq 0 \). Since \( M \) is primitive, we may assume that all entries of \( M^k v \) are positive. By assertion (iii) of Perron-Frobenius (Theorem 2.4.14), the vector \( \lim \lambda^{-k} M^k v = (tz) \cdot v = t(z \cdot v) \) where \( t \) is a positive right eigenvector of \( M \) relative to \( \lambda \) such that \( z \cdot t = 1 \). This implies that \( z \cdot v > 0 \).

Conversely, if \( z \cdot v > 0 \), the vector \( \lim \lambda^{-k} M^k v \) has all its components positive and thus there exists \( k \geq 0 \) such that \( M^k v \in \mathbb{Z}_d^+ \). This shows that \( v \in \Delta^+_M \).

\[ \square \]
Example 3.3.6 Consider the primitive matrix \( M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \). Since \( M \) is invertible, we have \( R_M = \mathbb{R}^2 \) and \( \Delta_M = \mathbb{Z}^2 \). Next, the dominant eigenvalue of \( M \) is \( \lambda = (1 + \sqrt{5})/2 \), and a corresponding row eigenvector is
\[
z = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}.
\]
Thus, one has \( \Delta_M^+ = \{ v \in \mathbb{Z}^2 \mid z \cdot v \geq 0 \} \). The map \( (\alpha, \beta) \mapsto \lambda \alpha + \beta \) is a positive isomorphism morphism from \( (\Delta_M, \Delta_M^+) \) to the group of algebraic integers \( \mathbb{Z}[\lambda] = \mathbb{Z} + \lambda \mathbb{Z} \). The order unit \( 1_M = (1, 1) \) is mapped to \( \lambda + 1 \).

This shows that the direct limit of the sequence \( \mathbb{Z} \xrightarrow{M} \mathbb{Z} \xrightarrow{M} \cdots \) is isomorphic to the group of algebraic integers \( \mathbb{Z} + \lambda \mathbb{Z} \) with the order induced by the reals and \( 1 + \lambda \) as ordered unit. One can normalize the order unit to be 1 as follows. Using the map \( x + \lambda y \mapsto x\lambda^{-1} + y \) which amounts to dividing by \( \lambda \), we send our group to \( \mathbb{Z}[\lambda^{-1}] \) with unit 1. But since \( (1 + \lambda)^{-1} = 2 - \lambda \), we have \( \mathbb{Z}[\lambda^{-1}] = \mathbb{Z}[\lambda] \).

Thus we find that the group \( (\Delta_M, \Delta_M^+, 1_M) \) is isomorphic to \( \mathbb{Z}[\lambda] \).

We next give an example with a non primitive matrix.

Example 3.3.7 Consider now \( M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). We have again \( \Delta_M = \mathbb{Z}^2 \) but this time \( \Delta_M^+ = \{ (x, y) \mid y > 0 \} \cup \{ (x, 0) \mid x \geq 0 \} \). Thus we find that ordered group of the matrix \( M \) is \( \mathbb{Z}^2 \) with the lexicographic order.

Note that there can be nonzero vectors \( v \) in \( \Delta_M \) such that \( z \cdot v = 0 \) and thus that such a vector cannot be in \( \Delta_M^+ \) (see Example 3.5.3).

Let \( (H, H^+, 1) \) be a unital ordered group and for every \( n \geq 1 \), let
\[
j_n : (G_n, G_n^+, 1_n) \to (H, H^+, 1)
\]
be a morphism such that \( j_{n+1} \circ i_{n+1,n} = j_n \) for every \( n \).

The following result will be used later (see Lemma 3.3.2).

**Proposition 3.3.8** There exists a unique morphism of unital ordered groups \( j : (G, G^+, 1_G) \to (H, H^+, 1_H) \) such that \( j \circ i_n = j_n \) for every \( n \). It is surjective if \( \cup_n j_n(G_n) = H \) and it is injective if \( \ker(j_n) \subset \ker(i_n) \) for every \( n \).

**Proof.** Let \( g = i_n(g_n) \in G \). Set \( j(g) = j_n(g_n) \). Then \( j \) is a well-defined morphism from \( G \) into \( H \) which is a morphism of unital ordered groups. Conversely, if \( j \) is such that \( j \circ i_n = j_n \) for every \( n \) and if \( g = i_n(g_n) \), then \( j(g) = j \circ i_n(g_n) = j_n(g_n) \). The last assertions follow easily. \( \blacksquare \)

### 3.4 Dimension groups

A *dimension group* is a direct limit
\[
\mathbb{Z}^{k_1} \xrightarrow{M_{i_1}} \mathbb{Z}^{k_2} \xrightarrow{M_{i_2}} \mathbb{Z}^{k_3} \cdots
\]
of groups $\mathbb{Z}^{k_i}$ with $k_i \geq 1$ ordered in the usual way and with order unit $(1, \ldots, 1)$, with the morphisms defined by nonnegative matrices $M_i$. Thus a dimension group is a unital ordered group.

The definition of dimension groups implies some properties of these groups, which hold in any group $\mathbb{Z}^d$ with the natural order.

First of all, a dimension group $(G, G^+)$ is unperforated. Let indeed $g$ be a nonzero element of $G = \lim_{\rightarrow} \mathbb{Z}^{k_n}$ and assume that $ng \in G^+$ for some $n > 0$. Let $m \geq 1$ be such that $g = i_m(x)$ for $x \in \mathbb{Z}^{k_m}$. Then $nx > 0$ implies $x > 0$ and thus $g \in G^+$. Next, Dimension groups satisfy the Riesz interpolation property that we introduce now.

### 3.4.1 Riesz groups

An ordered group $G$ satisfy the Riesz interpolation property if for any $x_1, x_2, y_1, y_2 \in G$ such that $x_1 \leq y_1, y_2$ and $x_2 \leq y_1, y_2$, there exists some $z \in G$ such that $x_1, x_2 \leq z \leq y_1, y_2$.

This property is equivalent to the Riesz decomposition property, requiring that given $x_1, x_2, y_1, y_2 \in G^+$ if $x_1 + x_2 = y_1 + y_2$, then there exists $z_{ij} \in G^+$ with $1 \leq i, j \leq 2$ such that $x_1 = \sum_j z_{ij}$ and $y_j = \sum_i z_{ij}$ (Exercise 3.17).

As a variant of the interpolation property, we have that for every $x, y_1, \ldots, y_k$ in $G^+$ such that $x \leq y_1 + \ldots + y_k$, there are $x_1, \ldots, x_k$ in $G^+$ such that $x = x_1 + \ldots + x_k$ and $x_i \leq y_i$ for $1 \leq i \leq k$ (see Exercise 3.18).

An ordered group $G$ is said to be a Riesz group if it satisfies the Riesz interpolation property.

The groups $\mathbb{Z}$ and $\mathbb{R}$ clearly have the Riesz interpolation property as any totally ordered group. More generally, any group in which two elements have a least upper bound is a Riesz group. Next, we have the following more subtle example.

**Example 3.4.1** Any dense subgroup of $\mathbb{R}^2$ is a Riesz group.

![Figure 3.4.1: The Riesz interpolation property](image)

Indeed, let $x_1, x_2, y_1, y_2 \in \mathbb{R}^2$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ as in Figure 3.4.1. The set of points $z$ such that $x_1, x_2 \leq z \leq y_1, y_2$ is the central rectangle.

**Example 3.4.2** Let $G$ be the quotient of $\mathbb{Z}^4$ by the subgroup generated by $(1, 1, -1, -1)$ and the order induced by the natural order. Denote by $[x]$ the projection on $G$ of $x \in \mathbb{Z}^4$. The group $G$ is not a Riesz group. Indeed, one has, with $x = (1, 0, 0, 0), y = (0, 1, 0, 0), z = (0, 0, 1, 0)$ and $t = (0, 0, 0, 1)$ the
inequality \(|x| \leq |z| + |t|\) since \(|x| + |y| = |z| + |t|\). However, \(|x|\) cannot be written as a sum of two positive smaller elements and thus the decomposition property fails to hold.

### 3.4.2 The Effros-Handelman-Shen Theorem

The following important theorem characterizes dimension groups among countable ordered groups.

**Theorem 3.4.3 (Effros, Handelman, Shen)** A countable ordered group is a dimension group if and only if it is an unperforated directed Riesz group.

Theorem 3.4.3 gives a much easier way to verify that an ordered group is a dimension group than using the definition, since it does not require to find an infinite sequence of morphisms. For example, it shows directly that any countable dense subgroup of \(\mathbb{R}^2\) is a dimension group since it is an unperforated Riesz group (see Example 3.4.1).

The essential step of the proof is the following lemma. In the proof, we will find it convenient to identify the group \(\mathbb{Z}^n\) with the free abelian group on a set \(A\) with \(n\) elements, denoted \(\mathbb{Z}(A)\). The elements of \(\mathbb{Z}(A)\) have the form \(\sum_{a \in A} x_a a\) with \(x_a \in \mathbb{Z}\). This amounts to identify the set \(A\) with the canonical basis of \(\mathbb{Z}^n\).

**Lemma 3.4.4** Let \((G, G^+)\) be an unperforated Riesz group and let \(n \geq 1\). Let \(\alpha : (\mathbb{Z}^n, \mathbb{Z}^n_+) \to (G, G^+)\) be a morphism and let \(x \in \mathbb{Z}^n\) be such that \(\alpha(x) = 0\).

There is an integer \(m \geq 1\), a surjective morphism \(\eta : (\mathbb{Z}^n, \mathbb{Z}^n_+) \to (\mathbb{Z}^m, \mathbb{Z}^m_+)\) and a morphism \(\beta : (\mathbb{Z}^m, \mathbb{Z}^m_+) \to (G, G^+)\) such that \(\eta(x) = 0\) and \(\beta \circ \eta = \alpha\) (see the diagram below).

![Figure 3.4.2: The diagram of Lemma 3.4.4](image)

**Proof.** We first remark that if we can prove the statement with a morphism \(\eta : (\mathbb{Z}^n, \mathbb{Z}^n_+) \to (\mathbb{Z}^m, \mathbb{Z}^m_+)\) which is not surjective, we may replace \((\mathbb{Z}^m, \mathbb{Z}^m_+)\) by the ordered \(\eta(\mathbb{Z}^m, \mathbb{Z}^m_+)\), which is isomorphic to \((\mathbb{Z}^m', \mathbb{Z}^m'_+)\) (by Exercise 3.3) and thus \(\eta\) becomes surjective.
Identify as above \( \mathbb{Z}^n \) and \( \mathbb{Z}(A) \). For \( x = \sum_{a \in A} x_a a \in \mathbb{Z}(A) \), set \( \|x\| = \max\{|x_a| \mid a \in A\} \) and \( m(x) = \text{Card}\{a \in A \mid |x_a| = \|x\|\} \). Let

\[
A_+ = \{a \in A \mid x_a > 0\} \quad \text{and} \quad A_- = \{a \in A \mid x_a < 0\}.
\]

We will use an induction on the pairs \((\|x\|, m(x))\) ordered lexicographically. Suppose first that \(\|x\| = 0\). Then \(x = 0\) and there is nothing to prove.

Assume next that \(\|x\| > 0\). Since \(\alpha(x) = 0\), we may assume that \(A_+ \) and \(A_-\) are both nonempty. Changing if necessary \(x\) into \(-x\), we may assume that \(\|x\| = \max\{|x_a| \mid a \in A_+\}\). Choose \(a_0 \in A_+\) such that \(x_{a_0} = \|x\|\). Since \(\alpha(x) = 0\), we have

\[
x_{a_0}\alpha(a_0) \leq \sum_{a \in A_+} x_a \alpha(a) = \sum_{a \in A_+} (-x_a)\alpha(a) \leq x_{a_0} \sum_{a \in A_-} \alpha(a).
\]

Since \(G\) is unperforated, we derive that \(\alpha(a_0) \leq \sum_{a \in A_-} \alpha(a)\). Since \(G\) is a Riesz group, there are some \(g_a \in G^+\), for each \(a \in A_-\), such that \(\alpha(a_0) = \sum_{a \in A_-} g_a\) with \(g_a \leq \alpha(a)\) for all \(a \in A_-\).

Consider the set \(B = (A \setminus \{a_0\}) \cup C\) where \(C = \{a' \mid a \in A_-\}\) is a copy of \(A_-\). We define two positive morphisms \(\eta : \mathbb{Z}(A) \to \mathbb{Z}(B)\) and \(\beta : \mathbb{Z}(B) \to G\) by

\[
\eta(a) = \begin{cases} 
\sum_{a_- \in A_-} a_- & \text{if} \quad a = a_0 \\
a & \text{if} \quad a \in A \setminus (A_+ \cup \{a_0\}) \\
a + a' & \text{if} \quad a \in A_-
\end{cases}
\]

and

\[
\beta(b) = \begin{cases} 
\alpha(b) & \text{if} \quad b \in A_+ \setminus \{a_0\} \\
\alpha(b) - g_b & \text{if} \quad b \in A_- \\
g_a & \text{if} \quad b = a' \in C.
\end{cases}
\]

It is easy to verify that \(\alpha = \beta \circ \eta\). Next, we claim that \(y = \eta(x)\) is such that \((\|y\|, m(y)) < (\|x\|, m(x))\).

Indeed, we have

\[
y = \sum_{a \neq a_0} x_a a + \sum_{a \in A_-} (x_a + x_{a_0})a'.
\]  

(3.4.1)

For every \(a \in A_-\), we have \(-x_{a_0} \leq x_a < 0\) and thus \(0 \leq x_a + x_{a_0} < x_{a_0}\). This shows, by inspection of the right-hand side of Equation (3.4.1), that \(\|\eta(x)\| \leq \|x\|\). In the case of equality, we clearly have less terms with maximal absolute value since there is no term \(a_0\). Thus \(m(y) < m(x)\).

This allows us to apply the induction hypothesis to the morphism \(\beta : (\mathbb{Z}(B), \mathbb{Z}(B^+)) \to (G, G^+)\) and \(y \in \mathbb{Z}(B)\). The solution is a pair of morphisms \(\eta' : \mathbb{Z}(B) \to \mathbb{Z}(B')\) and \(\beta' : \mathbb{Z}(B') \to G\) such that \(\eta'(y) = 0\) with the diagram of Figure [3.4.3] being commutative. Since \(\eta' \circ \eta(x) = 0\), the pair \((\eta \circ \eta', \beta')\) is a solution.

We prove a second lemma using iteratively the first one.
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Lemma 3.4.5 Let \((G, G^+)\) be an unperforated Riesz group and let \(n \geq 1\). Let \(\alpha : (\mathbb{Z}^n, \mathbb{Z}_+^n) \to (G, G^+)\) be a morphism. Then there is \(m \geq 1\) a surjective morphism \(\eta : (\mathbb{Z}^n, \mathbb{Z}_+^n) \to (\mathbb{Z}^m, \mathbb{Z}_+^m)\) and a morphism \(\beta : (\mathbb{Z}^m, \mathbb{Z}_+^m) \to (G, G^+)\) such that \(\ker \eta = \ker \alpha\) and \(\beta \circ \eta = \alpha\) (see the diagram 3.4.3).

Proof. Since \(\ker \alpha\) is a subgroup of \(\mathbb{Z}^n\), it is finitely generated (see Exercise 3.3). Let \(x_1, x_2, \ldots, x_k\) be a set of generators of \(\ker \alpha\). We use induction on \(k\). If \(k = 0\), there is nothing to prove. Otherwise, by Lemma 3.4.4 we find an integer \(m \geq 1\), a surjective morphism \(\eta : (\mathbb{Z}^n, \mathbb{Z}_+^n) \to (\mathbb{Z}^m, \mathbb{Z}_+^m)\) and a morphism \(\beta : (\mathbb{Z}^m, \mathbb{Z}_+^m) \to (G, G^+)\) such that \(\eta(x_1) = 0\) and \(\alpha = \beta \circ \eta\). Then \(\ker \beta\) is generated by \(\eta(x_2), \ldots, \eta(x_k)\). By induction hypothesis, there is an integer \(m', \) a surjective morphism \(\eta' : (\mathbb{Z}^m, \mathbb{Z}_+^m) \to (\mathbb{Z}^{m'}, \mathbb{Z}_+^{m'})\) and a morphism \(\beta' : (\mathbb{Z}^{m'}, \mathbb{Z}_+^{m'}) \to (G, G^+)\) such that \(\ker \eta' = \ker \beta\). Then \(\alpha = \beta' \circ \eta \circ \eta'\) (see Figure 3.4.3) and \(\ker \alpha = \ker \eta \circ \eta'\), whence the conclusion.

We prove a third lemma. We will only need one direction of the equivalence but we state the full result.

Lemma 3.4.6 (Shen) A directed ordered countable group \((G, G^+)\) is a dimension group if and only if for every morphism \(\alpha : (\mathbb{Z}^n, \mathbb{Z}_+^n) \to G\) there is an integer \(m \geq 1\) and morphisms \(\eta : (\mathbb{Z}^n, \mathbb{Z}_+^n) \to (\mathbb{Z}^m, \mathbb{Z}_+^m)\) and \(\beta : (\mathbb{Z}^m, \mathbb{Z}_+^m) \to (G, G^+)\) with \(\eta\) surjective such that \(\alpha = \beta \circ \eta\) with \(\ker \alpha = \ker \eta\).

Proof. Assume first that \((G, G^+) = \lim (\mathbb{Z}^{k_n}, \mathbb{Z}_+^{k_n})\). Let \(\alpha : (\mathbb{Z}(A), \mathbb{Z}_+(A)) \to (G, G^+)\) be a morphism. Choosing \(n\) large enough, we can find in \(\mathbb{Z}_+^{k_n}\) elements \(x_\alpha\) such that \(i_n(x_\alpha) = \alpha(a)\). Set \(\eta(a) = x_\alpha\). Let \(u_1, \ldots, u_k\) be a set of generators of \(\ker \alpha\). Choosing \(n\) large enough, we will have \(\eta(u_1) = \ldots = \eta(u_k) = 0\) and thus \(\ker \alpha = \ker \eta\). Thus the morphisms \(\eta\) and \(i_\alpha\) are a solution.

Let us prove the converse. Since \(G\) is countable, \(G^+\) is also countable. Let \(S = \{g_0, g_1, \ldots\}\) with \(g_n \in G^+\) be a set of generators of \(G^+\) and consider a set \(A = \{a_0, a_1, \ldots\}\) in bijection with \(S\).

We are going to build a sequence \((A_0, B_0, A_1, B_1, \ldots)\) and morphisms \(\alpha_n, \beta_n, \theta_n, \eta_n\) with \(\ker \alpha_n = \ker \eta_n\) and \(\eta_n\) surjective, as in Figure 3.4.4. Set \(A_0 = B_0 = \{a_0\}\) and define \(\alpha_0(a_0) = \beta_0(a_0) = g_0\) while \(\eta_0\) is the identity.

Assume that \(A_n, B_n, \beta_n, \eta_n\) are already defined. Set \(A_{n+1} = B_n \cup \{a_{n+1}\}\) and let \(\theta_n\) be the natural inclusion of \(\mathbb{Z}(B_n)\) into \(\mathbb{Z}(A_{n+1})\). Define \(\alpha_{n+1}: \)}
CHAPTER 3. ORDERED GROUPS

\[ \begin{array}{cccccc}
Z(A_0) & \xrightarrow{\eta_0} & Z(B_0) & \xrightarrow{\theta_0} & Z(A_1) & \xrightarrow{\eta_1} & Z(B_1) & \rightarrow & \cdots \\
\downarrow{\alpha_0} & & \downarrow{\beta_0} & & \downarrow{\alpha_1} & & \downarrow{\beta_1} & & \downarrow{G}
\end{array} \]

Figure 3.4.4: The construction of \((A_i, B_i)\).

\[ Z(A_{n+1}) \to G \]

by

\[ \alpha_{n+1}(a) = \begin{cases} 
  g_{n+1} & \text{if } a = a_{n+1} \\
  \beta_n(a) & \text{otherwise}
\end{cases} \]

By the hypothesis applied to the morphism \(\alpha_{n+1}\), we obtain \(B_{n+1}\), \(\eta_{n+1}\) and \(\beta_{n+1}\). Thus the iteration can continue indefinitely (unless \(S\) is finite, in which case we stop).

Consider the sequence

\[ Z(B_0) \xrightarrow{\gamma_0} Z(B_1) \xrightarrow{\gamma_1} Z(B_2) \cdots \]

with \(\gamma_n = \eta_{n+1} \circ \theta_n\), which is obtained by telescoping the top line of Figure 3.4.4.

Let \((H, H^+) = \lim \longrightarrow Z(B_n)\) be its direct limit. In case \(S\) is finite, we take for \(H\) the last \(Z(B_n)\) instead of the direct limit.

Let \(i_n : Z(B_n) \to H\) be the natural morphism. By the universal property of direct limits, there is a morphism \(h : (H, H^+) \to (G, G^+)\) such that \(h \circ i_n = \beta_n\).

The morphism \(h\) is injective. Indeed, since \(\alpha_n = \eta_n \circ \beta_n\) with \(\ker \alpha_n = \ker \eta_n\), \(\beta_n\) is injective.

Finally, \(h\) is surjective. Indeed, let \(g \in S\). Then \(g = g_n\) for some \(n\) and thus \(g = \alpha_n(a_n)\), which implies that \(g\) is in the image of \(h\). Since \(G\) is directed, it is generated by \(G^+\) and thus by \(S\). Therefore the image of \(h\) is \(G\).

The proof of Theorem 3.4.3 is now reduced to a concluding sentence.

Proof of Theorem 3.4.3: We have already seen that a dimension group is a countable directed Riesz group. Conversely, let \(G\) be a countable unperforated directed Riesz group. By Lemma 3.4.5 the condition of Shen’s Lemma (Lemma 3.4.6) is satisfied. Thus \(G\) is a dimension group.

Let us illustrate the proof on the example of the group \(G = \mathbb{Z}[1/2]\) of dyadic rationals. The submonoid \(G^+\) is generated by \(S = \{1, 1/2, 1/4, \ldots\}\). We start with \(A_0 = B_0 = \{a_0\}\) and \(\alpha_0(a_0) = 1\). Next, we find \(A_1 = \{a_0, a_1\}\) with \(\alpha_1(a_1) = 1/2\). Three iterations of the proof of Lemma 3.4.5 give \(B_1 = \{b_0\}\) with \(\eta(a_0) = 2b_0\) and \(\eta(a_1) = b_0\). Continuing in this way (Exercise 3.19), we obtain \(G\) as the direct limit of the sequence

\[ \mathbb{Z} \xrightarrow{\eta_0} \mathbb{Z} \xrightarrow{\eta_1} \mathbb{Z} \xrightarrow{\eta_2} \cdots \]

which was expected (see Example 3.3.1).
3.5 Stationary systems

Let us consider in more detail the dimension groups obtained as the direct limit of a sequence

\[ \mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \xrightarrow{M} \mathbb{Z}^d \ldots \]

where at each step the matrix \( M \) is a fixed nonnegative integer matrix. Such a sequence is called a stationary system.

The direct limit \( G_M = (\Delta_M, \Delta_M^+, 1_M) \) is a dimension group which has properties closely related to algebraic number theory. Indeed, the largest eigenvalue \( \lambda \) of \( M \) is an algebraic integer since it is a root of the polynomial \( \det(xI - M) \). Consequently, all components of the corresponding eigenvector are in the algebraic field \( \mathbb{Q}[\lambda] \).

We begin with the following simple property.

**Proposition 3.5.1 (Elliott)** A dimension group \( G = (\Delta_M, \Delta_M^+, 1_M) \) with \( M \) primitive is simple and has a unique state.

**Proof.** We have seen already that when \( M \) is primitive, we have \( \Delta_M^+ = \{ x \in \Delta_M \mid v^t \cdot x \geq 0 \} \) where \( v \) is a row eigenvector of \( M \) relative to the maximal eigenvalue. We may further assume that \( v^t \cdot 1_M = 1 \). Since \( v \) has all its components positive, the group is simple. This implies, by Proposition 3.2.8 that \( x \rightarrow x \cdot v \) is the unique state of \( G \).

We will now suppose that \( M \) is unimodular. This means that \( M \) has determinant \( \pm 1 \) or, equivalently that \( M \) is an element of the group \( GL(n, \mathbb{Z}) \) of integer matrices with integer inverse. The dominating eigenvalue \( \lambda \) of \( M \) is then a unit of the ring \( \mathbb{Z}[\lambda] \). Indeed, set \( \det(xI - M) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0 \). Then \( \lambda(\lambda^{d-1} + a_{d-1}\lambda^{d-2} + \cdots + a_1) = -a_0 \). Since \( a_0 = \det(M) = \pm 1 \), we conclude that \( \lambda \) is invertible.

**Proposition 3.5.2** Let \( M \in GL(n, \mathbb{Z}) \) be a primitive unimodular matrix with dominating eigenvalue \( \lambda \). The group \( G_M/\text{Inf}(G_M) \) is isomorphic to \( \mathbb{Z}[\lambda] \) with order unit the projection of \( u = [1 \ 1 \ \ldots \ 1]^t \) in the quotient.

**Proof.** Let \( p(x) \) be the minimal polynomial of \( \lambda \). Set \( \det(xI - M) = p(x)q(x) \) and let \( E = \ker p(M) \), \( F = \ker q(M) \). The subgroups \( E \) and \( F \) are invariant by \( M \). Since \( p \) is irreducible and \( \lambda \) has multiplicity 1, \( p \) and \( q \) are relatively prime. This implies that \( \mathbb{Z}^n = E \oplus F \). Indeed, let \( a(x), b(x) \) be such that \( a(x)p(x) + b(x)q(x) = 1 \). For every \( w \in \mathbb{Z}^n \), we have \( w = a(M)p(M)w + b(M)q(M)w \). Since \( p(M)q(M) = 0 \), the first term is in \( F \) and the second one in \( E \). Let \( v \) be a row eigenvector of \( M \) corresponding to \( \lambda \). We have for every \( w \in F \)

\[
\begin{align*}
v \cdot w &= v \cdot a(M)p(M)w \\
&= a(\lambda)p(\lambda)v \cdot w \\
&= 0
\end{align*}
\]
On the other hand, for every nonzero vector \( w \in E \), we have \( v \cdot w \neq 0 \) since otherwise \( E \) would contain an invariant subgroup, in contradiction with the fact that \( p \) is irreducible. Thus \( E \) is the infinitesimal subgroup and \( F \) is isomorphic with \( \mathbb{Z}[\lambda] \) via the map \( w \mapsto v \cdot w \).

Example 3.5.3 Let

\[
M = \begin{bmatrix}
1 & 1 & 0 \\
2 & 1 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

The eigenvalues of \( M \) are \( \lambda = (3 + \sqrt{5})/2, (3 - \sqrt{5})/2 \) and \(-1\). Thus, we have \( p(x) = x^2 - 3x + 1 \) and \( q(x) = x + 1 \). A left eigenvector of \( M \) for \( \lambda \) is \( [2\lambda - 2, \lambda, 1] \).

In the basis formed of the three vectors

\[
\begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
-2 \\
2
\end{bmatrix},
\]

the matrix \( M \) takes the form

\[
\begin{bmatrix}
2 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Since \( v \cdot u = 3\lambda - 1 = \lambda^2 \), the group is isomorphic to \( \mathbb{Z}[\lambda] \times \mathbb{Z} \) with unit \((1, 0)\).

It is a natural question to ask when the groups \( \mathcal{G}_M \) is isomorphic, as a unital group, to \( \mathbb{Z}[\lambda] \times \mathbb{Z}^k \) with unit \((1, 0)\). Actually, this happens if and only if \( v \cdot u \) is unit, as in the above example.

### 3.6 Exercises

**Section 3.1**

3.1 Let \( S \) be a submonoid of an abelian group \( G \). Show that the subgroup generated by \( S \) is the set \( S - S = \{ s - t \mid s, t \in S \} \).

3.2 Show that an ordered group \((G, G^+)\) is simple if and only if every nonzero element of \( G^+ \) is an order unit.

3.3 Show that a subgroup of \( \mathbb{Z}^n \) can be generated by at most \( n \) elements. Hint: use induction on \( n \).

**Section 3.2**

3.4 Show that the definition of an infinitesimal element in an unperforated ordered group does not depend on the choice of the order unit.
3.5 Let $G = (G, G^+ u)$ be a unital unperforated ordered group. Show that $g$ is infinitesimal if and only if $-\varepsilon u \leq g \leq \varepsilon u$ for every $\varepsilon \in \mathbb{Q}^+$. 

3.6 Let $G = (G, G^+, u)$ be a simple unital group. Denote by $\dot{g}$ the image of $g$ in $G/\text{Inf}(G)$. Show that $G/\text{Inf}(G)$ has a natural ordering defined by $\dot{g} \leq 0$ if $g + h \geq 0$ for some infinitesimal $h$, that $G/\text{Inf}(G)$ is simple and that the infinitesimal subgroup of $G/\text{Inf}(G)$ is trivial.

3.7 Let $G = (G, G^+, u)$ be a unital ordered group. Show that for every $g \in G^+$, one has the following minimax principle.

$$\inf\{p(g) \mid p \in S(G)\} = \sup\{\varepsilon \in \mathbb{Q}^+ \mid \varepsilon u \leq g\}$$

(hint: set

$$f_\ast(g) = \sup\{\varepsilon \in \mathbb{Q}^+ \mid \varepsilon u \leq g\},$$

$$f^\ast(g) = \inf\{\varepsilon \in \mathbb{Q}^+ \mid g \leq \varepsilon u\}.$$

and show that $\alpha = f_\ast(g)$, $\beta = f^\ast(g)$ where $\alpha, \beta$ are as in Lemma 3.2.5).

Section 3.3

3.8 Let $(G_n)_{n \geq 0}$ be a sequence of abelian groups with connecting morphisms $i_{n+1,n} : G_n \to G_{n+1}$. For $m \leq n$, set $i_{n,m} = i_{n,n-1} \circ \cdots \circ i_{m+1,m}$, with convention that $i_{n,n}$ is the identity. Let $G$ be the quotient of the disjoint union of the $G_n$ by the equivalence generated by the pairs $(g, i_{n+1,n}(g))$ for all $g \in G_n$ and all $n \geq 0$. Denote by $[g]$ the class of $g \in \bigcup G_n$.

Show that $G$ is a group for the operation

$$[g + h] = [g + i_{n,m}(h)]$$

where $g \in G_n$ and $h \in G_m$ with $m < n$.

Show that $G$ is isomorphic with the direct limit of the sequence $(G_n)_{n \geq 0}$.

3.9 Prove the universal property of direct limits.

3.10 Let $(G_n)_{n \geq 0}$ be a sequence of unital ordered groups $G_n = (G_n, G_n^+, 1_n)$ with connecting morphisms $i_{n+1,n}$. Let $G$ be the quotient of the disjoint union of the union of the $G_n$ by the equivalence generated by the pairs $(g, i_{n+1,n}(g))$ for all $g \in G_n$ and all $n \geq 0$, as in Exercise 3.8. Let $G^+$ be the set of classes of the elements of $\bigcup G_n^+$ and let $1$ be the class of $1_0$. Show that $(G, G^+, 1)$ is isomorphic to the direct limit of the sequence $G_n$.

3.11 Let

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Show that $\Delta_M \sim \mathbb{Z}[1/2]$ and $\Delta_M^+ \sim \mathbb{Z}_+[1/2]$. 

3.12 Two integral square matrices \(M, N\) are \textit{shift equivalent over} \(\mathbb{Z}\) with lag \(\ell\), written \(M \sim_{\mathbb{Z}} N\), if there are rectangular integral matrices \(R, S\) such that
\[
MR = RN, \quad SM = NS,
\]
and
\[
M^\ell = RS, \quad N^\ell = SR.
\]
We denote this situation by \((R, S) : M \sim_{\mathbb{Z}} N\) (lag \(\ell\)). When \(M, N\) are nonnegative, the matrices are \textit{shift equivalent}, written \(M \sim N\), if \(R, S\) can be chosen to be nonnegative integral matrices. We then denote \((R, S) : M \sim N\) (lag \(\ell\)). Show that shift equivalence over \(\mathbb{Z}\) (and shift equivalence) is an equivalence relation.

3.13 Show that two matrices which are shift equivalent over \(\mathbb{Z}\) have the same nonzero eigenvalues.

3.14 For a square integral matrix \(M\), denote by \(\delta_M\) the restriction of \(M\) to \(\Delta_M\). Show that \(M \sim_{\mathbb{Z}} N\) if and only if \((\Delta_M, \delta_M) \simeq (\Delta_N, \delta_N)\) (the latter means that there is a linear isomorphism \(\theta : \Delta_M \rightarrow \Delta_N\) such that \(\delta_N \circ \theta = \theta \circ \delta_M\)).

3.15 Show that two nonnegative integral matrices are shift equivalent if and only if \((\Delta_M, \Delta_M^+, \delta_M) \simeq (\Delta_N, \Delta_N^+, \delta_N)\) (in the sense that there is a linear isomorphism \(\theta : \Delta_M \rightarrow \Delta_N\) such that \(\delta_N \circ \theta = \theta \circ \delta_M\)).

Section 3.4

3.16 A submonoid of an abelian group is \textit{simplicial} if its positive cone is generated, as a monoid, by a finite independent set. Show that an ordered group is isomorphic to \(\mathbb{Z}^n\) with the natural order if and only if its positive cone is simplicial.

3.17 Show that an abelian group \(G\) satisfies the Riesz interpolation property if an only if it satisfies the Riesz decomposition property.

3.18 Let \((G, G^+)\) be an ordered group which satisfies the Riesz interpolation property. Show that for all \(x, y_1, \ldots, y_k \in G^+\) such that \(x \leq y_1 + y_2 + \ldots + y_k\), there exist \(x_1, x_2, \ldots, x_k \in G^+\) such that \(x = x_1 + x_2 + \ldots + x_k\) and \(x_i \leq y_i\) for \(1 \leq i \leq k\).

3.19 Let \(G = \mathbb{Z}[1/2]\) be the group of dyadic rationals. Show that the proof of Theorem 3.4.3 gives successively the following values for \(A_n, B_n\) and the morphisms \(\alpha_n, \beta_n, \eta_n\).
\[
A_0 = \{a_0\}, \quad A_i = \{b_{i-1}, a_i\} \text{ with } i > 0, \quad B_i = \{b_i\}
\]
\[
\alpha_i(b_{i-1}) = \frac{1}{2^{i-1}}, \quad \alpha_i(a_i) = \frac{1}{2^i}, \quad \eta_i(b_{i-1}) = 2b_i, \quad \eta_i(a_i) = b_i, \quad \beta_i(b_i) = \frac{1}{2^i}.
\]
3.7 Solutions

Section 3.1

3.1 The set $S - S$ contains 0 and is closed under addition because

$$(s - t) + (s' - t') = (s + s') - (t + t').$$

It is also closed by taking inverses because $-(s - t) = t - s$. Thus it is a subgroup of $G$. Since any subgroup containing $S$ contains $S - S$, the statement is proved.

3.2 If $G$ is simple, consider $u \in G^+$. By Proposition 3.1.7, the set $J = [u] - [u]$ is an ideal such that $J \cap G^+ = [u]$ Thus $J = G$ and $[u] = G^+$, which implies that $u$ is an order unit. Conversely, if every nonzero element of $G^+$ is an order unit, consider an ideal $J$ of $G$ not reduced to 0. Let $u \in J^+$ with $u \neq 0$. Since $u$ is an order unit, we have $[u] = G^+$ and thus $J = J^+ - J^+ = G^+ - G^+ = G$.

3.3 We use an induction on $n$. For $n = 1$, the result is true since a subgroup of $\mathbb{Z}$ is cyclic. Next, assume the result true for $n - 1$ and consider a subgroup $H$ of $\mathbb{Z}^n$. Let $\pi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the projection on the first $n - 1$ components. Thus $\pi(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1})$. By induction hypothesis, the group $\pi(H)$ is generated by $k$ elements $\pi(h_1), \ldots, \pi(h_k)$ with $k \leq n - 1$. On the other hand, $\ker(\pi)$ is isomorphic to $\mathbb{Z}$ and thus $H \cap \ker(\pi)$ is cyclic. Let $h_{k+1}$ be a generator of $\ker(\pi)$. For every $h \in H$, we have $\pi(h) = \sum_{i=1}^k n_i \pi(h_i)$ for some $n_i \in \mathbb{Z}$. Since $\pi(h - \sum_{i=1}^k n_i h_i) = 0$, we have $h - \sum_{i=1}^k n_i h_i = nk_{k+1} = n_{k+1}h_{k+1}$ and thus $h = \sum_{i=1}^{k+1} n_i h_i$. This shows that $H$ is generated by $h_1, h_2, \ldots, h_{k+1}$.

Section 3.2

3.4 Let $g$ be an infinitesimal element. Let $v \in G^+$ be another order unit. By definition, there is an integer $m$ such that $u \leq mv$. Since $g$ is infinitesimal, we have $mng \leq u$ for any $n \in \mathbb{Z}$ and thus $mng \leq mv$ which implies $ng \leq v$.

3.5 Assume first that $g$ is infinitesimal and consider $\varepsilon \in \mathbb{Q}_+$. Set $\varepsilon = p/q$ with $p, q \geq 0$. We have $qg \leq u \leq pq$ and thus $g \leq \varepsilon u$. Similarly, $qg \leq u$ implies $-pu \leq -u \leq -qg$ and thus $-\varepsilon u \leq g$.

Conversely if $n \geq 0$, $g \leq (1/n)u$ implies $ng \leq u$ and if $n \leq 0$, $-(1/n)u \leq g$ implies also $ng \leq u$.

3.6 Set $H = G/H$ and $H^+ = \{ \dot{g} \mid g + h \geq 0 \text{ for some } h \in \text{Inf}(G) \}$. If $\dot{g} \in H^+ \cap (-H^+)$ we have $g + h \geq 0$ and $-g - h' \geq 0$ for some $h, h' \in \text{Inf}(G)$. Then $h - h' \geq 0$ implies $h = h'$ since $\mathcal{G}$ is simple. We conclude that $g + h = 0$ and thus that $\dot{g} = 0$.

If $g + h > 0$, there is since $\mathcal{G}$ is simple an integer $n > 0$ such that $u \leq n(g + h)$. Then $\dot{u} \leq n\dot{g}$ since $nh \in \text{Inf}(G)$. Thus $(H, H^+)$ is simple.
Let $g \in G$ be such that $\dot{g} \in \text{Inf}(H)$. Then for every $\varepsilon \in \mathbb{Q}_+$, we have $\dot{g} \leq 2\varepsilon u$ and thus $g + h \leq 2\varepsilon u$ for some $h \in \text{Inf}(G)$. Since $-\varepsilon u \leq h$ it follows that $g \leq \varepsilon u$. A similar argument shows that $-\varepsilon u \leq g$. Thus $g \in \text{Inf}(G)$ and finally $\dot{g} = 0$.

### 3.7
Set $H = \mathbb{Z}u$ with $H^+ = \mathbb{Z}_+u$. There is a unique state $q$ on $(H, H^+, u)$ given by $q(nu) = n$.

Note first that since $0u \leq 1g$, we have $f_*(g) \geq 0$. Next, for every $n \geq 0$ such that $nu \leq mg$, we have $n/m = p(nu)/m \leq \alpha$ showing that $f_*(g) \leq \alpha$. Next, consider $x \in H$ and $m > 0$ such that $x \leq mg$. Set $x = nu$ with $n \in \mathbb{Z}$. If $n < 0$, then $p(x)/m = n/m < 0 \leq f_*(g)$. Next if $n \geq 0$, we have $p(x)/m = n/m \leq f_*(g)$. Thus $\alpha \leq f_*(g)$. We conclude that $\alpha = f_*(g)$. The proof that $f^*(g) = \beta$ is similar.

By Lemma 3.2.5 this shows that

$$0 \leq f_*(g) \leq f^*(g) < \infty$$

and that for every trace $p$ on $H + \mathbb{Z}t$ one has $f_*(g) \leq p(g) \leq f^*(g)$.

We finally claim that if $f_*(g) \leq \gamma \leq f^*(g)$ there is a state $p \in S(G)$ such that $p(g) = \gamma$. Indeed, by Lemma 3.2.5 (3), there is a state $r$ on $H + \mathbb{Z}q$ such that $r(g) = \gamma$. By Lemma 3.2.6 $r$ extends to a state $p$ on $G$. This proves the claim.

This shows that $\inf\{p(g) \mid p \in S(G)\} = f_*(g)$.

### Section 3.3

#### 3.8
It is easy to verify that $G$ is a group with neutral element $[0]$ since the operation is the unique operation on $G$ which extends the operations of the $G_n$. Consider the map $\pi : \cup_{n \geq 0} G_n \to \Delta$ which sends $g \in G_n$ to $\pi(g) = (g, i_{n+1,n}(g), \ldots)$. Since $\pi^{-1}(\Delta^0) = [0]$, the map $\pi$ it induces an isomorphism from $G$ onto $\lim\nrightarrow G_n$.

#### 3.9
Let $(G_n)$ be a sequence of abelian groups with connecting morphisms $i_{n+1,n} : G_n \to G_{n+1}$ and let $\alpha_n : G_n \to H$ be for each $n \geq 0$ a morphism such that $\alpha_n = \alpha_{n+1} \circ i_{n+1,n}$. Let $(g_n) \in \Delta$ be such that $g_{m+1} = i_{m+1,m}(g_m)$ for every $m \geq n$. Then $\alpha_{m+1}(g_{m+1}) = \alpha_{m+1} \circ i_{m+1,m}(g_m) = \alpha_n(g_m)$. Thus there is a morphism $h : \Delta \to H$ such that $h(g_0, g_1, \ldots) = \alpha_m(g_m)$ for all $m \geq n$. Since $\Delta^0 \subset \ker h$, the morphism $h$ induces a morphism $\varphi : G \to H$ such that $\alpha_n = \varphi \circ i_n$.

#### 3.10
We have seen in Exercise 3.8.4 that the map $\pi$ which sends $g \in G_n$ to $(g, i_{n+1,n}(g), \ldots)$ induces an isomorphism from $G$ onto the direct limit of the $G_n$. Since $\pi(g) \in \Delta^+$ if and only if $g \in \cup G_n$ and since $\pi(1_n) = (1_n)_{n \geq 0}$, the triple $(G, G^+, 1)$ is a unital ordered group isomorphic to the direct limit of the $G_n$. 
3.11  This follows from
\[ \mathcal{R}_M = \{ \begin{bmatrix} x \\ x \end{bmatrix} \mid x \in \mathbb{R} \}. \]

3.12  We have to prove the transitivity. Assume that \((R, S) : M \sim_Z N \) (lag \(\ell\)) and \((T, U) : N \sim_Z P \) (lag \(k\)). Then \((RT, US) : M \sim_Z P \) (lag \(k + \ell\)). Indeed, \(MRT = RNT = RTP, USM = UNS = PUS\) and
\[ M^{\ell+k} = RSM^k = RN^kS = RTUS, \quad P^{\ell+k} = P^\ell UT = UN^\ell T = USRT. \]

3.13  Assume that \((R, S) : M \sim_Z N \) (lag \(\ell\)). Let \(\lambda\) be a nonzero eigenvalue of \(M\) and let \(v \neq 0\) be a corresponding eigenvector. Set \(w = Sv\). We have
\[ Rw = RSv = M^{\ell}v = \lambda^\ell v \]
and thus \(w \neq 0\). Next
\[ Nw = NSv = SMv = \lambda Sv = \lambda w. \]
Thus \(\lambda\) is an eigenvalue of \(N\). The proof that every nonzero eigenvalue of \(N\) is an eigenvalue of \(M\) is similar.

3.14  First suppose that \((R, S) : M \sim_Z N \) (lag \(\ell\)). Denote by \(m, n\) the sizes of \(M, N\). Let \(\tilde{R}\) and \(\tilde{S}\) be the maps defined respectively on \(\mathcal{R}_N\) and \(\mathcal{R}_M\) by \(\tilde{R}(v) = Rv\) and \(\tilde{S}(w) = Sw\). It follows from (3.6.1) and (3.6.2) that \(\tilde{R}\) and \(\tilde{S}\) are mutually inverse linear isomorphisms between \(\mathcal{R}_M\) and \(\mathcal{R}_N\). Suppose that \(w \in \Delta_N\) and let \(k \geq 1\) be such that \(N^k w \in \mathbb{Z}^n\). Then, since \(\tilde{R}\) is integral,
\[ M^k(Rw) = RN^k w \in \mathbb{Z}^m \]
showing that \(\tilde{R}(w) \in \Delta_M\). Thus \(\tilde{R}(\Delta_N) \subset \Delta_M\). Similarly \(\tilde{S}(\Delta_M) \subset \Delta_N\).

By (3.6.2), we have the commutative diagrams below and this shows that \((\Delta_M, \delta_M) \simeq (\Delta_N, \delta_N)\).

\[
\begin{array}{ccc}
\Delta_N & \xrightarrow{\tilde{R}} & \Delta_M \\
\downarrow{\delta_N} & & \downarrow{\delta_M} \\
\Delta_N & \xrightarrow{\tilde{S}} & \Delta_N
\end{array}
\]  \quad (3.7.1)

Conversely, suppose that \(\theta : \Delta_M \to \Delta_N\) is a linear isomorphism such that \(\delta_N \circ \theta = \theta \circ \delta_M\). Since \(\theta(\Delta_M) \subset \Delta_N\), there is, for every \(v \in \mathbb{Z}^m\), a \(k \geq 1\) such that \(N^k \theta(M^m v) \in \mathbb{Z}^n\). Let \(S\) be the matrix of the linear map \(v \mapsto N^k \theta(M^m v)\) from \(\mathbb{R}^m\) to \(\mathbb{R}^n\). Then for every \(v \in \mathbb{R}^m\)
\[ SMv = N^k \theta(M^{m+1} v) = N^{k+1} \theta(M^m v) = NSv. \]
and thus $SM = NS$. Similarly, there is an $\ell$ such that $M^\ell \theta^{-1}(N^\ell w) \in \mathbb{Z}^n$ for every $w \in \mathbb{Z}^n$. Let $R$ be the matrix of the linear map $w \mapsto M^\ell \theta^{-1}(N^\ell w)$. Then we have as above $MR = RN$. Since finally $RS = M^{k+\ell+m+n}$ we conclude that $M$ and $N$ and shift equivalent over $\mathbb{Z}$.

Let $(R, S) : M \sim N$ (lag $\ell$) with $R, S$ nonnegative. Then the isomorphism $\tilde{S} : \Delta_M \to \Delta_N$ defined in the solution of Exercise 3.14 maps $\Delta_M^+ \to \Delta_N^+$. Similarly $\tilde{R}(\Delta_M^+) \subset \Delta_M^+$. But $\tilde{R} \circ \tilde{S} = \delta_M^+$ maps $\Delta_M^+$ onto itself, so that $\tilde{S}$ maps $\Delta_M^+$ onto $\Delta_N^+$. Hence $(\Delta_M, \Delta_M^+, \delta_M^+) \simeq (\Delta_N, \Delta_N^+, \delta_N)$.

Conversely, assume that $(\Delta_M, \Delta_M^+, \delta_M^+) \simeq (\Delta_N, \Delta_N^+, \delta_N)$, it is easy to verify that the matrices $R, S$ defined in the solution of Exercise 3.14 are nonnegative.

**Section 3.4**

Assume first that $(G, G^+)$ is isomorphic to $\mathbb{Z}^n$ with the usual order. Let $\alpha : \mathbb{Z}^n \to G$ be an isomorphism such that $\alpha(\mathbb{Z}^n) = G^+$. Then $G^+$ is generated by the images of the elementary basis vectors, which form an independent set.

Conversely, assume that $S \subset G^+$ is a finite independent set which generates $G^+$ as a semigroup. Set $S = \{s_1, s_2, \ldots, s_n\}$ and let $\alpha : \mathbb{Z}^n \to S$ be the linear map sending the $i$th elementary basis vector to $s_i$. Since $G = G^+ - G^+$, the map $\alpha$ is surjective. It is injective since $S$ is independent. Finally $\alpha(\mathbb{Z}^n) = G^+$ and thus $(G, G^+)$ is isomorphic to $(\mathbb{Z}^n, \mathbb{Z}^n)$.

Assume first that $G$ satisfies the Riesz interpolation property and consider $x_1, x_2, y_1, y_2 \geq 0$ such that $x_1 + x_2 = y_1 + y_2$. Since $0 \leq x_1 \leq y_1 + y_2$, we have $0, x_1 - y_2 \leq x_1, y_1$ and thus by the interpolation property, there is some $z_{11}$ such that $0, x_1 - y_2 \leq z_{11} \leq x_1, y_1$. Set

$$z_{12} = x_1 - z_{11}, \ z_{21} = y_1 - z_{11}, \ z_{22} = y_2 - z_{12}.$$ 

These elements are all positive and $x_1 = z_{11} + z_{12}, y_1 = z_{11} + z_{21}, y_2 = z_{12} + z_{22}$. Similarly $z_{21} + z_{22} = y_1 - z_{11} + y_2 - z_{12} = y_1 + y_2 - x_1 = x_2$.

Conversely, assume that $G$ satisfies the decomposition property. By subtraction, it is enough to prove the interpolation property for $0, x \leq y_1, y_2$. We then have $0 \leq y_1 \leq y_1 + (y_2 - x) = y_2 + (y_1 - x)$. Let $z \geq 0$ be such that $y_1 + z = y_2 + (y_1 - x)$. By the decomposition property, there are $z_{ij} \geq 0$ such that $y_1 = z_{11} + z_{12}, y_2 = z_{11} + z_{21}$ and $y_1 - x = z_{12} + z_{22}$. Then $x = z_{11} - z_{22}$ and thus $0, x \leq z_{11} \leq y_1, y_2$.

We use an induction on $k$. The property holds trivially for $k = 1$. Next, assume that $x, y_1, y_2, \ldots, y_{k+1} \in G^+$ satisfy the hypothesis. Since $0, x - y_{k+1} \leq x, y_1 + \ldots + y_k$ there is by Riesz interpolation some $z \in G$ such that $0, x - y_{k+1} \leq z \leq x, y_1 + \ldots + y_k$. By induction hypothesis, there are $x_1, \ldots, x_k$ such that $z = x_1 + \ldots + x_k$ with $x_i \leq y_i$ for $1 \leq i \leq k$. But then $x_1, \ldots, x_k, x_{k+1} = x - z$ are a solution.
Indeed, we have
\[
\beta_i(\eta_i(b_{i-1})) = \alpha_i(b_{i-1}) = \frac{1}{2^{i-1}}, \quad \beta_i(\eta_i(a_i)) = \alpha_i(a_i) = \frac{1}{2^i}
\]
and
\[
\ker(\eta_i) = \ker(\alpha_i) = \mathbb{Z}(b_{i-1} - 2a_i)
\]

### 3.8 Notes

Ordered algebraic structures are a classical subject of which ordered groups (and a fortiori ordered abelian groups) are a particular case. See Fuchs (1963) for a general introduction to ordered groups. Many authors assume ordered groups to be directed (see for example Putnam (2018)). This simplifies the presentation but has the drawback of complicating the definition of subgroups of ordered groups. We follow the choice of Goodearl and Handelman (1976). This does not make any difference in the sequel since dimension groups are directed.

#### 3.8.1 States

The notion of state is closely related with the notion of trace in an algebra (see Davidson (1996)). Theorem 3.2.4 is a Hahn Banach type existence theorem due to Goodearl and Handelman (1976).

Proposition 3.2.8 is (Effros, 1981, Corollary 4.2) (it is stated there for a simple dimension group but actually holds in this slightly more general case).

The definition of the infinitesimal group is from Giordano et al. (1995).

#### 3.8.2 Direct limits

The notion of direct limit is classical and can be formulated for other categories than groups, in particular for algebras, as we shall see in Chapter 10.

The group \( \Delta_M \) defined by Equation (3.3.2) is called in Lind and Marcus (1995) the dimension group of \( M \). It is actually a dimension group, in the sense of the definition given in Section 3.3. Proposition 3.3.4 is essentially Theorem 7.5.13 in Lind and Marcus (1995).

#### 3.8.3 Dimension groups

The definition of dimension groups (Section 3.3) was introduced by G. Elliott in Elliott (1976). Simplicial semigroups (Exercise 3.16) are taken from Elliott (1979).

A group with the Riesz (interpolation or decomposition) property was called a Riesz group in Fuchs (1965). Some authors add the requirement that the group is unperforated (see Davidson (1996) for example).
The Effros, Handelman and Shen Theorem (Theorem 3.4.3) is from [Effros et al., 1980], (see also the expositions in [Effros, 1981, Theorem 3.1], [Davidson, 1996, Section IV.7] or [Putnam, 2018, Chapter 8]). Part of the proof is already in [Shen, 1979]. In particular, Lemma 3.4.5 is [Shen, 1979, Theorem 3.1] and the argument of Lemma 3.4.6 is already in [Elliott, 1979].

The unimodular conjecture proposed by Effros and Shen [Effros and Shen, 1979] asks whether any dimension group $G$ can be obtained as a direct limit

$$
\mathbb{Z}^k \xrightarrow{M_1} \mathbb{Z}^k \xrightarrow{M_2} \mathbb{Z}^k \ldots
$$

where $k \geq 1$ and all $M_n$ are unimodular matrices. It was proved to hold when $G$ is simple and has one state [Riedel, 1981a] but disproved in the general case [Riedel, 1981b].

### 3.8.4 Exercises

The minimax principle on ordered groups (Exercise 3.7) is due to [Goodearl and Handelman, 1976].

Our treatment of shift equivalence (Exercises 3.12, 3.14 and 3.15) follows [Lind and Marcus, 1995] where the pair $(\Delta_M, \delta_M)$ is called the dimension pair of $M$ and the triple $(\Delta_M, \Delta_M^+, \delta)$ is called the dimension triple of $M$. The statement of Exercise 3.15 can be expressed by the property that the dimension pair is a complete invariant of shift equivalence over $\mathbb{Z}$ and the dimension triple a complete invariant of shift equivalence, a result of [Krieger, 1980a].

Note that shift equivalence is decidable, that is, given two matrices $M, N$, one can effectively decide whether there are $U, V$ and $\ell$ such that $(U, V) : M \equiv N$ (lag $\ell$) [Kim and Roush, 1988] (see also [Kim and Roush, 1979]). Strong shift equivalence obviously implies shift equivalence. After remaining many years as a conjecture, known as Williams Conjecture, the converse was disproved by [Kim and Roush, 1992], even for irreducible matrices [Kim and Roush, 1997].
Chapter 4

Ordered cohomology

In this chapter, we define coboundaries and cohomologous functions in the space of continuous integer valued functions on a topological dynamical system. These terms are used in homological algebra and are the dual notion of boundaries and homologous elements. We briefly explain these terms in the elementary setting of graphs.

Assume that $G = (V, E)$ is a graph on a set $V$ of vertices with a set $E$ of edges. Each edge $e$ has its origin $\alpha(e)$ and its end $\omega(e)$. One defines the boundary operator $\partial : \mathbb{Z}(E) \to \mathbb{Z}(V)$ from the free abelian group on $E$ to the free abelian group on $V$ by $\partial(e) = \omega(e) - \alpha(e)$. The elements of $\text{Im}(\partial)$ are called boundaries and the elements of $\text{ker}(\partial)$ are the cycles. Elements of $\mathbb{Z}(E)$ equivalent modulo $\text{ker}(\partial)$ are said to be homologous.

Identifying $\mathbb{Z}^V$ to $\text{Hom}(\mathbb{Z}(V), \mathbb{Z})$, by duality, we have a coboundary operator $\partial^t : \mathbb{Z}^V \to \mathbb{Z}^E$ (note that it operates in the reverse way). It is such that for $\phi \in \mathbb{Z}^V$

$$\partial^t \phi(e) = \phi(\omega(e)) - \phi(\alpha(e)).$$

Indeed, by definition of the dual operator, we have

$$\partial^t \phi(e) = \phi(\partial(e)) = \phi(\omega(e) - \alpha(e)) = \phi(\omega(e)) - \phi(\alpha(e)).$$

The elements of $\text{Im}(\partial^t)$ are called coboundaries and the elements of $\text{ker}(\partial^t)$ cocycles. In what follows, we will choose for $V$ a topological space $X$, use the graph with edges $(x, Tx)$ for a transformation $T$ on $X$ and use the topological dual $C(X, \mathbb{Z})$ instead of $\mathbb{Z}^V$.

The chapter is organized as follows. In Section 4.1, we define the coboundary operator on continuous functions from a space $X$ to $\mathbb{R}$. We prove in the next section an important result, due to Gottschalk and Hedlund, characterizing coboundaries in a minimal system (Theorem 4.2.3). It is used in the next section (section 4.3) to define the ordered cohomology group $K^0(X, T)$ of a
system \((X,T)\). In Sections 4.6 and 4.7 we relate the group \(K^0(X,T)\) to the operations of conjugacy and induction.

In Section 4.8 we introduce invariant probability measures on topological dynamical systems. We recall the basic notions of invariant measure and of ergodic measure. We prove the unique ergodicity of primitive substitution shifts (Theorem 4.8.9). We relate in Section ?? coboundaries with invariant probability measures and the notion of state. We prove the important result stating that, for a minimal Cantor system, the states of the dimension group are in one-to-one correspondance with invariant probability measures (Theorem 4.9.3). We finally use this result to give a description of the dimension groups of Sturmian shifts (Theorem 4.9.4).

4.1 Coboundaries

Let \((X,T)\) be a topological dynamical system. We denote by \(C(X,\mathbb{R})\) the group of real valued continuous functions on \(X\) and by \(C(X,\mathbb{Z})\) the group of integer valued continuous functions. We denote by \(C(X,\mathbb{R}_+)\) and \(C(X,\mathbb{Z}_+)\) the corresponding sets of non-negative functions.

As any function with values in a discrete space, an integer valued function \(f\) on \(X\) is continuous if and only if it is locally constant, that is, for every \(x\in X\), there is a neighborhood of \(x\) on which \(f\) is constant. When \(X\) is a Cantor set, this neighborhood can be chosen clopen. When \(X\) is a space without non trivial clopen sets, like the torus or any non trivial closed interval of \(\mathbb{R}\), then \(C(X,\mathbb{Z})\) consists of constant functions.

Since \(X\) is compact, a function \(f\in C(X,\mathbb{Z})\) takes only a finite number of values. Indeed, the family \((f^{-1}(\{n\}))_{n\in\mathbb{Z}}\) is a covering of \(X\) by open sets, which has a finite subcover.

For every \(f\in C(X,\mathbb{R})\), we define the coboundary of \(f\) as the function

\[
\partial_T f = f \circ T - f.
\]

Clearly, the map \(f\mapsto \partial_T f\) is an endomorphism of the group of both \(C(X,\mathbb{Z})\) and \(C(X,\mathbb{R})\). Note that the operator \(\partial_T\) is the coboundary operator (as introduced above) related to the graph with \(X\) as set of vertices and edges from each \(x\in X\) to \(Tx\).

A function \(f\in C(X,\mathbb{R})\) is a coboundary if there is a function \(g\in C(X,\mathbb{R})\) such that \(f = \partial_T g\). Two functions \(f,f'\) are cohomologous if \(f-f'\) is a coboundary.

Note that if \(f\in C(X,\mathbb{R})\) is a coboundary, then \(f\circ T\) is also a coboundary. Indeed, if \(f = g \circ T - g\), then \(f \circ T = g \circ T^2 - g \circ T = \partial_T (g \circ T)\).

Example 4.1.1 Let \(A = \{a,b\}, \delta \in \mathbb{R}\) and \((A^\mathbb{Z},S)\) be the full shift on \(A\). The continuous function \(f\) defined, for \(x\in A^\mathbb{Z}\) by

\[
f(x) = \begin{cases} 
\delta & \text{if } x \in [ab], \\
-\delta & \text{if } x \in [ba], \\
0 & \text{otherwise},
\end{cases}
\]
is a coboundary. Indeed, it is the coboundary of any function \( g \) defined for \( x \in A^Z \) by
\[
g(x) = \begin{cases} 
\alpha & \text{if } x \in [a], \\
\beta & \text{if } x \in [b], 
\end{cases}
\]
for \( \beta - \alpha = \delta \).

We now give a natural example of a real valued continuous function on a Cantor set.

**Example 4.1.2** Let \( X = \{0, 1\}^\mathbb{N} \) be the one sided full shift on 0,1. To every \( x \in X \), we associate the real number
\[
f(x) = \sum_{n \geq 0} x_n 2^{-n-1}
\]
which is the value of \( x \) considered as an expansion \( 0.x_0x_1\cdots \) in base 2. The map \( f \) is continuous and we have
\[
\partial f(x) = \begin{cases} 
f(x) & \text{if } x \leq 1/2 \\
f(x) - 1 & \text{otherwise}
\end{cases}
\]

**Proposition 4.1.3** Let \((X, T)\) be a minimal dynamical system and \( f \in C(X, \mathbb{R}) \). Then
1. One has \( \partial T f = 0 \) if and only if \( f \) is constant.
2. If \( f \in C(X, \mathbb{Z}) \) is a coboundary, it is the coboundary of some \( h \in C(X, \mathbb{Z}) \).

*Proof.* 1. Suppose \( \partial T f = 0 \). For \( c \in \mathbb{R} \), the set \( Y = f^{-1}(\{c\}) \) is closed. Assume that \( Y \) is nonempty. Since \( \partial T f = 0 \), the set \( Y \) is invariant by \( T \). Hence, \((X, T)\) being minimal, this forces \( Y = X \).

2. Assume that \( f = \partial T g \) with \( g \in C(X, \mathbb{R}) \). Let \( \tau : \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) be the natural projection onto the torus \( T = \mathbb{R}/\mathbb{Z} \). Since \( g \circ T - g \) belongs to \( C(X, \mathbb{Z}) \), we have \( \tau \circ (g \circ T - g) = 0 \) and thus \( \partial T (\tau \circ g) = \tau \circ g \circ T - \tau \circ g = 0 \). Since \( \tau \) is continuous, the same argument as above implies that \( \tau \circ g \) is constant. Thus there exists \( c \in \mathbb{R} \) such that \( h(x) = g(x) - c \) is an integer for all \( x \in X \). Since \( \partial T h = \partial T g \) we obtain the conclusion. \( \blacksquare \)

For \( n > 0 \), we set
\[
f^{(n)} = f + f \circ T + \cdots + f \circ T^{n-1}
\]
with \( f^{(0)} = 0 \). The family \( f^{(n)} \) for \( n \geq 0 \) is called the *cocycle* associated to \( f \). One has for all \( n, m \geq 0 \), the relation
\[
f^{(m+n)} = f^{(m)} + f^{(n)} \circ T^m \tag{4.1.1}
\]
called the \textit{cocycle relation}. Indeed, we have
\[
\begin{align*}
f^{(m+n)} &= (f + f \circ T + \cdots + f \circ T^{m-1}) + (f \circ T^m + \cdots + f \circ T^{m+n-1}) \\
&= f^{(m)} + (f + f \circ T + \cdots + f \circ T^{n-1}) \circ T^m \\
&= f^{(m)} + f^{(n)} \circ T^m.
\end{align*}
\]

The following formula will be used often.

**Proposition 4.1.4** Let \((X,T)\) be a dynamical system and \(g \in C(X, \mathbb{R})\). If \(f = \partial_T g\), then we have for all \(n \geq 0\),
\[
f^{(n)} = g \circ T^n - g. \tag{4.1.2}
\]

**Proof.** We have
\[
\begin{align*}
f^{(n)} &= g \circ T - g + g \circ T^2 - g \circ T + \cdots + g \circ T^n - g \circ T^{n-1} \\
&= g \circ T^n - g.
\end{align*}
\]

\[
\square
\]

### 4.2 Gotschalk and Hedlund Theorem

We first prove the following simple property.

**Proposition 4.2.1** Let \((X,T)\) be a topological dynamical system. If \(f \in C(X, \mathbb{R})\) is a coboundary, then the sequence \((f^{(n)})\) is bounded uniformly.

**Proof.** Let \(f = \partial_T g\) for some \(g \in C(X, \mathbb{R})\), then, using Proposition 4.1.4, \(f^{(n)} = g \circ T^n - g\) and thus all \(|f^{(n)}|\) are bounded by \(2 \sup |g|\).

The following consequence will be used in the next section.

**Corollary 4.2.2** Let \((X,T)\) be a recurrent topological dynamical system. If \(f \in C(X, \mathbb{R})\) is a non-negative coboundary, it is identically zero.

**Proof.** Let \(x_0 \in X\) be a recurrent point. By Proposition 4.2.1, the sequence \(f^{(n)}(x_0)\) is bounded. Since \(x_0\) is recurrent, \(f(T^n x_0) \geq 1/2 \sup f\) for infinitely many values of \(n\). Thus \(f^{(n)}(x_0) \to \infty\) as \(n \to \infty\) unless \(\sup f = 0\).

In the minimal case there is a converse of Proposition 4.2.1 by the following result. We will use it several times.

**Theorem 4.2.3 (Gottschalk, Hedlund)** Let \((X,T)\) be a minimal topological dynamical system and \(f \in C(X, \mathbb{R})\). The following are equivalent.

1. \(f\) is a coboundary.
2. The sequence \((f^{(n)})\) is bounded uniformly.

3. There exists \(x_0 \in X\) such that the sequence \((f^{(n)}(x_0))_{n \geq 0}\) is bounded.

Proof. We have already seen that 1 implies 2. Assertion 2 clearly implies 3.

The proof that 3 implies 1 is in three steps.

**Step 1** For each clopen neighborhood \(U\) of \(x_0\), we set
\[
\Lambda(U) = \{f^{(n)}(x_0) \mid n \geq 0, T^n x_0 \in U\} \quad \text{and} \quad \Lambda = \bigcap U \Lambda(U)
\]

\(U\) running over all the clopen neighborhoods of \(x_0\). These sets are bounded and contain 0. We claim that \(\Lambda = \{0\}\). To prove it, we show that for any \(a \in \Lambda\), we have \(2a \in \Lambda\), which will imply \(a = 0\) since \(\Lambda\) is bounded. Suppose indeed that \(a\) belongs to \(\Lambda\). Let \(U\) be a clopen neighborhood of \(x_0\) and \(\varepsilon > 0\). Since \(a \in \Lambda(U)\), there exists \(n \geq 0\) such that \(T^n x_0 \in U\) and \(|f^{(n)}(x_0) - a| < \varepsilon\). Since \(T^n\) and \(f^{(n)}\) are continuous, there is a clopen neighborhood \(V \subset U\) of \(x_0\) such that \(T^n y \in V\) and \(|f^{(n)}(y) - a| < 2\varepsilon\) for all \(y \in V\). Since \(a\) belongs to \(\Lambda(V)\), there exists \(m \geq 0\) with \(T^m x_0 \in V\) and \(|f^{(m)}(x_0) - a| < \varepsilon\). As \(T^m x_0\) belongs to \(V\), we have \(T^{n+m} x_0 \in U\), \(f^{(n+m)}(x_0) \in \Lambda(U)\) and \(|f^{(n)}(T^n x_0) - a| < 2\varepsilon\). By the cocycle relation (4.1.1), we obtain
\[
|f^{(n+m)}(x_0) - 2a| \leq |f^{(n)}(T^n x_0) - a| + |f^{(m)}(x_0) - a| \\
< 3\varepsilon.
\]

Since \(\varepsilon\) is arbitrary, this implies that \(2a\) belongs to \(\Lambda(U)\) and \(U\) being an arbitrary neighborhood of \(x_0\), it implies that \(2a\) also belongs to \(\Lambda\), which proves the claim.

**Step 2** Because each \(\Lambda(U)\) is compact and \(\Lambda(U \cap U') \subset \Lambda(U) \cap \Lambda(U')\), for every \(\varepsilon > 0\), there exists a neighborhood \(U_\varepsilon\) of \(x_0\) such that \(\Lambda(U_\varepsilon) \subset [-\varepsilon, \varepsilon]\).

We claim that there exists a function \(g \in C(X, \mathbb{R})\) such that \(g(T^n(x_0)) = f^{(n)}(x_0)\) for all \(n \geq 0\). For this, it is enough to prove that for every \(x \in X\) and every sequence \(n_i \to \infty\) of integers such that \(T^{n_i} x_0 \to x\), the sequence \(f^{(n_i)}(x_0)\) converges. We then define \(g(x)\) as this limit. Fix \(\varepsilon > 0\). By minimality, there exists \(n \geq 0\) such that \(T^n x_0\) and \(T^n x\) are in \(U_\varepsilon\).

Let \(W\) be a neighborhood of \(x\) such that \(T^n y \in U_\varepsilon\) and \(|f^{(n)}(y) - f^{(n)}(x)| < \varepsilon\) for all \(y \in W\). For \(i\) large enough, \(y = T^{n_i} x_0\) is in \(W\). Then \(T^n y = T^{n + n_i} x_0 \in U_\varepsilon\) and consequently \(|f^{(n_i+n)}(x_0)| < \varepsilon\). Moreover, we have \(|f^{(n)}(y) - f^{(n)}(x)| = |f^{(n)}(T^{n_i} x_0) - f^{(n)}(x)| < \varepsilon\).

Thus
\[
|f^{(n_i)}(x_0) + f^{(n)}(x)| \leq \varepsilon + |f^{(n_i)}(x_0) + f^{(n)}(T^{n_i} x_0)| = \varepsilon + |f^{(n_i+n)}(x_0)| \leq 2\varepsilon.
\]

For large enough \(n_i, n_j\), we obtain
\[
|f^{(n_i)}(x_0) - f^{(n_j)}(x_0)| \leq |f^{(n_i)}(x_0) - f^{(n)}(x)| + |f^{(n)}(x) - f^{(n_j)}(x_0)| \leq 4\varepsilon.
\]

It follows that \((f^{(n_i)}(x_0))\) is a Cauchy sequence and converges.
Step 3  Since \((X,T)\) is minimal, the orbit of \(x_0\) is dense. For any \(x = T^nx_0\) in this set, we have by construction
\[
\partial_T g(x) = g \circ T^{n+1}(x_0) - g \circ T^n(x_0) = f^{(n+1)}(x_0) - f^{(n)}(x_0) = f \circ T^n(x_0) - f(x)
\]
By continuity, this extends to any \(x \in X\) and this proves that \(f\) is a coboundary.

It can be shown that the function \(g\) such that \(f = \partial g\) is determined uniquely up to a constant (Exercise 4.8).

We prove the following additional result. The proof uses the Baire Category Theorem (see Appendix A).

**Proposition 4.2.4** Let \((X,T)\) be a minimal topological dynamical system. The following conditions are equivalent for \(f \in C(X,\mathbb{Z})\).

(i) There exists \(g \in C(X,\mathbb{Z})\) such that \(f + \partial_T g \geq 0\).

(ii) There exists \(g \in C(X,\mathbb{R})\) such that \(f + \partial_T g \geq 0\).

(iii) The family of functions \((f^{(n)})_{n \geq 0}\) is uniformly bounded from below.

(iv) For every \(x \in X\), the family of numbers \((f^{(n)}(x))_{n \geq 0}\) is bounded from below.

**Proof.** (i) \(\Rightarrow\) (ii) is obvious.

(ii) \(\Rightarrow\) (iii) If \(f + \partial_T g \geq 0\), then \((f + \partial_T g)^{(n)} = f^{(n)} + (\partial_T g)^{(n)} \geq 0\). Hence, by Equation (4.1.2), \(f^{(n)} + g \circ T^n - g \geq 0\) and thus \(f^{(n)} \geq g - g \circ T^n\) is bounded by below.

(iii) \(\Rightarrow\) (iv) is trivial.

(iv) \(\Rightarrow\) (i). For each \(n \geq 0\), let \(g_n\) and \(g\) be defined by \(g_n(x) = \inf \{f^{(k)}(x) \mid 0 \leq k \leq n\}\) and \(g(x) = \inf \{f^{(n)}(x) \mid n \geq 0\} = \inf \{g_n(x) \mid n \geq 0\}\). For each \(n \geq 0\) and \(k \geq 1\), we obtain \(g_{n+k}(x) = \inf \{g_{k-1}(x), f^{(k)}(x) + g_n \circ T^k(x)\}\) and \(g(x) = \inf \{g_{k-1}(x), f^{(k)}(x) + g \circ T^k(x)\}\). In particular, \(g(x) = \inf \{0, f(x) + g \circ T(x)\} \leq f(x) + g \circ T(x)\), which implies \(f + \partial_T g \geq 0\). Because each \(g_n\) belongs to \(C(X,\mathbb{Z})\), it is sufficient to prove that \(g = g_n\) for some \(n \geq 0\).

For each \(n \geq 0\), let \(K_n = \{x \in X \mid g(x) = g_n(x)\}\). For all \(x \in X\), as each \(g_n\) takes only integer values, there exists an \(n \geq 0\) such that \(g(x) = g_n(x)\) and thus \(x \in K_n\). Consequently, \(X = \cup_{n \geq 0} K_n\). Since each \(K_n\) is closed, by Baire Category Theorem, there exists \(m \geq 0\) such that \(K_m\) has a nonempty interior. By minimality and compactness, there exists \(p \geq 1\) such that \(\cup_{1 \leq k \leq p} T^{-k}K_m = X\). Let \(x \in X\) and \(1 \leq k \leq p\) be such that \(T^kx \in K_m\). We obtain
\[
g(x) = \inf \{g_{k-1}(x), f^{(k)} + g \circ T^k(x)\} = \inf \{g_{k-1}, f^{(k)} + g_m \circ T^k(x)\} = g_{k+m}(x)
\]
and thus \(x \in K_{k+m} \subset K_{p+m}\). We conclude that \(K_{p+m} = X\) and \(g = g_{p+m}\).
4.3 Ordered cohomology group of a dynamical system

Let \((X,T)\) be a topological dynamical system. Since the coboundary operator on \(C(X,\mathbb{Z})\) is a group morphism, its image \(\partial T C(X,\mathbb{Z})\) is a subgroup. We denote by \(H(X,T,\mathbb{Z})\) the quotient group

\[
H(X,T,\mathbb{Z}) = C(X,\mathbb{Z})/\partial T C(X,\mathbb{Z}).
\]

and by \(H^+(X,T,\mathbb{Z})\) the image of \(C(X,\mathbb{Z}^+)\) in this quotient.

We denote by \(K_0^0(X,T)\) the triple

\[
K_0^0(X,T) = (H(X,T,\mathbb{Z}), H^+(X,T,\mathbb{Z}), 1_X).
\]

where \(1_X\) is the image in \(H(X,T,\mathbb{Z})\) of the constant function with value 1 on \(X\).

The proof of the following result uses Gottschalk and Hedlund Theorem (more precisely Corollary 4.2.2).

**Proposition 4.3.1** For any recurrent topological dynamical system \((X,T)\), the triple \(K_0^0(X,T)\) is a unital ordered group.

**Proof.** Clearly \(H^+(X,T,\mathbb{Z})\) is a subsemigroup which generates \(H(X,T,\mathbb{Z})\) since both properties hold for \(C(X,\mathbb{Z}^+)\). Finally assume that \(f, f' \in C(X,\mathbb{Z}^+)\) are such that \(f\) is cohomologous to \(-f'\). Then \(f + f'\) is a nonnegative coboundary. By Corollary 4.2.2 we have \(f + f' = 0\) and thus \(f = f' = 0\). Thus \(H^+(X,T,\mathbb{Z}) \cap -H^+(X,T,\mathbb{Z}) = \{0\}\).

Following the term introduced by Boyle and Handelman, the group \(K_0^0(X,T)\) is called the ordered cohomology group of the topological dynamical system \((X,T)\).

We give now a very simple example of computation of \(K_0^0(X,T)\).

**Example 4.3.2** Let \((X,S)\) be the shift space formed of the two infinite sequences \(x = (\cdots abab.abab\cdots)\) and \(y = (\cdots baba.baba\cdots)\). Obviously, \(Sx = y\) and \(Sy = x\). The characteristic functions of \(x\) and \(y\) are also exchanged by \(S\). Thus \(H(X,S,\mathbb{Z}) = \mathbb{Z}\) and \(K_0^0(X,S) = (\mathbb{Z}, \mathbb{Z}^+, 1)\).

We can generalize the last example by considering the case of a finite set \(X\) and a permutation \(T\) on \(X\). Then \(H(X,T,\mathbb{Z}) = \mathbb{Z}^d\) where \(d\) is the number of orbits of the permutation \(T\).

**Example 4.3.3** We now give an elementary argument to show that \(H(X,S,\mathbb{Z})\) is isomorphic to \(\mathbb{Z}^2\) when \((X,S)\) is a Sturmian shift like the Fibonacci shift.

We will see later (using return words, in Section 5.4) how this can be done by more general methods. We denote by \(\chi_{[w]}\) the characteristic function of the cylinder set \([w]\).
Set $A = \{a, b\}$. We show by induction on $|w|$ that $\chi_{[w]}$ is cohomologous to an element of the subgroup $G$ generated by $\chi_{[a]}$ and $\chi_{[b]}$. This is true if $|w| = 1$. Assume that it holds for words of length $n$ and consider a word $w$ of length $n + 1$. Then $w = ux$ for some nonempty word $u$ of length $n$ and some letter $x \in \{a, b\}$. If $u$ is not right-special, we have $uA^N = uxA^N$ and thus $\chi_{[ux]}$ is cohomologous to an element of $G$ by induction hypothesis. Otherwise, we have

$$\chi_{[u]} = \chi_{[ua]} + \chi_{[ub]}.$$  \hfill (4.3.1)

As $w$ is either $ua$ or $ub$, it suffices to show that $\chi_{[ua]}$ and $\chi_{[ub]}$ are cohomologous to some elements in $G$. Set $u = yv$ with $y$ a letter and $v$ a word. Then $va$ and $vb$ cannot be both left-special. Assume that $va$ is not left-special. Then $\chi_{[ua]} = \chi_{[yva]} = \chi_{[va]} \circ T$. By the induction hypothesis, $\chi_{[va]}$ is cohomologous to an element of $G$ and thus also the map $\chi_{[ua]}$. By Equation (4.3.1), this implies that $\chi_{[ub]}$ is also cohomologous to an element of $G$.

Since $C(X, \mathbb{Z})$ is generated by the functions $\chi_{[w]}$ (see Proposition 4.4.1 below), this shows that $H(X, S, \mathbb{Z})$ is generated by the projections of $\chi_{[a]}$ and $\chi_{[b]}$. Thus the morphism sending $(\alpha, \beta)$ to the class of $\alpha \chi_{[a]} + \beta \chi_{[b]}$ is surjective from $\mathbb{Z}^2$ to $H(X, S, \mathbb{Z})$. It is injective because if $f = \alpha \chi_{[a]} + \beta \chi_{[b]}$ with $\alpha$ or $\beta$ nonzero, then $f^{(n)}$ is not bounded and thus $f$ is not a coboundary by Theorem 4.2.3.

We may also consider the group $H(X, T, \mathbb{R}) = C(X, \mathbb{R})/\partial_T C(X, \mathbb{R})$ and define $H^+(X, T, \mathbb{R})$ as the image of $C(X, \mathbb{R})$ in $H(X, T, \mathbb{R})$. By Corollary 4.2.2, the pair $(H(X, T, \mathbb{R}), H^+(X, T, \mathbb{R}))$ is an ordered group. We note the following relation between $H^+(X, T, \mathbb{Z})$ and $H^+(X, T, \mathbb{R})$.

**Corollary 4.3.4** $H(X, T, \mathbb{Z})$ is a subgroup of $H(X, T, \mathbb{R})$ and

$$H^+(X, T, \mathbb{Z}) = H(X, T, \mathbb{Z}) \cap H^+(X, T, \mathbb{R}).$$  \hfill (4.3.2)

**Proof.** The group $C(X, \mathbb{Z})$ is included in $C(X, \mathbb{R})$ and if $f \in C(X, \mathbb{Z})$ is in $\partial_T C(X, \mathbb{R})$, then by Proposition 4.4.2, it is in $\partial_T C(X, \mathbb{Z})$. Thus the inclusion of $C(X, \mathbb{Z})$ in $C(X, \mathbb{R})$ defines an injection of $H(X, T, \mathbb{Z})$ in $H(X, T, \mathbb{R})$.

Next the left side of (4.3.2) is clearly included in the right side. Suppose conversely that $f = \partial_T g + h$ with $f \in C(X, \mathbb{Z})$, $g \in C(X, \mathbb{R})$ and $h \in C(X, \mathbb{R}_+)$. For every $n \geq 0$, we have by Equation (4.3.2)

$$f^{(n)}(x) = g(T^n x) - g(x) + h^{(n)}(x)$$

and the family $(f^{(n)})_{n \geq 0}$ is uniformly bounded by below. By Proposition 4.2.4, $f$ is cohomologous in $C(X, \mathbb{Z})$ to some $f' \in C(X, \mathbb{Z}_+)$, which proves 4.3.2. \hfill ■

The next proposition characterizes functions which are cohomologous to integer valued functions. Before we need a classical lemma. We provide a proof for sake of completeness. Let $\tau : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection from $\mathbb{R}$ onto the torus $T = \mathbb{R}/\mathbb{Z}$.
Lemma 4.3.5 Let $X$ be a Cantor space and $f : X \to \mathbb{T}$ be a continuous map. For every $\epsilon > 0$, there exists a continuous function $h_\epsilon : X \to [-\epsilon, 1]$ such that $f = \tau \circ h_\epsilon$.

Proof. Let $\epsilon > 0$. For each $x \in X$, there exist a neighbourhood $U_x$ and a continuous map $E_x : U_x \to \mathbb{R}$ such that $E_x(U_x)$ is included in $[E_x(x) - \epsilon, E_x(x) + \epsilon]$ and $f(y) = \tau \circ E_x(y)$, for all $y \in U_x$. Since $X$ is a Cantor space, we can suppose $U_x$ is a clopen set. Compactness of $X$ yields points $x_1, \ldots, x_n$ such that $U_{x_1}, \ldots, U_{x_n}$ is a finite clopen covering of $X$. Then, for each $i \in [1, n]$ there exists a (possibly empty) clopen set $V_i$ included in $U_{x_i}$ such that $V_1, \ldots, V_n$ is a partition of $X$. Let $I = \{i \mid E_{x_i}(x_i) + \epsilon > 0\}$. Then the map $F_\epsilon = \sum_{i \in I} 1_{V_i} E_{x_i} + \sum_{i \in I} 1_{V_i}(E_{x_i} - 1)$ fulfills the requirement.

Proposition 4.3.6 Let $U$ be a non empty clopen set in $X$, and $f \in C(X, \mathbb{R})$.

1. If $f^{(n)}(x) \in \mathbb{Z}$ for every $x \in U$ such that $T^n x \in U$, then $f$ is cohomologous to some $g \in C(X, \mathbb{Z})$.

2. If $f^{(n)}(x) \in \mathbb{Z}_+$ for every $x \in U$ such that $T^n x \in U$, then $f$ is cohomologous to some $g \in C(X, \mathbb{Z}_+)$.

Proof. 1. Using the same method as the step 2 in the proof of Theorem 4.2.3, we get that the sequence $(\tau(f^{(n)}(x_0)))$ converges in $\mathbb{T}$ whenever $n_i \to \infty$ and $T^n x_0$ converges. Thus there exists a continuous function $u : X \to \mathbb{T}$ such that $u(T^n x_0) = \tau(f^{(n)}(x_0))$ for every $n \geq 0$. If $x = T^n x_0$ for some $n \geq 0$, then

$$u(T x) - u(x) = u(T^{n+1} x_0) - u(T^n x_0) = \tau(f^{(n+1)}(x_0) - f^{(n)}(x_0)) = \tau(f T^n x_0) = \tau(f(x)).$$

By density, the same is true for every $x \in X$. By Lemma 4.3.5 there exists $h \in C(X, \mathbb{R})$ such that $\tau \circ h = u$. The function $g = f - \partial_T h$ belongs to $C(X, \mathbb{Z})$ and is a coboundary of $f$.

2. The family $(f^{(n)})$ is uniformly bounded by below and the result follows from Proposition 4.2.4.

4.4 Cylinder functions

We consider in this section the case of shift spaces, in which the cylinder functions play an important role.

Let $(X, S) \subseteq \mathbb{A}^Z$ be a shift space. Recall that we denote by $\mathcal{L}_n(X)$ the set words of length $n$ in $\mathcal{L}(X)$. We denote by $R_n(X)$ the group of maps from $\mathcal{L}_n(X)$ into $\mathbb{R}$, by $Z_n(X)$ the group of maps from $\mathcal{L}_n(X)$ into $\mathbb{Z}$ and by $Z_n^+(X)$ the corresponding subset of non negative maps.

For $\phi \in R_n(X)$, the function $\phi : X \to \mathbb{R}$ given by

$$\phi(x) = \phi(x_{[0, n-1]})$$
is called the \textit{cylinder function} associated to $\phi$. It belongs to $C(X,\mathbb{Z})$ when $\phi$ belongs to $\mathbb{Z}_n(X)$, and, to $C(X,\mathbb{Z}_+)$ when $\phi$ belongs to $\mathbb{Z}_n^+(X)$.

A function $f \in C(X,\mathbb{Z})$ is a cylinder function if and only if there exists $n \geq 1$ such that $f(x)$ depends only on $x_{[0,n-1]}$.

**Proposition 4.4.1** Let $(X,S)$ be a subshift. Every function in $C(X,\mathbb{Z})$ (resp. $C(X,\mathbb{Z}_+)$) is cohomologous to some cylinder function (resp. non-negative cylinder function).

**Proof.** Let $f \in C(X,\mathbb{Z})$. Since $f$ is locally constant, there exists $n$ such that $f(x)$ depends only on $x_{[-n,n]}$. Then $f(T^n x)$ depends only on $x_{[0,2n]}$ and $f \circ T^n$ is a cylinder function. Since $f \circ T^n - f = (\partial T f)^{(n)}$ by Equation (4.1.2), $f$ is cohomologous to $f \circ T^n$ and the conclusion follows. Finally, if $f \in C(X,\mathbb{Z}_+)$, then $f \circ T^n$ is non-negative.

**Proposition 4.4.2** Let $(X,S)$ be a subshift. If a cylinder function with integer values is a coboundary, it is the coboundary of some cylinder function.

**Proof.** Let $g \in C(X,\mathbb{Z})$ and suppose that $f = \partial_T g$ is a cylinder function. We may choose $n$ large enough so that simultaneously $g(x)$ depends only on $x_{[-n,n]}$ and $f(x)$ depends only on $x_{[0,n]}$. Assume that $y_{[0,2n]} = z_{[0,2n]}$. Since $f(x)$ depends only on $x_{[0,n]}$, the value $f^{(n)}(x)$ depends only on $x_{[0,2n]}$ and thus $f^{(n)}(y) = f^{(n)}(z)$. Similarly, since $(T^n y)_{[-n,n]} = (T^n z)_{[-n,n]}$, we have $g(T^n y) = g(T^n z)$. Thus, by Equation (4.1.2), we have

$$g(y) = g(T^n y) - f^{(n)}(y) = g(T^n z) - f^{(n)}(z) = g(z)$$

and thus $g$ is a cylinder function.

### 4.5 Ordered group of a recurrent shift space

In this section, we show how to compute the ordered cohomology group of a recurrent shift space using the cylinder functions introduced in Section 4.4. This description will be used in the next chapters. We will first define an ordered group $G_n(X)$ associated with cylinder functions corresponding to words of length $n$ (Proposition 4.5.1). In a second part we show that the direct limit of these groups is the cohomology group $K^0(X,S)$ of the shift space $(X,S)$ (Proposition 4.5.2).

**4.5.1 Groups associated with cylinder functions**

For a word $w = a_1 a_2 \cdots a_n$ of length $n \geq 1$ with $a_i \in A$, we set $p_n(w) = a_1 \cdots a_{n-1}$ and $s_n(w) = a_2 \cdots a_n$. Next, given a shift space $(X,S)$, recall that
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\( R_n(X) \) denotes the group of maps from the set \( \mathcal{L}_n(X) \) of words of length \( n \) in \( \mathcal{L}(X) \) into \( \mathbb{R} \). We define, for \( n \geq 1 \), three group morphisms

\[
p^*_n, s^*_n, \partial_{n-1} : R_{n-1}(X) \rightarrow R_n(X)
\]

by

\[
p^*_n(\phi) = \phi \circ p_n, \quad s^*_n(\phi) = \phi \circ s_n,
\]

and

\[
\partial_{n-1}(\phi) = s^*_n(\phi) - p^*_n(\phi) = \phi \circ s_n - \phi \circ p_n
\]

for every \( \phi \in R_{n-1}(X) \). These morphisms map \( Z_{n-1}(X) \) into \( Z_n(X) \) (recall that \( Z_n(X) \) denotes the group of functions from \( \mathcal{L}_n(X) \) into \( \mathbb{Z} \)). Moreover, \( p^*_n \) and \( s^*_n \) are injective and positive.

Note that for every \( \phi \in R_{n-1}(X) \), the cylinder functions associated to \( p^*_n(\phi) \) and to \( \phi \) are the same, that is

\[
p^*_n(\phi) = \phi. \tag{4.5.1}
\]

Note also that for \( \phi \in R_{n-1}(X) \) and \( \psi \in R_n(X) \) one has

\[
\psi = \partial_{n-1}(\phi) \iff \psi = \partial_n \phi \tag{4.5.2}
\]

and thus the cylinder function \( \psi \) associated to \( \psi \in \partial_{n-1}R_{n-1}(X) \) is a coboundary. We denote

\[
G_n(X) = Z_n(X)/\partial_{n-1}Z_{n-1}(X) \tag{4.5.3}
\]

the quotient of the group \( Z_n(X) \) by its subgroup \( \partial_{n-1}Z_{n-1}(X) \), we denote by \( G^+_n(X) \) the image in \( G_n(X) \) of \( Z^+_n(X) \) and by \( 1_n(X) \) the image in \( G_n(X) \) of the constant function \( 1 \in Z_n(X) \).

**Proposition 4.5.1** For every recurrent shift space \((X, S)\), the triple

\[
\mathcal{G}_n(X) = (G_n(X), G^+_n(X), 1_n(X))
\]

is a unital ordered group.

**Proof.** The set \( G^+_n(X) \) is a submonoid of \( G_n(X) \) because \( Z^+_n(X) \) is a submonoid of \( Z_n(X) \) and it generates \( G_n(X) \) because \( Z^+_n(X) \) generates \( Z_n(X) \). To prove that \( G^+_n(X) \cap (-G^+_n(X)) = \{0\} \), we have to prove that if \( \phi \in Z^-_n(X) \) is such that \( \partial_{n-1}\phi \in Z^+_n(X) \), then \( \partial_{n-1}\phi = 0 \). Let \( u, v \in L_{n-1}(X) \). Since \((X, S)\) is recurrent, there exists \( m \geq n \) and \( w \in L_m(X) \) such that \( u \) is a prefix of \( w \) and \( v \) is a suffix of \( w \). Then, since \( u = w|_{[1,m-1]} \) and \( v = w|_{[m-n+2,m]} \), we have, because the first sum is telescopic,

\[
\phi(v) - \phi(u) = \sum_{i=1}^{m-n+1} \phi(w|_{[i+1,i+n-1]}) - \phi(w|_{[i,i+n-2]})
\]

\[
= \sum_{i=1}^{m-n+1} (\partial_{n-1}\phi)(w|_{[i,i+n-1]}) \geq 0.
\]
Since this is true for every \( u, v \), it implies that \( \phi \) is constant and thus that \( \partial_{n-1} \phi = 0 \). □

Moreover, since \( \partial_n \circ p^*_{n-1} = p^*_n \circ \partial_{n-1} \), the morphism \( p^*_n \) induces a morphism of the quotient groups

\[ i_{n+1,n} : G_n(X) \to G_{n+1}(X) \]

which is morphism of unital ordered groups.

**Example 4.5.2** Let \( (X, S) \) be the Fibonacci shift. We have \( L_1(X) = \{a, b\} \) and \( L_2(X) = \{aa, ab, ba\} \). Since \( \partial_1(\chi_{[a]}) = \chi_{[ba]} - \chi_{[ab]} \), the projections of \( \chi_{[ba]} \) and \( \chi_{[ab]} \) in \( G_2(X) \) are equal and \( G_2(X) \simeq \mathbb{Z}^2 \). The matrix of the projection of \( Z_2(X) \) on \( G_2(X) \) is

\[ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \]

The matrix of the morphism \( i_{2,1} \) is

\[ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \]

since for example

\[ i_{2,1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = P(p^*_1 \chi_{[a]}) = P \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \]

### 4.5.2 Ordered cohomology group

We now prove the following result which describes the ordered cohomology group \( K^0(X, S) \) of a recurrent shift space \( (X, S) \).

**Proposition 4.5.3** For every recurrent shift space \( (X, S) \), the unital ordered group \( K^0(X, S) \) is the inductive limit of the family \( \mathcal{G}_n(X) \) with the morphisms \( i_{n+1,n} \).

**Proof.** To every \( \phi \in Z_n(X) \) we can associate the corresponding cylinder function \( \phi \) and its projection \( \pi(\phi) \) in \( H(X, S, \mathbb{Z}) \). When \( \phi \) is in \( \partial_n(Z_n-1(X)) \), the cylinder function \( \phi \) is a coboundary and \( \pi(\phi) = 0 \). If \( \phi \) belongs to \( Z_n^+(X) \), then \( \phi \) is in \( C(X, \mathbb{Z}_+) \) and the cylinder function associated with the constant function equal to 1 on \( L_n(X) \) is the constant function equal to 1 on \( X \).

Thus we have defined a morphism of unital ordered groups

\[ j_n : \mathcal{G}_n \to K^0(X, S) \]

and clearly \( j_{n+1} \circ i_{n+1,n} = j_n \). Therefore the sequence \( (j_n)_{n \geq 1} \) induces a morphism \( j \) from the inductive limit of the \( \mathcal{G}_n \) to the group \( K^0(X, S) \). Let us show that \( j \) is an isomorphism.
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Let $\phi \in Z_n(X)$, let $\alpha$ be its projection in $G_n(X)$ and suppose that $j_n \alpha = 0$. Then $\phi$ is a coboundary and, by Proposition 4.4.1, it is the coboundary of some cylinder function $\psi$. Let $m \geq n$ be such that $\psi \in Z_m(X)$. Since $p_m^* \circ \cdots \circ p_n^* \phi$ belongs to $Z_{m+1}(X)$, we have by Equation (4.5.1)

$$\phi = p_m^* \circ \cdots \circ p_n^* \phi.$$ 

Now, since $\phi = \partial_T \psi$, we have by (4.5.2)

$$p_m^* \circ \cdots \circ p_n^* \phi = \partial_T \psi \implies p_m^* \circ \cdots \circ p_n^* \phi = \partial_m \psi \quad \text{which implies} \quad p_m^* \circ \cdots \circ p_n^* \phi \in \partial_m(Z_m(X)) \quad \text{and} \quad i_{m,m-1} \circ \cdots \circ i_{n+1,n} \alpha = 0. \quad \text{Thus the image of } \alpha \text{ in the inductive limit is 0. This shows that } j \text{ is injective.}$$

Moreover, every $f \in C(X, Z)$ (resp. $C(X, Z^+)$) is cohomologous to a cylinder function (resp. to a nonnegative cylinder function). It follows that $j$ is onto and maps the positive cone of the inductive limit onto $H^+(X, S, Z)$.

We now present an example which shows that the ordered cohomology group of a shift of finite type may fail to be a dimension group.

**Example 4.5.4** Let $(X, T)$ be the set of two-sided infinite paths in the graph represented in Figure 4.5.1.

![Figure 4.5.1: A shift of finite type](image)

Since $a$ can be preceded by $a, d$ or $e$, we have

$$\chi_{[a]} \circ T = \chi_{[aa]} + \chi_{[da]} + \chi_{[ea]}.$$ 

Similarly, since $a$ can be followed by $a, b$ or $c$, we have

$$\chi_{[a]} = \chi_{[aa]} + \chi_{[ab]} + \chi_{[ac]}.$$ 

Thus $\partial \chi_{[a]} = \chi_{[da]} + \chi_{[ea]} - \chi_{[ab]} - \chi_{[ac]}$. We have then in $H(X, S, Z)$ the equality

$$\chi_{[d]} + \chi_{[e]} = \chi_{[da]} + \chi_{[ea]} + \sum_{x \in \{d, e\}, y \in \{b, c\}} \chi_{[xy]} \approx \chi_{[ab]} + \chi_{[ac]} + \sum_{x \in \{c, d\}, y \in \{b, c\}} \chi_{[xy]} \approx \chi_{[b]} + \chi_{[c]}.$$
4.6 Factor maps and conjugacy

Let \((X, T)\) and \((X', T')\) be two topological dynamical systems and let \(\phi\) be a factor map from \((X, T)\) to \((X', T')\). If \(f \in C(X', \mathbb{Z})\) is the coboundary of \(g\), then \(f \circ \phi\) is the coboundary of \(g \circ \phi\). Therefore, the group homomorphism \(f \mapsto f \circ \phi\) from \(C(X', \mathbb{Z})\) to \(C(X, \mathbb{Z})\) induces a group homomorphism

\[\phi^*: H(X', T', \mathbb{Z}) \to H(X, T, \mathbb{Z}).\]

Clearly, \(\phi^*\) is a morphism of ordered groups with order units since \(\phi^*(H^+(X', T', \mathbb{Z}))\) is a subset of \(H^+(X, T, \mathbb{Z})\) and \(\phi^*(1_{X'}) = 1_X\).

**Proposition 4.6.1** If \((X, T)\) is minimal, the morphism \(\phi^*\) is injective and

\[\phi^*(H^+(X', T', \mathbb{Z})) = \phi^*(H(X', T', \mathbb{Z})) \cap H^+(X, T, \mathbb{Z}).\]

**Proof.** If \(f \in C(X', \mathbb{Z})\) is such that \(f \circ \phi\) is a coboundary, then by Theorem 4.2.3 the sequence of functions \((f \circ \phi)^{(n)}\) is uniformly bounded. But, for all \(n \geq 0\), \((f \circ \phi)^{(n)} = f^{(n)} \circ \phi\) and since \(\phi\) is onto, \(\|f^{(n)} \circ \phi\|_\infty = \|f^{(n)}\|_\infty\). Thus the sequence \((f^{(n)})\) is uniformly bounded and, by Theorem 4.2.3 again, \(f\) is a coboundary in \(C(X', \mathbb{Z})\). This shows that \(\phi^*\) is injective.

To prove the second assertion, we first note that the inclusion from left to right is clear since \(\phi^*(H^+(X', T', \mathbb{Z})) \subset H^+(X, T, \mathbb{Z})\). Conversely, consider \(f \in C(X', \mathbb{Z})\) and \(g \in C(X, \mathbb{Z})\) such that \(f \circ \phi + \partial_T g \geq 0\). By Proposition 4.2.3.
the sequence of functions \((f \circ \phi)^{(n)}\) is bounded by below. By the same argument as in the proof of the first assertion, the sequence \((f^{(n)})\) is bounded by below. Thus, by Proposition 4.6.1, there is a function \(g' \in C(X', \mathbb{Z})\) such that \(f + \delta_f g' \geq 0\). Thus the class of \(f\) is in \(H^+(X', T', \mathbb{Z})\) and its image by \(\phi^*\) is in \(\phi^*(H^+(X', T', \mathbb{Z}))\).

We deduce from Proposition 4.6.1 that the ordered cohomology group is invariant under conjugacy.

**Corollary 4.6.2** If \(\phi\) is a conjugacy from \((X, T)\) onto \((X', T')\), then the map \(\phi^*\) from \(H(X', T', \mathbb{Z})\) to \(H(X, T, \mathbb{Z})\) is an isomorphism.

Note that Proposition 4.4.2 can be deduced from Proposition 4.6.1. Indeed, let \((Y, S)\) be the one-sided shift space associated with \((X, S)\) and \(\theta : (X, S) \to (Y, S)\) be the natural morphism. Assume that \(f \in C(X, \mathbb{Z})\) is a cylinder function with integer values. Then \(f = k \circ \theta\) for some \(k \in C(Y, \mathbb{Z})\). If \(f\) is a coboundary, by Proposition 4.6.1 \(k\) is a coboundary. Thus \(k = \delta_S h\) for some \(h \in C(Y, S)\). Then \(f = \delta_S h \circ \theta = h \circ S \circ \theta - h \circ \theta = h \circ \theta \circ T - h \circ \theta\). Thus \(f\) is the coboundary of \(h \circ \theta\) which is a cylinder function.

Note also that this argument shows that

\[\theta^* : (H(Y, S, \mathbb{Z}), H_+(Y, S, \mathbb{Z}), 1_Y) \to (H(X, T, \mathbb{Z}), H_+(X, T, \mathbb{Z}), 1_X)\]

is an isomorphism of unital ordered groups.

### 4.7 Groups of induced systems

Let \((X, T)\) be a minimal topological dynamical system and let \(U\) be a nonempty clopen subset of \(X\). We have already seen the notion of induced system \((U, T_U)\). Recall that \(T_U(x) = T^{n(x)}(x)\) where \(n(x) = \inf\{n > 0 \mid T^n x \in U\}\). The system \((U, T_U)\) is again minimal.

As an example, which will be considered often in the next chapters, let us consider a minimal shift space \((X, S)\), a word \(u \in \mathcal{L}(X)\) and the clopen set \(U = [u]\). Let \(R_X(u)\) be the set of left return words to \(u\) and let \(\varphi : B \to R'_X(u)\) be a bijection extended to a morphism from \(B^*\) to \(R'_X(u)^*\). Every \(x \in U\) can be written in a unique way as \(x = \varphi(y)\) for some \(y \in B^\mathcal{Z}\) (see Proposition 2.4.23 if you are not entirely convinced). Then \(Y = \varphi^{-1}(U)\) is a shift space on \(B\) and \((U, S_U)\) is isomorphic to \((Y, T)\) where \(T\) is the shift of \(B^\mathcal{Z}\). Indeed, by definition of return words, for every \(x \in [u]\), the integer \(n(x)\) is the length of the unique prefix of \(x^+\) in \(R_X(u)\). Thus

\[S_U \circ \varphi(y) = S^{n(x)}(\varphi(y)) = \varphi \circ T(y)\]

showing that \(\varphi\) is a conjugacy from \((Y, T)\) onto \((U, S_U)\). The shift \((Y, T)\) is called the derivative shift of \(X\) with respect to \([u]\).
Example 4.7.1 Let X be the Fibonacci shift, which is the substitution shift generated by \( \varphi : a \to ab, b \to a \). The system induced on \( U = [a] \) is isomorphic to X. Indeed, the set of left return words to a is \( R^1(a) = \{ab, a\} \) and thus \( \varphi \) is an isomorphism from X onto \((U, T_U)\).

Let \( I_U : C(X, \mathbb{Z}) \to C(U, \mathbb{Z}) \) and \( R_U : C(U, \mathbb{Z}) \to C(X, \mathbb{Z}) \) be the morphisms of ordered groups defined by

\[
(I_U f)(x) = f^{(n(x))}(x) \text{ for all } x \in U,
\]

and let \( R_U f \) be the map equal to \( f \) on \( U \) and equal to 0 elsewhere.

Proposition 4.7.2 The maps \( I_U, R_U \) induce reciprocal isomorphisms of ordered groups from \( K^0(X, T) \) onto \( K^0(U, T_U) \).

Proof. Let \( f \in C(X, \mathbb{Z}) \) be the coboundary of \( g \in C(X, \mathbb{Z}) \). Then for every \( x \in U \), by Equation (4.1.2), we have \( f^{(n(x))} = g \circ T^{n(x)} - g \) and thus \( I_U f \) is the coboundary of the restriction of \( g \) to \( U \). This shows that \( I_U \) induces a morphism

\[
i_U : H(X, T, \mathbb{Z}) \to H(U, T_U, \mathbb{Z}).
\]

Since \( I_U \) maps \( C(X, \mathbb{Z}_+) \) to \( C(U, \mathbb{Z}_+) \), it is a morphism of ordered groups from \( K^0(X, T) \) to \( K^0(U, T_U) \).

Let us now show that \( R_U \) induces a map of ordered groups from \( K^0(U, T_U) \) to \( K^0(X, T) \). For every \( x \in X \), let \( m(x) = \inf \{ m \geq 0 \mid T^m x \in U \} \). Then the function \( x \mapsto m(x) \) is continuous, coincides with \( n(x) \) outside \( U \) and is zero on \( U \). Suppose that \( f \in C(U, \mathbb{Z}) \) is the coboundary in \((U, T_U)\) of some \( g \in C(U, \mathbb{Z}) \). Let \( h(x) = g(T^{m(x)} x) \). We have

\[
h(Tx) - h(x) = \begin{cases} f(x) & \text{if } x \in U \\ 0 & \text{if } x \notin U. \end{cases}
\]

Indeed, if \( x \in U \), then \( m(x) = 0 \), \( m(Tx) = n(x) - 1 \) and

\[
h(Tx) - h(x) = g(T^{m(x)}) - g(x) = g(T_U x) - g(x) = f(x)
\]

and if \( x \notin U \), then \( m(Tx) = m(x) - 1 \) and \( h(Tx) = h(x) \). This shows that \( \partial_{T} h = R_U f \) and thus that \( R_U \) maps coboundaries of \((U, T_U)\) to coboundaries of \((X, T)\) and thus induces a map

\[
r_U : H(U, T_U, \mathbb{Z}) \to H(X, T, \mathbb{Z}).
\]

Finally, for any \( f \in C(U, \mathbb{Z}) \) and \( x \in U \), we have

\[
I_U \circ R_U f(x) = (R_U f)^{(n(x))}(x) = f(x) + R_U f(Tx) + \ldots + R_U f(T^{n(x)-1} x) = f(x).
\]

Thus \( I_U \circ R_U f = f \) and \( I_U \circ R_U \) is the identity. Next for every \( f \in C(X, \mathbb{Z}) \), we have

\[
R_U \circ I_U f - f = \partial_T k \quad (4.7.1)
\]
where \( k(x) = f^{(m(x))}(x) \). Indeed, for every \( x \in X \), the above equation can be rewritten

\[
(RUf)^{(n(x))}(x) - f(x) = f^{(m(Tx))}(Tx) - f^{(m(x))}(x).
\]

If \( x \in U \), the value of the right-hand side is \( f^{(m(Tx))}(Tx) - f^{(m(x))}(x) = f^{(n(x)-1)}(Tx) - f^{(0)}(x) = f^{(n(x)-1)}(Tx) = f^{(n(x))}(x) - f(x) \). Next, if \( x \not\in U \), the value of the left-hand side is \( -f(x) \) and the value of the right-hand side is \( f^{(n(x)-1)}(Tx) - f^{(n(x))}(x) = -f(x) \). Equation (4.7.1) shows that \( RU \circ IU \) is the coboundary of \( k \) and thus that \( RU \circ IU \) is the identity on \( H(X,T,Z) \).

Note that the isomorphism from \( K^0(X,T) \) onto \( K^0(U,T,U) \) is not an isomorphism of unital ordered groups because the units are not the same. Indeed, the image by \( RU \) of \( \chi_U \) is the characteristic function of \( U \) and not the constant function \( \chi_X \).

### 4.8 Invariant probability measures

Let \((X,T)\) be a topological dynamical system. There is an important connexion between the ordered cohomology group \( H(X,T,Z) \) of \((X,T)\) and the invariant probability measures on \((X,T)\).

#### 4.8.1 Invariant measures

Recall (see Appendix B) that a Borel probability measure on a topological space \( X \) is a Borel measure \( \mu \) such that \( \mu(X) = 1 \).

Consider a shift space \((X,S)\) on the alphabet \( A \). A Borel probability measure \( \mu \) on \( X \) determines a map \( \pi : \mathcal{L}(X) \to [0,1] \) by

\[
\pi(u) = \mu([u]_X) \quad (4.8.1)
\]

where \([u]_X = \{ x \in X \mid x_{[0,|u|-1]} = u \} \) is the cylinder defined by \( u \). This map satisfies the compatibility conditions \( \pi(\varepsilon) = 1 \) and

\[
\sum_{a \in A} \pi(ua) = \pi(u) \quad (4.8.2)
\]

for every \( u \in (X) \). Conversely for any map satisfying these conditions defines a unique Borel probability measure satisfying (4.8.1) by the Caratheodory extension theorem (see Appendix B).

Let now \((X, T)\) be a topological dynamical system. A probability measure \( \mu \) on \( X \) is said to be invariant if one has \( \mu(T^{-1}U) = \mu(U) \) for every Borel subset \( U \) of \( X \). The set of invariant probability measures is convex and closed for the weak-star topology.

**Theorem 4.8.1 (Krylov, Bogolyubov)** Any topological dynamical system has at least one invariant probability measure.
Proof. Let \((X, T)\) be a topological dynamical system and let \(x \in X\). For every \(n \geq 0\), \(\mu_n = \delta_{T^n x}\), where \(\delta_y\) denotes the Dirac measure at the point \(y\), is a probability measure on \(X\) and so is

\[
\mu_N = \frac{1}{N} \sum_{n < N} \delta_{T^n x},
\]

By Theorem 4.8.1, the sequence \((\mu_N)\) has a cluster point \(\mu\) for the weak-star topology. Let \((\mu_{N_i})\) be a the subsequence converging to \(\mu\). For every Borel subset \(U\) of \(X\),

\[
\mu(U) = \lim_{N_i} \frac{1}{N_i} \text{Card}\{n < N_i \mid T^n x \in U\}
= \lim_{N_i} \frac{1}{N_i} \text{Card}\{n < N_i \mid T^{n+1} x \in U\} = \mu(T^{-1} U).
\]

Thus \(\mu\) is an invariant probability measure.

A different proof uses the Markov-Kakutani fixed point Theorem (see Exercise 4.12).

A measure-theoretic dynamical system is a system \((X, T, \mu)\) where \(X\) is a compact metric space, \(\mu\) is a Borel probability measure on \(X\) and \(T : X \to X\) is a measurable map that preserves the measure \(\mu\), that is, such that \(T^{-1}(U)\) is a Borel set with measure \(\mu(T^{-1}(U)) = \mu(U)\) for every Borel set \(U \subset X\).

Thus, Theorem 4.8.1 shows that every topological dynamical system may be considered as a measure-theoretic one. Although this book is devoted to topological dynamical systems, we will have occasions to meet measure-theoretic ones.

A basic result concerning mesure-theoretic dynamical systems is Poincaré Recurrence Theorem, which we will use in Chapter 7.

Theorem 4.8.2 (Poincaré) Let \((X, T, \mu)\) be a measure-theoretic dynamical system with \(\mu\) a finite measure. For every Borel set \(U \subset X\) such that \(\mu(U) > 0\), the set of points \(x \in U\) such that \(T^n x \notin U\) for all \(n \geq 1\) has measure 0.

Proof. Let \(N\) be the set of \(x \in U\) such that \(T^n x \notin U\) for all \(n \geq 1\). It is a Borel set because

\[
N = U \cap \bigcap_{n \geq 1} T^{-n}(X \setminus U)
\]

Then \(T^{-n} N \cap N = \emptyset\) for all \(n \geq 1\), which implies that the sets \(N, T^{-1} N, \ldots\) are disjoint. Therefore

\[
\mu(X) \geq \mu\left( \bigcup_{n \geq 0} T^{-n} N \right) = \sum_{n \geq 0} \mu(T^{-n} N) = \sum_{n \geq 0} \mu(N).
\]

Since \(\mu(X)\) is finite, this implies that \(\mu(N) = 0\).
Let $\mu$ be an invariant Borel probability measure on $(X, T)$. The associated map $\pi : L(X) \to [0, 1]$ such that $\pi(u) = \mu([u]_X)$ satisfies in addition to the compatibility conditions (4.8.2), the symmetric conditions
\[ \pi(u) = \sum_{a \in A} \pi(au). \tag{4.8.3} \]

Indeed, $\sum_{a \in A} \pi(au) = \sum_{a \in A} \mu([au]_X) = \mu(T^{-1}[u]_X) = \mu([u]_X) = \pi(u)$. Conversely, for every map $\pi : L(X) \to [0, 1]$ satisfying the compatibility conditions (4.8.2) and (4.8.3), there is by Carathéodory Theorem (see Appendix B) a unique invariant Borel probability measure $\mu$ such that $\mu([u]_X) = \pi(u)$.

### 4.8.2 Ergodic measures

Recall that, for a topological dynamical system $(X, T)$, a subset $U$ of $X$ is said to be invariant if $T^{-1}U = U$.

An invariant probability measure on $(X, T)$ is ergodic whenever $\mu(U)$ equals 0 or 1 for every invariant Borel subset $U$ of $X$. One also says that the transformation $T$ is ergodic with respect to $\mu$ or that the triple $(X, T, \mu)$ is ergodic.

A real valued measurable function $f$ on $X$ is invariant if $f = f \circ T$. Thus a set $U$ is invariant if and only if its characteristic function is invariant.

For two sets $U, V$ we write $U = V$ mod $\mu$ if $U, V$ differ by sets of measure 0. The following statement gives a useful variant of the definition of an ergodic measure.

**Proposition 4.8.3** The following conditions are equivalent for an invariant Borel probability measure $\mu$ on $(X, T)$.

(i) $\mu$ is ergodic.

(ii) Every Borel set $U$ such that $T^{-1}U = U$ mod $\mu$ is such that $\mu(U) = 0$ or $\mu(U) = 1$.

The proof is left as Exercise 4.13.

Ergodicity is, as we shall see, closely related with minimality. We first note the following relation between ergodicity and recurrence.

**Proposition 4.8.4** Let $(X, T)$ be a topological dynamical system and let $\mu$ be an invariant Borel probability on $(X, T)$. Assume that $\mu(U) > 0$ for every nonempty open set $U$. If $\mu$ is ergodic, then $(X, T)$ is recurrent.

The proof is left as an exercise (Exercise 4.14).

Recall from Appendix B that a Borel probability measure $\nu$ is absolutely continuous with respect to $\mu$, denoted $\nu \ll \mu$, if for every Borel set $U \subset X$ such that $\mu(U) = 0$ one has $\nu(U) = 0$.

The following result shows that the ergodic measures are the minimal invariant measures with respect to the preorder $\ll$. 


Proposition 4.8.5 Let $\mu, \nu$ be invariant Borel probability measures on $(X, T)$. If $\mu$ is ergodic and $\nu \ll \mu$, then $\mu = \nu$.

Proof. Since $\nu \ll \mu$, by the Radon-Nikodym Theorem (see Appendix [B]), there is a nonnegative $\mu$-integrable function $f$ such that $\nu(U) = \int_U f \, d\mu$ for every Borel set $U \subset X$.

Consider the Borel set $B = \{x \in X \mid f(x) > 1\}$. We will prove that $B$ is invariant modulo a set of $\mu$-measure zero. Note first that $\mu(T^{-1}B \setminus B) = \mu(B \setminus T^{-1}B)$ and $\nu(T^{-1}B \setminus B) = \nu(B \setminus T^{-1}B)$ since $\mu, \nu$ are invariant measures. Assume that $\mu(T^{-1}B \setminus B) = \mu(B \setminus T^{-1}B) > 0$. Then we have

$$\mu(B \setminus T^{-1}B) < \int_{B \setminus T^{-1}B} f \, d\mu = \nu(B \setminus T^{-1}B) = \nu(T^{-1}B \setminus B) = \int_{T^{-1}B \setminus B} f \, d\mu \leq \mu(T^{-1}B \setminus B) = \mu(B \setminus T^{-1}B)$$

which is absurd and thus $\mu(T^{-1}B \setminus B) = \mu(B \setminus T^{-1}B) = 0$. This in turn implies that $B = T^{-1}B$ modulo a set of $\mu$-measure zero. By Proposition 4.8.3 since $\mu$ is ergodic, this implies that $\mu(B) = 0$ or 1. If $\mu(B) = 1$, then $\nu(X) > 1$ which is absurd. Thus $\mu(B) = 0$. Since $\nu(X) = \int_X f \, d\mu = 1$, this implies that $f = 1$ almost everywhere or equivalently $\mu = \nu$.

An extreme point of a convex set $K$ is a point which does not belong to any open line segment in $K$. By the Krein-Millman Theorem (see Appendix [B]), the set $\mathcal{M}(X, T)$ of invariant probability measures on $(X, T)$ is the convex hull of its extreme points.

Proposition 4.8.6 The ergodic measures are the extreme points of the set of invariant probability measures.

Proof. Let $\mu$ be an extreme point of $\mathcal{M}(X, T)$. If $\mu$ is not ergodic, there is an invariant Borel set $U$ such that $0 < \mu(U) < 1$. The complement $V$ of $U$ is also invariant and

$$\mu_1(W) = \frac{1}{\mu(U)} \mu(W \cap U), \quad \mu_2(W) = \frac{1}{\mu(U)} \mu(W \cap V)$$

are both invariant probability measures. But then

$$\mu = \mu(U) \mu_1 + (1 - \mu(U)) \mu_2$$

shows that $\mu$ is not an extreme point.

Let now $\mu$ be ergodic. Then $\mu = \alpha \mu_1 + (1 - \alpha) \mu_2$ for $0 \leq \alpha \leq 1$ and some extreme points $\mu_1, \mu_2 \in \mathcal{M}(X, T)$. Assume $\alpha > 0$. For any Borel set $U \subset X$, one has that $\mu(U) = 0$ implies $\mu_1(U) = 0$, that is $\mu_1$ is absolutely continuous with respect to $\mu$. By Proposition 4.8.5 this implies that $\mu = \mu_1$. Thus $\mu$ is an extreme point.

In particular, we have the following important case.

Corollary 4.8.7 If there is a unique invariant probability measure, it is ergodic.
4.8. INVARIANT PROBABILITY MEASURES

4.8.3 Unique ergodicity

In view of Corollary 4.8.7, a system with a unique invariant probability measure is called uniquely ergodic. It is easy to give examples of a system which is not uniquely ergodic when it is not minimal. For example, \( \{0^\infty\} \cup \{1^\infty\} \) has clearly two ergodic measures. An example of a minimal system which is not uniquely ergodic is given in Exercise 4.16.

Theorem 4.8.8 (Oxtoby) Let \((X,T)\) be a topological dynamical system and \(\mu\) be an invariant probability measure on \((X,T)\). The following conditions are equivalent.

(i) \((X,T)\) is uniquely ergodic.

(ii) \(f_n(x) = \frac{1}{n}f^{(n)}(x)\) converges uniformly on \(X\) to \(\int f \, d\mu\) for every \(f \in C(X,\mathbb{R})\).

(iii) \(f_n(x) = \frac{1}{n}f^{(n)}(x)\) converges pointwise to \(\int f \, d\mu\) for every \(f \in C(X,\mathbb{R})\).

Proof. (i)\(\Rightarrow\)(ii). Suppose that (ii) does not hold. We can find \(\varepsilon > 0\), a map \(g \in C(X,\mathbb{R})\) and a sequence \((x_n)\) of points of \(X\) such that

\[ |g_n(x_n) - \int g \, d\mu| > \varepsilon. \]

For every \(n \geq 0\), the sum \(\delta_{x,n} = \frac{1}{n}\delta^{(n)}\), where \(\delta_x\) is the Dirac function, is a Borel probability measure and such that

\[ \int f \, d\delta_{x,n} = f_n(x) \]

for every \(f \in C(X,\mathbb{R})\) and \(x \in X\). Let \(\nu\) be a cluster point of the sequence \(\delta_{x,n}\) for the weak-star topology. Refining \((x_n)\) if necessary, we have

\[ \lim f_n(x_n) = \int f \, d\nu \]

for every \(f \in C(X,\mathbb{R})\). Hence

\[ |\int g \, d\mu - \int g \, d\nu| > \varepsilon \]

showing that \(\mu, \nu\) are distinct invariant measures. Thus (i) does not hold.

(ii)\(\Rightarrow\)(iii). We have

\[ \int f_n \, d\mu = \frac{1}{n} \sum_{k<n} \int f(T^k x) \, d\mu = \frac{1}{n} \sum_{k<n} \int f \, d\mu = \int f \, d\mu. \]

Thus \(\lim f_n(x) = \int f \, d\mu\).
(iii)⇒ (i). For any invariant probability measure \( \nu \), we have
\[
\int f_n \, d\nu = \frac{1}{n} \sum_{i<n} \int f \circ T^i \, d\nu = \int f \, d\nu.
\] (4.8.4)

On the other hand, by the Dominated Convergence Theorem, we have
\[
\lim \int f_n \, d\nu = \int f \, d\mu
\] (4.8.5)

Thus, by Equations (4.8.4) and (4.8.5), we conclude that \( \mu = \nu \).

Theorem 4.8.8 is a refinement for uniquely ergodic probability measures of Birkhoff’s ergodic Theorem. This theorem asserts that, given an ergodic measure \( \mu \), for every integrable function \( f \) on \( X \), the sequence \( f_n \) converges almost everywhere to \( \int f \, d\mu \). Thus, in the uniquely ergodic case and for a continuous function \( f \), the convergence is everywhere. The functions \( f_n \) are sometimes called the Birkhoff averages.

4.8.4 Unique ergodicity of primitive substitution shifts

In the case of substitution shifts, we have the following important result.

**Theorem 4.8.9** Primitive substitution shifts are uniquely ergodic.

For two word \( u, v \), we denote \( |u|_v \) the number of factors of \( u \) equal to \( v \). By convention, \( |u|_\varepsilon = |u| \). We say that an infinite word \( x \in A^\mathbb{Z} \) has uniform frequencies if for every factor \( v \) of \( x \), all sequences
\[
(f_{k,v}(x))_n = \frac{|x_k \cdots x_{k+n-1}|_v}{n}
\]
tend to the same limit \( f_v(x) \) when \( n \to \infty \), uniformly in \( k \).

Note that \( x \) has uniform frequencies if and only if for every \( v \in \mathcal{L}(x) \) there is an \( f_v \) such that for every \( \varepsilon \) there is an \( N \geq 1 \) such that for every \( u \in \mathcal{L}(x) \) of length at least \( N \) we have \( |u|_v / |u| - f_v| \leq \varepsilon \). We express this property by saying that the frequency of \( v \) in \( u \) tends to \( f_v \) when \( |u| \to \infty \) uniformly in \( u \).

**Proposition 4.8.10** The following conditions are equivalent for a minimal shift space \( (X, S) \).

(i) Every \( x \in X \) has uniform frequencies.

(ii) Some \( x \in X \) has uniform frequencies.

(iii) \( (X, S) \) is uniquely ergodic.

Moreover, in this case the unique invariant measure is given by \( \mu(|v|) = f_v(x) \) for every \( x \in X \).
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Proof. (i) implies (ii) is clear. Assume (ii). For \( v \in \mathcal{L}(X) \), set \( \pi(v) = f_v(x) \). This map satisfies clearly the compatibility conditions (4.8.2) and (4.8.3). Thus there is a unique invariant Borel probability measure \( \mu \) such that \( \mu([v]) = \pi(v) \). Set \( g_{k,v} = \chi[v] \circ S^k \). Since \( |x_k \ldots x_{k+n}|_v = \sum_{j<n-|v|} \chi[v](S^{j+k}x) = (g_{k,v})^{(n-|v|)}(x) \), we have

\[
(g_{k,v})^{(n)} \to \mu([v]) = \int \chi[d\mu]
\]

uniformly in \( k \). But the family of linear combinations all \( g_{k,v} \) is dense in \( C(X, \mathbb{R}) \).

Hence \( \frac{1}{n}g^{(n)} \to \int g \, d\mu \) for every continuous function \( g \). By Oxtoby Theorem 4.8.8, this implies that \( (X, S) \) is uniquely ergodic.

Finally, by Oxtoby Theorem again, (iii) implies (i).

Let \( \varphi : A \to A^* \) be a primitive substitution. Let \( M \) be the incidence matrix of \( \varphi \). Since \( M \) is a primitive matrix, by Perron-Frobenius Theorem (Theorem 2.4.14), the matrix \( M \) has a dominant eigenvalue \( \lambda_M \) and positive left and right eigenvectors \( x = (x_a) \), \( y = (y_b) \) relative to \( \lambda_M \). We may assume that \( \sum_{a \in A} x_a = 1 \) and that \( xy = \sum_{a \in A} x_a y_a = 1 \). With this notation, we prove two lemmas.

**Lemma 4.8.11** For every \( a, b \in A \), the sequence

\[
\frac{|\varphi^n(a)|_b}{|\varphi^n(a)|}
\]

converges at geometric rate to \( x_b \).

**Proof.** We have

\[
\frac{|\varphi^n(a)|_b}{|\varphi^n(a)|} = \frac{M^n_{a,b}}{\sum_{a \in A} M^n_{a,b}} \to \frac{y_a x_b}{\sum_{a \in A} y_a x_b} = x_b
\]

and the result is proved since the convergence is at geometric rate by Theorem 2.4.14 (iii).

We will now extend Lemma 4.8.11 to arbitrary words \( u, v \in \mathcal{L}(X) \). For this, set \( \ell = |v| \). We consider the alphabet \( A_\ell \) in one-to-one correspondence with \( \mathcal{L}_\ell(X) \) via \( f : \mathcal{L}_\ell(X) \to A_\ell \) and the \( \ell \)-block presentation \( \varphi_\ell \) of \( \varphi \) (see Section 2.4). Recall that, by Proposition 2.4.34, \( \varphi_\ell \) is primitive.

Let \( M_\ell \) be the incidence matrix of \( \varphi_\ell \). Recall that \( M_\ell \) has dominant eigenvalue \( \lambda_M \) (Proposition 2.4.35). Let \( x^{(\ell)} \) be the left eigenvector of \( M_\ell \) relative to \( \lambda_M \) with coefficients of sum 1.

**Lemma 4.8.12** For every \( u, v \in \mathcal{L}(X) \), the sequence

\[
\frac{|\varphi^n(u)|_b}{|\varphi^n(u)|}
\]

converges at geometric rate to \( \pi(v) = x^{(\ell)}_b \) where \( b = f(v) \).
Proof. Suppose first that \( u = a \in A \). Let \( c \in A_\ell \) be such that \( w = f^{-1}(c) \) begins with \( a \). We have by \( \text{(4.8.10)} \) \(|\varphi^n(a) = |\varphi^n(c)|\) for all \( n \geq 1 \). We extend \( f \) naturally to a map from \( L_{\geq \ell}(X) \) to \( L(X^{(\ell)}) \). By Proposition \( \text{(4.8.33)} \), \( \varphi^n(c) \) is the prefix of length \(|\varphi^n(a)|\) of \( f(\varphi^n(w)) \). Thus there is a word \( v \) of length \( \ell - 1 \) such that \( \varphi^n(\ell) = f(\varphi^n(a)r) \). This shows that \(|\varphi^n(a)|_v \) and \(|\varphi^n(c)|_b \) with \( b = f(v) \) differ by a constant. Thus

\[
\frac{|\varphi^n(a)|_v}{|\varphi^n(a)|} \sim \frac{|\varphi^n(c)|_b}{|\varphi^n(c)|}.
\]

By Lemma \( \text{4.8.11} \) the right hand side tends at geometric rate to \( x_b^{(\ell)} \), whence the conclusion in this case.

Consider finally arbitrary \( u, v \in L(X) \). We use induction on \(|u|\). The result is true for \(|u| = 1 \) by the previous case. Consider now \( u = u'a \) with \( u' \in L(X) \) and \( a \in A \). By induction hypothesis, we have \(|\varphi^n(u')|_v - \pi(v)|\varphi^n(a)| \leq cr^n|\varphi^n(u')| \) and \(|\varphi^n(a)|_v - \pi(v)|\varphi^n(a)| \leq cr^n|\varphi^n(a)| \) for some \( c > 0 \) and \( r < 1 \).

Every occurrence of \( v \) in \( \varphi^n(u) \) is either an occurrence in \( \varphi^n(u') \) or in \( \varphi^n(a) \) except \( \theta \leq |v| \) occurrences which begin in \( \varphi^n(a) \) and end in \( \varphi^n(a) \). Thus

\[
|\varphi^n(u)|_v = |\varphi^n(u')|_v + \theta + |\varphi^n(a)|_v
\]

with \( \theta \leq |v| \). This implies

\[
|\varphi^n(u)|_v - \pi(v)|\varphi^n(u)| \leq |\varphi^n(u')|_v - \pi(v)|\varphi^n(u')| + \theta + |\varphi^n(a)|_v - \pi(v)|\varphi^n(a)| \leq cr^n|\varphi^n(u')| + cr^n|\varphi^n(a)| + |v| \leq cr^n|\varphi^n(u)| + |v|
\]

and thus the conclusion since \(|v|/|\varphi^n(u)|\) tends to 0 at geometric rate.

Recall from Chapter 2 that, for a morphism \( \varphi : A \rightarrow A^* \), we denote \(|\varphi| = \max_{a \in A}|\varphi(a)|\). The following lemma is interesting in itself.

**Lemma 4.8.13** For every nonempty word \( u \in L(X) \), there is some \( m \geq 0 \) and words \( v_i, w_i \in L(X) \) for \( 0 \leq i \leq m \), with \( v_m \) nonempty such that

\[
\text{(4.8.6)} \quad u = v_0\varphi(v_1)\cdots\varphi^{m-1}(v_{m-1})\varphi^m(v_m)\varphi^{m-1}(w_{m-1})\cdots\varphi(w_1)w_0,
\]

with \(|v_i|, |w_i| \leq |\varphi|\).

**Proof.** We use induction on \(|u|\). The result is true if \(|u| < |\varphi| \) choosing \( m = 0 \) and \( v_0 = u \). Otherwise, by definition of \( L(X) \), there exists a nonempty word \( u' \in L(X) \) such that \( u = v_0\varphi(u')w_0 \). Choosing \( u' \) of maximal length, we have moreover \(|v_0|, |w_0| \leq |\varphi| \). By induction hypothesis, we have a decomposition \( \text{(4.8.30)} \) for \( u' \), that is

\[
u' = v_0'\varphi(v'_1)\cdots\varphi^{m-1}(v'_{m-1})\varphi^m(v'_m)\varphi^{m-1}(w'_{m-1})\cdots\varphi(w'_1)w'_0.
\]
In this way, we obtain
\[ u = v_0 \phi(u') w_0 = v_0 \phi(v_0') \cdots \phi^m(v_{m-1}') \phi^{m+1}(v_m') \phi^m(w_{m-1}') \cdots \phi(w_0') w_0. \]
which is of the form (4.8.6).

**Proof of Theorem 4.8.9.** Let \( x \in X \). By Proposition 4.8.10 it is enough to prove that \( x \) has uniform frequencies. We will prove that \( |u_k \cdots u_{k+n-1}|_v/n \) converges to \( \pi(v) \) uniformly in \( k \), where \( \pi(v) \) is as in Lemma 4.8.12. Set \( u = u_k \cdots u_{k+n-1} \). By Lemma 4.8.13, we have
\[ |u|_v = \sum_{i \leq m} |\phi^i(v_i)|_v + \sum_{i < m} |\phi^i(w_i)|_v + \theta \]
where \( \theta \) is the number of occurrences of \( v \) which overlap more than one \( v_i, w_i \). Thus \( \theta \leq 2m|v| \). By Lemma 4.8.12 there are \( c > 0 \) and \( r < 1 \) such that
\[ ||\phi^i(v_i)|_v - \pi(v)|\phi^i(v_i)|| \leq cr^i \]
(4.8.7) for \( 0 \leq i \leq m \) (and the analogue inequality for the \( w_i \)). Since
\[ n = \sum_{i \leq m} |\phi^i(v_i)| + \sum_{i < m} |\phi^i(w_i)| \]
we obtain
\[ ||u|_v - \pi(v)n| = ||u|_v - \pi(v)(\sum_{i \leq m} |\phi^i(v_i)| + \sum_{i < m} |\phi^i(w_i)|)|| \]
\[ \leq \sum_{i \leq m} ||\phi^i(v_i)|_v - \pi(v)|\phi^i(v_i)|| \]
\[ + \sum_{i < m} ||\phi^i(w_i)|_v - \pi(v)|\phi^i(w_i)|| + 2m|v| \]
\[ \leq 2c \sum_{i \leq m} r^i + 2m|v| \]
\[ \leq dr^m + 2m|v|. \]
But \( m/n \) tends to 0 when \( n \) tends to infinity. Indeed, by (2.4.6), there is a constant \( e > 0 \) such that \( |\phi^n(u)| \geq e|u|\lambda_M^n \) for \( n \) large enough. This implies that \( n \geq |\phi^m(v_m)| \geq e|v_m|\lambda_M^m \). Hence \( m/n \leq m/e\lambda_M^m |v_m| \) which tends to 0.
This shows that \( |u|_v/n \) converges to \( \pi(v) \) when \( n \) tends to infinity independently of \( u \), which concludes the proof.

### 4.8.5 Computation of the unique invariant probability measure

The computation of the unique invariant probability measure can be done using the matrices \( M_\ell \) of \( \ell \)-block presentations of the substitution, as we shall see below.
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Let $\varphi : A \to A^*$ be a primitive substitution and let $(X, S)$ be the associated shift space. Let $\mu$ be the unique invariant probability measure on $(X, S)$. For $\ell \geq 1$, let $A_\ell$ be an alphabet in bijection with $L_\ell(X)$ via $f : L_\ell(X) \to A_\ell$ and let $x^{(\ell)}$ be the unique positive row eigenvector of the matrix $M_\ell$ corresponding to the maximal eigenvalue $\lambda$ and such that the sum of its components is 1.

**Proposition 4.8.14** For every $v \in L_\ell(X)$,

$$\mu([v]) = x^{(\ell)}_c$$

with $c = f(v)$.

**Proof.** By Theorem 4.8.10, $\mu([v])$ is equal to the frequency $f_v(x)$ of the word $v$ in any $x \in X$. By Lemma 4.8.12, the frequency of $v$ in $\varphi^n(a)$ tends to $x^{(\ell)}_c$ when $n \to \infty$ and thus we have $f_v(x) = x^{(\ell)}_c$.

We develop below the cases of the Fibonacci and of the Morse substitutions. For the Fibonacci shift, we already now that it is uniquely ergodic since this is true of every Sturmian shift (Exercise 4.15).

**Example 4.8.15** Let $(X, S)$ be the Fibonacci shift. Since the Fibonacci substitution is primitive, there is a unique invariant probability measure on $(X, S)$. Its values on the cylinder $[w]$ defined by words $w$ of length at most 4 are shown on Figure 4.8.1 with $\rho = (\sqrt{5} - 1)/2$.

This is consistent with the value of the eigenvector of $M_2$

$$v^{(2)} = \begin{bmatrix} 2\rho - 1 & 1 - \rho & 1 - \rho \end{bmatrix}$$

Note that the Fibonacci shift is Sturmian of slope $\alpha = (3 - \sqrt{5})/2$. The unique invariant probability measure $\mu$ on the Fibonacci shift can also be computed using the natural representation $\gamma_\alpha$ (Exercise 4.15) giving $\mu([a]) = 1 - \alpha$. Consistently with the above, $\rho = 1 - \alpha$.

**Example 4.8.16** Consider the Morse substitution. The matrix $M_2$ is

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

The unique invariant probability measure on the Morse shift is shown in Figure 4.8.2. The values on the words of length 2 is consistent with the value of the eigenvector of $M_2$ which is

$$v^{(2)} = \begin{bmatrix} 1/6 & 1/3 & 1/3 & 1/6 \end{bmatrix}$$
4.9 Invariant measures and states

We now come to an essential point and relate invariant measures and coboundaries.

**Proposition 4.9.1** Let \( \mu \) be an invariant measure on the dynamical system \((X, T)\). For every coboundary \( f \in \partial_T C(X, \mathbb{R}) \), one has \( \int f d\mu = 0 \).

**Proof.** When \( \mu \) is a \( T \)-invariant probability measure we have \( \mu \circ T^{-1} = \mu \). Thus, for \( f = \partial_T g \), by the change of variable formula (see Appendix B), \( \int \partial_T g d\mu = \int g \circ T d\mu - \int g d\mu = 0 \).

Proposition 4.9.1 implies that for all \( T \)-invariant probability measures \( \mu \) on \((X, T)\) the map \( f \mapsto \int f d\mu \) defines a group homomorphism

\[ \alpha_\mu : H(X, T, \mathbb{Z}) \to \mathbb{R} \]

**Proposition 4.9.2** The map \( \alpha_\mu \) is a morphism of unital ordered groups from \( K^0(X, T) \) to \((\mathbb{R}, \mathbb{R}_+, 1)\).

**Proof.** By definition of \( \int f d\mu \), we have \( \alpha_\mu(H(X, T, \mathbb{Z}_+)) \subset \mathbb{R}_+ \). Moreover, \( \alpha_\mu(1_X) = \mu(X) = 1 \) since \( \mu \) is a probability measure.

The proof of the following statement uses the Carathéodory extension Theorem (see Appendix B).

**Theorem 4.9.3** Let \((X, T)\) be a minimal Cantor system. The map \( \mu \mapsto \alpha_\mu \) is a bijection from the space of \( T \)-invariant probability measures on \((X, T)\) to the set of states of \( K^0(X, T) \).
Proof. Let $\alpha$ be a state on $K^0(X,T)$. For a clopen set $U \subset X$, we set $\phi(U) = \alpha(\chi_U)$ where $\chi_U$ is the image in $H(X,T,\mathbb{Z})$ of the characteristic function of $U$. Thus $\phi(U) \geq 0$ for every clopen set $U$, and $\phi(U \cup V) = \phi(U) + \phi(V)$ if $U,V$ are disjoint clopen sets. Since $X$ is a Cantor space, its topology is generated by the clopen sets. Thus, there is, by the Carathéodory Theorem, a unique probability measure $\mu$ on $X$ such that $\mu(U) = \phi(U)$ for every clopen set $U$. This already shows that $\mu \mapsto \alpha_\mu$ is one-to-one. For every clopen set $U$, the difference between the characteristic functions of $U$ and $T^{-1}U$ is a coboundary. Thus these functions have the same image in $H(X,T,\mathbb{Z})$ and $\phi(U) = \phi(T^{-1}U)$. It follows that $\mu$ is $T$-invariant. Moreover, by construction $\alpha = \alpha_\mu$ since $\alpha_\mu(\chi_U) = \int \chi_U d\mu = \mu(U) = \alpha(\chi_U)$ for every clopen set $U$. This shows that the map $\mu \mapsto \alpha_\mu$ is onto. 

Observe that Theorem 4.9.3 allows us to give a very simple proof of Theorem 4.8.9. Indeed, if $(X,S)$ is a primitive substitution shift, then $K^0(X,S)$ is the limit of a stationary system defined by a primitive matrix and thus has a unique state by Proposition 3.5.1. We conclude by Theorem 4.9.3 that $(X,S)$ is uniquely ergodic.
4.9. INARIANT MEASURES AND STATES

4.9.1 Dimension groups of Sturmian shifts

As an illustration of Theorem 4.9.3, let us prove the following statement, which gives the form of the cohomology group of Sturmian shifts. In the next statement the group $\mathbb{Z} + \alpha \mathbb{Z}$ of real numbers of the form $x + \alpha y$ is considered as a unital ordered group for the order induced by $\mathbb{R}$ and the order unit 1.

**Theorem 4.9.4** Let $(X, S)$ be a Sturmian shift of slope $\alpha$. Then $K^0(X, T) = \mathbb{Z} + \alpha \mathbb{Z}$.

**Proof.** We have already seen (Example 4.3.3) that $H(X, S, \mathbb{Z}) = \mathbb{Z}^2$ for every Sturmian shift. Let $\{0, 1\}$ be the alphabet of $X$. We can identify the group $H(X, S, \mathbb{Z})$ with the pairs $(x, y) \in \mathbb{Z}^2$ via the map $(x, y) \mapsto x[0] + y[1]$. We have seen (Exercise 4.15) that $(X, S)$ is uniquely ergodic and that the unique invariant probability measure $\mu$ is such that $\mu([0]) = 1 - \alpha$. By Proposition 4.9.3, the ordered group $K^0(X, S)$ has a unique state which is $\alpha \mu$. Thus, by Proposition 4.9.3, $H^+(X, S, \mathbb{Z}) = \{ (x, y) \in \mathbb{Z}^2 \mid (1 - \alpha)x + \alpha y > 0 \} \cup \{0\}$, the order unit being $(1, 1)$. Thus, the map $(x, y) \mapsto (x, y - x)$ identifies $K^0(X, T)$ and $\mathbb{Z} + \alpha \mathbb{Z}$.

**Example 4.9.5** Let $(X, S)$ be the Fibonacci shift. It is a Sturmian shift of slope $\alpha = (3 - \sqrt{5})/2$ (Example 2.5.7). Thus $K^0(X, T) = \mathbb{Z}[\alpha]$.

As an application of Theorem 4.9.4, we prove the following result.

**Theorem 4.9.6** Two Sturmian shifts are never conjugate unless they differ by a permutation of the two letters.

**Proof.** Let $(X, S)$ and $(Y, S)$ be Sturmian shifts of slopes $\alpha, \beta$ respectively. If they are conjugate, their cohomology groups are isomorphic and thus, by Theorem 4.9.4, the groups $\mathbb{Z} + \alpha \mathbb{Z}$ and $\mathbb{Z} + \beta \mathbb{Z}$ are isomorphic. Thus there are $a, b, c, d \in \mathbb{Z}$ such that $\beta = a + b\alpha$ and $\alpha = c + d\beta$. Then $\alpha = c + ad + bd\alpha$ implies $bd = 1$. Thus either $\alpha = \beta$ or $\alpha = 1 - \beta$.

This statement is a striking illustration of the power of the cohomology group. It would probably be very difficult to prove directly the above statement by a direct argument.

We can also use Proposition 4.9.3 to give the form of the image subgroup and the infinitesimal subgroup for a minimal Cantor system $(X, T)$.

The image subgroup of $K^0(X, T)$, denoted $I(X, T)$, is given by

$$I(X, T) = \cap_{\mu \in \mathcal{M}(X, T)} \left\{ \int f d\mu \mid f \in C(X, \mathbb{Z}) \right\}$$  (4.9.1)

where $\mathcal{M}(X, T)$ denotes the set of invariant probability measures on $(X, T)$. This follows directly from the definition of the image subgroup given in Equation 3.2.3 by Proposition 4.9.3.
The infinitesimal subgroup of $K^0(X,T)$, denoted $\text{Inf}(X,T)$, is given by

$$\text{Inf}(X,T) = \{ [f] \in H(X,T,\mathbb{Z}) \mid \int f d\mu = 0 \text{ for all } \mu \in \mathcal{M}(X,T) \}. \quad (4.9.2)$$

If $(X,T)$ is uniquely ergodic, we have $K^0(X,T)/\text{Inf}(K^0(X,T)) = I(X,T)$.

**Example 4.9.7** Let $X$ be the Fibonacci shift as in the previous example. Then $\text{Inf}(X,T) = \{0\}$ and $I(X,T) = \mathbb{Z} + \alpha\mathbb{Z}$.

### 4.10 Exercises

**Section 4.1**

4.1 Let $(X,T)$ be a topological dynamical system. For $f \in C(X,\mathbb{R})$ extend the definition of $f^{(n)}(x)$ to negative indexes by defining for $n \geq 1$ and $x \in X$

$$f^{(-n)}(x) = -f^{(n)}(T^{-n}x). \quad (4.10.1)$$

Show that, with this definition, the **cohomological equation**

$$f^{(n+m)}(x) = f^{(n)}(x) + f^{(m)}(T^n x) \quad (4.10.2)$$

holds for every $n \in \mathbb{Z}$ and $x \in X$.

4.2 A **homotopy** between topological spaces $X,Y$ is a family $f_t : X \to Y$ ($t \in [0,1]$) of maps such that the associated map $F : X \times [0,1] \to Y$ given by $F(x,t) = f_t(x)$ is continuous. One says that $f_0, f_1 : X \to Y$ are **homotopic** if there is a homotopy $f_t$ connecting them. Show that homotopy is an equivalence relation compatible with composition.

4.3 A continuous function is **nullhomotopic** if it is homotopy equivalent to a constant function. Show that any function $f : X \to [a,b]$ from $X$ to a closed interval $[a,b]$ of $\mathbb{R}$ is nullhomotopic.

4.4 A **continuous flow** is a pair $(X,(T_t)_{t \in \mathbb{R}})$ of a compact metric space $X$ and a family $(T_t)_{t \in \mathbb{R}}$ of homeomorphisms $T_t : X \to X$ such that

(i) the map $(x,t) \mapsto T_t(x)$ is continuous from $X \times \mathbb{R}$ to $X$.

(ii) $T_{t+s} = T_t \circ T_s$ for all $s,t \in \mathbb{R}$.

For every topological dynamical system $(X,T)$, one can build a continuous flow called the **suspension flow** over $(X,T)$ as follows. Consider the quotient $\tilde{X}$ of $X \times \mathbb{R}$ by the equivalence relation which identifies $(x,s+1)$ and $(Tx,s)$ for all $x \in X$ and $s \in \mathbb{R}$. Denote by $[(x,s)]$ the equivalence class of $(x,s)$. Then define $T_t$ on the quotient by

$$T_t([(x,s)]) = [(x,s+t)].$$

Show that we obtain in this way a continuous flow.
4.5 Show that the suspension flow of any periodic system can be identified with the torus \( T = \mathbb{R}/\mathbb{Z} \).

4.6 An equivalence between two continuous flows \((X, (T_t)_{t \in \mathbb{R}})\) and \((X', (T'_t)_{t \in \mathbb{R}})\) is a homeomorphism \( \pi : X \to X' \) which maps orbits of \( T_t \) to orbits of \( T'_t \) in an orientation preserving way, that is, for all \( x \in X \), \( \pi(T_t(x)) = T'_t(\pi(x)) \) for some monotonically increasing \( f_x : \mathbb{R} \to \mathbb{R} \). Two flows are equivalent if there is an equivalence from one to the other.

Two (ordinary) topological dynamical systems are flow equivalent if their suspension flows are flow equivalent. Show that two equivalent dynamical systems are flow equivalent but that the converse is false.

4.7 Let \((X, T)\) be a topological dynamical system and let \((\tilde{X}, (T_t)_{t \in \mathbb{R}})\) be its suspension flow (as defined in Exercise 4.4). The first Čech cohomology group of \( \tilde{X} \), denoted \( H^1(\tilde{X}, \mathbb{Z}) \), is the group of continuous maps from \( \tilde{X} \) to the torus \( T = \mathbb{R}/\mathbb{Z} \) modulo the group of nullhomotopic maps.

Show that \( H^1(\tilde{X}, \mathbb{Z}) \) is isomorphic to \( H(X, T, \mathbb{Z}) \) (hint: Consider the map \( \pi : C(X, \mathbb{Z}) \to C(\tilde{X}, \mathbb{Z}) \) which associates to \( f \in C(X, \mathbb{Z}) \) the map \( \pi(f) \in C(\tilde{X}, \mathbb{Z}) \) defined by \( \pi(f)(x, s) = f(T(x)s) \). Show that \( \pi \) induces an isomorphism from \( H(X, T, \mathbb{Z}) \) onto \( H^1(\tilde{X}, \mathbb{Z}) \).

Section 4.2

4.8 Show that if \((X, T)\) is a transitive system, then for any \( f \in C(X, \mathbb{R}) \) two solutions of the equation \( \partial g = f \) differ by a constant.

4.9 Let \((X, T)\) be a topological dynamical system. For \( f \in C(X, \mathbb{R}) \), assume that the sequence \((f^{(n)}(x))_{n \geq 0}\) is uniformly bounded. Set

\[
g(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x)
\]

where \( f^{(n)} \) is defined for all \( n \in \mathbb{Z} \) as in Exercise 4.1. Show that \( \partial g = f \).

4.10 Let \((X, T)\) be a minimal system. Let \( f \in C(X, \mathbb{R}) \) and \( x_0 \in X \) be such that \((f^{(n)}(x_0))_{n \geq 0}\) is bounded. Show that \( f^{(n)}(x) \) is bounded for all \( x \in X \) and that \( g(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x) \) is a continuous function such that \( \partial g = f \) (hint: use Exercise 4.10).

4.11 Let \((X, T)\) be a minimal dynamical system and \( f \in C(X, \mathbb{R}) \) be such that \((f^{(n)}(x_0))_{n \geq 0}\) is bounded for some \( x_0 \in X \).

Let \((Y, S)\) be the dynamical system formed of \( Y = X \times \mathbb{R} \) with \( S(x, t) = (Tx, t + f(x)) \). In this way, \( S^n(x, t) = (T^n x, t + f^{(n)}(x)) \) for all \( n \geq 0 \).

1. Show that the closure \( E \) of the set \( \{S^n(x_0, 0) \mid n \geq 0\} \) is compact and \( S \)-invariant.
2. Let $K$ be a minimal closed $S$-invariant nonempty subset of $E$. Show that $K = \{(x, g(x)) \mid x \in X\}$ for some $g \in C(X, \mathbb{R})$.

3. Show that $\partial g = f$.

Section 4.8

4.12 The Markov-Kakutani fixed point theorem states that if $V$ is a topological vector space and $K$ is a convex and compact subset of $V$, then every continuous linear map $T$ mapping $K$ into itself has a fixed point in $K$.

Use this theorem to give a proof of Theorem 4.8.1. Hint: Consider the map $T^*: \mathcal{M}_X \to \mathcal{M}_X$ defined by $T^*\mu = \mu \circ T^{-1}$.

4.13 Prove Proposition 4.8.3.

4.14 Prove Proposition 4.8.4 (hint: prove that the set of recurrent points has measure 1).

4.15 Use Oxtoby’s Theorem (Theorem 4.8.8) to prove that irrational rotations are uniquely ergodic. Conclude that Sturmian shifts are uniquely ergodic and that the unique invariant probability measure $\mu$ on a Sturmian shift of slope $\alpha$ satisfies $\mu([0]) = 1 - \alpha$. Hint: use the fact that trigonometric polynomials are dense in the space $C(S^1, \mathbb{C})$.

4.16 Let $(k_i)_{i \geq 0}$ be a sequence of positive integers such that $k_i$ divides $k_{i+1}$. Let $x \in \{0, 1\}^\mathbb{Z}$ be the Toeplitz sequence with periodic structure $(k_i)_{i \geq 0}$ defined as follows. For $i \geq 1$, let

$$E_i = \cup_{m \in \mathbb{Z}} \{n \in \mathbb{Z} \mid |n - mk_i| \leq k_{i-1}\}$$

For $n \in \mathbb{Z}$, let $p(n)$ be the least integer $p$ such that $n \in E_p$. Set $x_n \equiv p(n) \mod 2$. Show that if

$$\sum_{i \geq 1} \frac{k_{i-1}}{k_i} \leq \frac{1}{12} \quad (4.10.3)$$

then $x$ is not uniquely ergodic.

4.17 Show that if $(X, T)$ is a topological dynamical system and $\mu$ is an invariant Borel probability measure, then $(X, T, \mu)$ is a measure-theoretic dynamical system.

4.18 Let $(X, T, \mu)$ be a measure-theoretic dynamical system with $T$ invertible. Let $H = L^2(X)$ be the Hilbert space of real valued square integrable functions on $X$ (modulo a.e. vanishing functions).

Show that the operator $U$ defined by $Uf = f \circ T$ is a unitary operator from $H$ to itself.
4.19 Let $(X,T,\mu)$ be a measure-theoretic dynamical system. Show that an element $f$ of the Hilbert space $H = L^2(X)$ is a coboundary of the form $f = Ug - g$ for some $g \in H$, with $U$ as in Exercise 4.18, if and only if $\|f^{(n)}\|$ is bounded (hint: Assume $\|f^{(n)}\| \leq k$ and consider the closure of the set of convex linear combinations of the $U^nf$. Apply the Schauder-Tychonov fixed point theorem to the map $h \mapsto f + Uh$).

4.20 Let $\phi : A^* \to A^*$ be a primitive morphism and let $(X,S)$ be the associated shift space. We indicate here a method to compute the frequency of the factors of length $k$ in $L(X)$ by a faster method than by using Formula (4.8.8).

Let $M_k$ be the incidence matrix of $\phi_k$. Let $p$ be an integer such that $|\phi^p(a)| > k - 2$ for all $a \in A$. Let $U$ be the $L_2(X) \times L_k(X)$–matrix defined as follows. For $a,b \in A$ such that $ab \in L_2(X)$ and $y \in L_k(X)$, $U_{ab,y}$ is the number of occurrences of $y$ in $\phi^p(ab)$ that begin inside the prefix $\phi^p(a)$. Show that $UM_k = M_2U$,

and that if $v_2$ is a row eigenvector of $M_2$ corresponding to the common dominant eigenvalue $\rho$ of $M_2$ and $M_k$, then $v_k = v_2U$ is an eigenvector of $M_k$ corresponding to $\rho$.

4.21 Let $\mu : a \to ab, b \to ba$ be the morphism with fixpoint the Thue-Morse word. Show that for $k = 5, p = 3$, the matrix $U$ of the previous problem (with the 12 factors of length 5 of the Thue-Morse word listed in alphabetical order) is

$$U = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}$$

and that the vector $v_2U$ with $v_2 = [1 \ 2 \ 2 \ 1]$ is the vector with all components equal to 4. Deduce that the 12 factors of length 5 of the Thue-Morse word have the same frequency (see Example 4.8.16).

4.22 Let $k, p$ be as in the previous exercise. Let $V$ be the matrix of the map $\pi : L_k(X) \to L_2(X)$ which sends $a_0a_1\cdots a_{k-1}$ to $a_0a_1$. Show that $M_2$ is shift equivalent over $\mathbb{Z}$ to $M_k$ (see Exercise 3.12) and more precisely that $(U,V) : M_2 \sim M_k$ (lag $k$). Conclude that $M_2$ and $M_k$ have the same nonzero eigenvalues. Hint: use Exercise 3.13.

4.11 Solutions

Section 4.1

4.1 We have, extending the composition with $T$ to rational fractions in $T$,

$$f^{(n)} = f \circ \frac{1 - T^n}{1 - T}.$$
This holds also for $n < 0$ since
\[
\frac{1 - T^{-n}}{1 - T} = -\frac{1 - T^n}{1 - T} = -f^{(n)}(T^{-n}).
\]

The cohomological equation is then easy to verify since
\[
f^{(n)} + f^{(m)} \circ T^n = f \circ \frac{1 - T^n}{1 - T} + f \circ \frac{1 - T^m}{1 - T} \circ T^n
\[
= f \circ \frac{1 - T^n + T^n - T^{n+m}}{1 - T} = f \circ \frac{1 - T^{n+m}}{1 - T}
\]
\[
= f^{(n+m)}.
\]

4.2 We have to verify the transitivity. Assume that $f_0, f_1$ are connected by $f_t$ and that $g_0 = f_1, g_1$ are connected by $g_t$. Then $f_0, g_1$ are connected by $h_t$ where
\[
h_t(x) = \begin{cases} f_{2t}(x) & \text{if } 0 \leq t < 1/2 \\ g_{2t-1}(x) & \text{if } 1/2 < t \leq 1 \end{cases}
\]

Let $f_0, f_1 : X \to Y$ be connected by a homotopy $f_t$. For $g : Y \to Z$, the maps $h_0 = g \circ f_0$ and $h_1 = g \circ f_1$ are connected by $h_t = g \circ f_t$. Similarly, if $g_0, g_1 : X \to Z$ are connected by $g_t$, then for $f : X \to Y$, the maps $g_0 \circ f$ and $g_1 \circ f$ are connected by $g_t \circ f$. This proves the compatibility with composition.

4.3 Set $f_t(x) = ta + (1 - t)f(x)$.

4.4 First $\tilde{X}$ is compact because it can be identified with the quotient of $X \times [0, 1]$ by the equivalence which identifies $(x, 1)$ and $(Tx, 0)$ for all $x \in X$. As a continuous image of a compact metric space, it is metrizable (see Willard 2004). The map $(y, t) \mapsto T_t(y)$ is well defined and continuous since $(y, s) \mapsto (y, s + t)$ is continuous from $X \times \mathbb{R}$ to itself.

4.5 Set $X = \{0, 1, \ldots, n - 1\}$ with $T(i) = i + 1 \mod n$. Then $\tilde{X}$ can be identified with the torus $\mathbb{T}$ by the map $(i, t) \in X \times [0, 1] \mapsto (i + t)/n$.

4.6 Assume that $\varphi : (X, S) \to (Y, T)$ is an equivalence. Then the map $(x, t) \to (\varphi(x), t)$ induces an equivalence from the suspension flow over $(X, S)$ onto the suspension flow over $(Y, T)$. All periodic systems are flow equivalent by Exercise 4.5. Thus flow equivalence is weaker than equivalence.

4.7 Consider the map $\pi : C(X, Z) \to C(\tilde{X}, \mathbb{T})$ which associates to $f \in C(X, Z)$ the map $\pi(f) \in C(\tilde{X}, \mathbb{T})$ defined by $\pi(f)(x, s) = \tau(f(x)s)$.

We first show that $\pi$ is continuous on $C(X, Z)$ to $H^1(\tilde{X}, \mathbb{Z})$. Let $f : X \to \mathbb{T}$ be continuous. By Lemma 4.3.5 there is a continuous function $f : X \to \mathbb{R}$ such that $f(x, 0) = \tau(f(x))$. Set for $0 \leq s < 1$,
\[
h(x, s) = \tau(f(x)(1 - s) + f(Tx)s).
\]
4.11. SOLUTIONS

Then $h$ is nullhomotopic (by Exercise 4.3) because $f(x)(1 - s) + f(Tx)s$ extends to a continuous map from $X \times [0, 1]$ to $\mathbb{R}$ and $h(x, 0) = \hat{f}(x, 0)$. Therefore the map $g_1(x, s) = f(x, s)/h(x, s)$ is homotopic to $\hat{f}$ and $g_1(x, 0) = g_1(Tx, 1) = 0$ for all $x \in X$. For $x \in X$, let $r(x)$ be the number of times the loop $g_1(x, s)$ wraps around $\mathbb{T}$ as $s$ increases from 0 to 1. Then $r$ is continuous and, for each $x \in X$, $g(x, s) = \tau(r(x)s)$ wraps around $\mathbb{T}$ the same number of times as $g_1(x, s)$. Hence $g_1$ and $g$ are holomorphic. This shows that $\pi$ induces a surjective map.

Next, let $f, g \in C(X, \mathbb{Z})$ be such that $\pi(f)$ and $\pi(g)$ are in the same homotopy class. We may write for $0 \leq s < 1$, $\tau((f(x) - g(x))s) = \tau(r(x, s))$ for some continuous function $r$ from $X$ to $\mathbb{R}$ (because $\tau(r(x, s))$ is nullhomotopic). Then $(f(x) - g(x))s = r(x, s) + P(x, s)$ where $P(x, s) \in \mathbb{Z}$. But since $P(x, s)$ is continuous in $s$, this forces $P(x, s) = P(x, 0)$ for all $s$. Set $p(x) = P(x, 0)$. Then

$$p(Tx) = -r(Tx, 0) = -r(x, 1) = p(x) - (f(x) - g(x))$$

which shows that $f(x) - g(x)$ is a coboundary. Thus $\pi$ induces an injective map from $H(X, T, \mathbb{Z})$ to $H^1(X, \mathbb{Z})$.

Section 4.2

4.8 Let $g, g'$ be two solutions. Let $x$ be a recurrent point in $X$. Then for any $y = T^n x$, we have by 4.12

$$g(y) = f^{(n)}(x) - f(x) + g(x)$$

and thus

$$g(y) - g'(y) = g(x) - g'(x).$$

Since the positive orbit of $x$ is dense, this shows that $g - g'$ differ by a constant.

4.9 Since $f^{(n+1)}(x) = f(x) + f^{(n)}(Tx)$ for all $n \in \mathbb{Z}$, by Exercise 4.1 we have

$$g(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x) = \sup_{n \in \mathbb{Z}} f^{(n+1)}(x) = \sup_{n \in \mathbb{Z}} (f(x) + f^{(n)}(Tx))$$

$$= f(x) + g(Tx)$$

whence the result.

4.10 We first note that $|f^{(n)}(x)|$ is bounded for all $x \in X$. Indeed, set $M = \sup_{n \geq 0} |f^{(n)}(x_0)|$. If $|f^{(n)}(y)| > 2M$, the same inequality holds for any $z$ sufficiently close to $y$, in particular for some iterate $T^m(x_0)$. But then $2M < |f^{(n)}(T^m x_0)| \leq |f^{(n+m)}(x_0)| + |f^{(m)}(x_0)|$ contrary to the definition of $M$.

Thus $f^{(n)}(x)$ is bounded for all $n \in \mathbb{Z}$ and, by Exercise 4.9, the map $g(x) = \sup_{n \in \mathbb{Z}} f^{(n)}(x)$ is such that $\partial g = f$.

Define the oscillation of a real valued function $h : X \rightarrow \mathbb{R}$ defined on a metric space $X$ at a point $x$ as

$$\text{Osc}_n(x) = \lim_{\delta \rightarrow 0} \{\sup\{h(y) \mid d(x, y) < \delta \} - \inf\{h(y) \mid d(x, y) < \delta \}\}$$
Note that the oscillation of a function $h$ at $x$ vanishes if and only if $h$ is continuous at $x$. Since $f$ is continuous, we have $\text{Osc}_h(x) = 0$ for $h = \partial g$. This implies $\text{Osc}_{g \circ T}(x) = \text{Osc}_g(x)$ or $\text{Osc}_g \circ T = \text{Osc}_g$ and thus that the function $x \mapsto \text{Osc}_g(x)$ is invariant. For $\varepsilon > 0$, let $O_{\varepsilon,n}$ be the set of $x \in X$ such that $f(x) - f^{(n)}(x) \leq \varepsilon / 2$. Since $O_{\varepsilon,n}$ is closed, the set

$$\{x \in O_{\varepsilon,n} \mid \text{Osc}_g(x) \leq \varepsilon\}$$

is closed invariant and nonempty it is equal to $X$. Thus $g$ is continuous.

1. follows from the hypothesis that the sequence $f^{(n)}(x_0)$ is bounded.

2. Assume that $(x,u), (x,v) \in K$ for some $x \in X$ and $u \neq v$. The map $S$ commutes with the vertical translations $T_u : (x,t) \mapsto (x,t + u)$. Thus $T_{u-v}K = \{(x,t + u - v) \mid (x,t) \in K\}$ is also an $S$-invariant and minimal compact set. It intersects $K$ since $T_{u-v}K = (x,u)$. By minimality of $K$, this implies $T_{u-v}K = K$ which contradicts the fact that $K$ is bounded. This shows that $K$ is the graph of a function $g$. Its domain is $X$ since it is closed and $T$-invariant. The function $g$ is continuous since $K$ is compact.

3. Since $K$ is $S$-invariant, we have $S(x,g(x)) = (Tx, g(x) + f(x)) \in K$ for every $x \in X$. Thus $g(Tx) = g(x) + f(x)$.

Section 4.8

4.12 Let $(X,T)$ be a topological dynamical system. The set $\mathcal{M}_X$ is convex and compact by Theorem 3.2.3. The map $T^* : \mathcal{M}_X \rightarrow \mathcal{M}_X$ defined by $T^* \mu = \mu \circ T^{-1}$ is a continuous linear map. Thus, by the Markov-Kakutani Theorem, it has a fixed point $\mu$, which is obviously an invariant measure.

4.13 Assume that $\mu$ is ergodic. Let $U$ be a Borel set such that $T^{-1}U = U \mod \mu$. Set

$$V = \cap_{n \geq 0} \cup_{i \geq n} T^{-i}U.$$  

By construction, we have $T^{-1}V = V$. Next, since $\mu(\cap_{i \geq n} T^{-i}U) = \mu(U)$ for all $n \geq 0$, we have $\mu(V) = \mu(U)$. Thus $\mu(V) = 0$ or 1. This shows that (i) implies (ii). The converse is obvious.

4.14 Let $R$ be the set of recurrent points in $(X,T)$ (see Exercise 2.3). We show that $\mu(R) = 1$. This proves that $(X,T)$ is recurrent by Exercise 2.4

Let $(U_n)_{n \geq 1}$ be a countable basis of open sets of $X$ (this exists because $X$ is a metric space). Set

$$V_n = \cap_{k \geq 0} T^{-k}(X \setminus U_n).$$
We have $X \setminus R = \bigcup_{n \geq 1} V_n$. Since $T^{-1}V_n \subset V_n$ and $\mu(T^{-1}V_n) = \mu(V_n)$, it follows that $T^{-1}V_n = V_n \mod \mu$ and thus $\mu(V_n) = 0$ or 1 by Proposition 4.8.3 since $\mu$ is ergodic. But $U_n \subset X \setminus V_n$ implies $\mu(U_n) \leq \mu(X \setminus V_n)$. By our hypothesis, this implies $\mu(X \setminus V_n) > 0$ and thus $\mu(V_n) = 0$. This gives finally $\mu(R) = 1$.

By Weierstrass Theorem, the linear combinations of the functions $\chi_m(z) = z^m$ are dense in $C(S^1, \mathbb{C})$. Thus, by Theorem 4.8.3 it is enough to prove that for every $m \geq 0$, the averages $\chi_m^{(n)}$ converge uniformly to a constant. This is trivially true if $m = 0$. Otherwise, the rotation $R_{\alpha}(z) = \lambda z$ with $\lambda = e^{2i\pi \alpha}$ satisfies
\[
\frac{1}{n} \sum_{k=0}^{n-1} \chi_m(R_{\alpha}^k(z)) = \frac{1}{n} \sum_{k=0}^{n-1} e^{2i\pi km\alpha} = \frac{1 - e^{2i\pi n m\alpha}}{n(1 - e^{2i\pi m\alpha})} \leq \frac{2}{n|1 - e^{2i\pi m\alpha}|} \to 0.
\]

Every Sturmian shift $(X, S)$ of slope $\alpha$ is, by Proposition 2.5.5, the image by the natural coding $\gamma_\alpha$ of the rotation $T, R_\alpha$ of angle $\alpha$. Since $\gamma_\alpha$ is one-to-one except on a denumerable set (the orbit of 0 under $R_\alpha$), the invariant probability measures on $(T, R_\alpha)$ and on $(X, S)$ are exchanged by $\gamma_\alpha$. Thus $(X, S)$ is uniquely ergodic and its unique invariant measure $\mu$ is such that $\mu([0]) = 1 - \alpha$, $\mu([1]) = \alpha$.

It follows from the definition that for $1 \leq j \leq i$, the number of elements of $E_j$ in the interval $0 < n \leq k_i$ is exactly $(k_i/k_j)(2k_{j-1} + 1)$. Hence an upper bound to the number of elements of $E_1 \cup E_2 \cup \cdots \cup E_i$ in the interval $0 < n \leq k_i$ is
\[
\sum_{j=1}^{i} \frac{3k_i k_{j-1}}{k_j} \leq 3k_i \sum_{j=1}^{i} \frac{k_{j-1}}{k_j} < \frac{1}{4} k_i.
\]

It follows that $p(n) = i + 1$ for at least $3/4$ of the numbers $n$ in the interval $0 < n \leq k_i$. This implies that
\[
\left| \frac{1}{k_i} \sum_{n=1}^{k_i} x(n) - \frac{1}{k_{i+1}} \sum_{n=1}^{k_{i+1}} x(n) \right| \geq \frac{1}{2}.
\]

and thus the frequency of 1 is not defined.

Since $T$ is continuous, it is measurable and since $\mu$ is invariant, we have $\mu(T^{-1}V) = \mu(V)$ for every borel subset $V$ of $X$. Thus $T$ preserves $\mu$.

Set $(f, g) = \int fg d\mu$. We have, by change of variable (Equation (3.2.1)), for every $f, g \in L^2(X)$
\[
(U^*Uf, g) = \langle Uf, Ug \rangle = \int (Uf)(Ug) d\mu = \int U(fg) d\mu = \int fg d\mu = (f, g).
\]
Thus $U^*U = I$. Since $T$ is invertible, $U$ is surjective. Let $V$ be its right inverse, that is, such that $UV = I$. Then $V = U^*UV = U^*$. Thus $UU^* = I$. This shows that $U$ is unitary.

4.19 If $f = Ug - g$ with $g \in H$, then $\|f^{(n)}\| = \|U^n g - g\| \leq 2\|g\|$ and thus the sequence is bounded.

Conversely, assume that $\|f^{(n)}\| \leq k$ for $n \geq 1$. Let $S$ be the set of all convex linear combinations of $U^n f$ and let $\bar{S}$ be its closure in the weak topology of $H$. Then $\bar{S}$ is convex and as it is contained in the weakly compact set $\{h \in H \mid \|h\| \leq k\}$ it is weakly compact. Moreover $\bar{S}$ is invariant under the continuous affine map $h \mapsto f + Uh$. By the Schauder-Tychonoff Theorem this map has fixed point in $\bar{S}$, that is there is $g \in \bar{S}$ such that $g = x + Ug$ and thus $x = Ug - g$ is the coboundary of $g \in H$.

4.20 By the choice of $p$, the value of $\varphi_k^p$ is determined by the two first letters $a_0a_1$ of $x = a_0a_1 \cdots a_{k-1}$. Let $\alpha : \mathcal{L}_2(X)^* \to \mathcal{L}_k(X)^*$ be the morphism defined by $\alpha (a_0a_1) = \varphi_k^p (x)$. Then $U$ is the incidence matrix of $\alpha$. Since obviously $\varphi_k \circ \alpha = \alpha \circ \varphi_2$, we obtain $UM_k = M_2U$. Since

$$v_k M_k = v_2 UM_k = v_2 M_2 U = \rho v_2 U = \rho v_k$$

the last assertion follows.

4.21 We have $\mu^3(aa) = abbabaab \cdot abbabaab$ and thus

$$\mu_2^3(aa) = (abbab)(bbaba)(baba)(ababa) \cdots$$

whence the value of the first row of $U$ and similarly for the others.

4.22 The equalities

$$M_2 U = UM_k, \quad VM_2 = M_k V,$$

and

$$M_k^p = UV, \quad M_k^p = VU$$

result of the commutative diagram of Figure 4.11. The diagram on the left gives the equality $VM_2 = M_k V$ (see the diagram 2.4.15). The other three equalities result from the diagram on the right.

4.12 Notes

For a general introduction to cohomology, see for example [Hatcher, 2001].
4.12.1 Gottschalk and Hedlund Theorem

Gottschalk and Hedlund’s Theorem (Theorem 4.2.3) is from [Gottschalk and Hedlund, 1955]. A somewhat simpler, although incorrect, proof appears in [Katok and Hasselblatt, 1995, Theorem 2.9.4]. It is reproduced as Exercise 4.10 using a corrected version due to Petite (2019). Yet another proof (perhaps the most elegant one) is given in Exercise 4.11. According to Petite (2019) it should be credited to Michael Herman (through Sylvain Crovisier). Given \( f \in C(X,T) \), one may consider \( \partial g = f \) as a functional equation with \( g \) as unknown. More generally, an equation of the form

\[
\lambda g(Tx) - g(x) = f(x) \tag{4.12.1}
\]

where \( T : X \to X \) is a given map, \( f \) is a given scalar function on \( X \), \( \lambda \) is a given constant and \( g \) is an unknown scalar function is called in [Katok and Hasselblatt, 1995] a cohomological equation. In this context, given a dynamical system \((X,T)\), a map \( \alpha : \mathbb{Z} \times X \to \mathbb{R} \) is called a one-cocycle if it satisfies the identity

\[
\alpha(n + m, x) = \alpha(n, T^m x) + \alpha(m, x).
\]

Thus, for every map \( f \in C(X,\mathbb{R}) \), the map \((n, x) \mapsto f^{(n)}(x)\) is a one-cocycle (see Exercise 4.1).

The Baire Category Theorem, used in the proof of Proposition 4.2.4, can be found for example in [Willard, 2004, Theorem 25.3].

4.12.2 Ordered cohomology group

Boyle and Handelman [1976] have introduced the term ordered cohomology group of a topological dynamical system \((X,T)\) for the group \( K^0(X,T) \), which had before been only defined for minimal systems. They showed that this group is a complete invariant for flow equivalence of irreducible shifts of finite type.

The proof of Lemma 4.3.5 follows the lines of the proof given in [Giordano et al., 2018].

4.12.3 Ordered group of a recurrent shift space

Example 4.5.4 showing that the ordered cohomology group of a shift of finite type may fail to be a dimension group is from [Kim et al., 2001, Example 3.3].

![Figure 4.11.1: A commutative diagram](image-url)
4.12.4 Invariant measures and states

For an introduction to probability measures on topological dynamical systems, see Katok and Hasselblatt (1995); Berthé and Rigo (2010).

For an introduction to the notion of integral of a measurable function (Section 4.8), see Halmos (1974). The original reference for the Krylov-Bogolyubov Theorem (Theorem 4.8.1) is (Krylov and Bogolioubov, 1937). The notion of measure-theoretic dynamical systems is the central object of ergodic theory. See Katok and Hasselblatt (1995) for a systematic exposition. The original reference for Poincaré Recurrence Theorem is Poincaré (1890).

The original reference for Oxtoby’s Theorem (Theorem 4.8.8) is (Oxtoby, 1952). The example of a minimal non uniquely ergodic shift (Exercise 4.16) is from Oxtoby (1952). It is the first ever constructed sequence with this property.

The ergodic theorem, due to Birkhoff, is also called the pointwise ergodic theorem or strong ergodic theorem, in contrast with the mean ergodic theorem, due to von Neumann which states the weaker convergence in mean in an $L^2$-space. The pointwise ergodic theorem is sometimes also called the Birkhoff-Khinchin Theorem.

The fact that the shift associated with a primitive substitution is uniquely ergodic (Theorem 4.8.9) is due to Michel (1974). The computation of the invariant measure on the shift associated with a primitive substitution is developed in Queffélec (2010). Formula (4.8.8) is from (Queffélec, 2010, Corollary 5.14).

Proposition 4.9.3 is Theorem 5.5 in (Herman et al., 1992), where it is credited to Kerov. The computation of the cohomology group of Sturmian shifts is classical. It appears in particular in Dartnell et al. (2000).

4.12.5 Exercises

The equivalent definition of $H(X,T,Z)$ in terms of Čech cohomology (Exercise 4.17) is taken from Parry and Tuncel (1982). The definition of the Čech cohomology is not the classical one but is equivalent. According to Parry and Tuncel (1982), it defines the Brushlinski group.

The variant of Gottschalk Hedlund Theorem for $L^2$ presented in Exercise 4.19 is from Parry and Tuncel (1982, Proposition II.2.11). The Schauder-Tychonoff fixed-point Theorem can be found as (Dunford and Schwartz, 1988, Theorem V.10.5), while the Markov-Kakutani fixed point theorem appears as Theorem V.10.6 in (Dunford and Schwartz, 1988). The unique ergodicity of irrational rotations (Exercise 4.15) is known as the Kronecker-Weyl Theorem.
Chapter 5

Partitions in towers

In this chapter, we first present the important notion of partition in towers. It is the basis of many of the constructions that will follow. We prove the Theorem of Herman, Putnam and Skau asserting that every minimal Cantor system can be represented by a sequence of partitions in towers (Theorem 5.1.7).

We next show how partitions in towers allow to compute the ordered cohomology group of minimal Cantor systems. We first define in Section 5.2 the ordered group associated to a partition. The definition uses the notions on induction introduced in Section 4.7. In Section 5.3 we prove that the ordered cohomology group $K^0(X,T)$ is the direct limit of the sequence of ordered groups associated with a sequence of partitions in towers. This allows us to prove, as a main result of this chapter, the theorem of Herman, Putnam and Skau asserting that the ordered cohomology group of a minimal invertible Cantor system is a simple dimension group (Theorem 5.3.4).

We will use these results to determine the dimension groups of some minimal shift spaces and relate them to several notions such as return words or Rauzy graphs. We will in particular consider episturmian shifts (Proposition 5.4.3), which will illustrate the use of return words. Next, we will use Rauzy graphs to give a description of dimension groups of substitution shifts (Proposition 5.6.6). These methods will appear again in a new light in the next chapter when we consider Bratteli diagrams.

In this chapter all the dynamical systems $(X,T)$ we consider are invertible, that is, such that $T$ is a homeomorphism.

5.1 Partitions in towers

Let $(X,T)$ be an invertible Cantor system. Let $B_1,\ldots,B_m$ be a family of disjoint nonempty open sets and $h_1,\ldots,h_m$ be positive integers. Assume that the family

$$\mathcal{P} = \{T^jB_i \mid 1 \leq i \leq m, \ 0 \leq j < h_i\}.$$
is a partition of $X$. It implies that each element of $\mathcal{P}$ is a clopen set. We say that it is the clopen partition in towers (or Kakutani-Rohlin partition or KR-partition) of $(X,T)$ built on $B_1, \ldots, B_m$ with heights $h_1, \ldots, h_m$.

Observe that each atom of the partition is a clopen set. The number of towers is $m$, $\{T^j B_i \mid 0 \leq j < h_i\}$ is the $i$-th tower, $h_i$ is its height and $B_i$ its basis. The union $B(\mathcal{P}) = \cup_i B_i$ is the basis of the partition $\mathcal{P}$.

Since $T$ is bijective, the shift sends the elements at the top of the towers back to the bottom, that is $T^{h_i} B_i \subset B(\mathcal{P})$ for $1 \leq i \leq m$. As a consequence, when such a partition exists, the integers $h_i$ are unique.

Thus, informally speaking, a partition in towers gives an approximate description of the action of $T$ on $X$. Each tower can be seen as a stack of clopen sets. The transformation consists in climbing one step up the stack except at the top level where it goes back to the basis in some tower (see Figure 5.1.1).

![Figure 5.1.1: A partition in towers.](image)

### 5.1.1 Partitions and return words

We give two examples of partitions in towers of a shift space. The first example is related to return words. The partition relies on the location of two (possibly overlapping) occurrences of a word $w$, with the second one occurring at a strictly positive index.

**Proposition 5.1.1** Let $(X,S)$ be a minimal subshift and let $w \in \mathcal{L}(X)$. For every $w \in \mathcal{L}(X)$, the family

$$
\mathcal{P} = \{S^j[vw] \mid v \in R'_X(w), 0 \leq j < |v|\}
$$

is a partition in towers with basis the cylinder $[w]$.

**Proof.** Let $x \in X$. Since $(X,S)$ is minimal, there is a smallest integer $i > 0$ such that $x_{[i,i+|w|−1]} = w$. By definition of a left return word, there is a unique integer $j \geq 0$ such that $x_{[−j,i−1]}$ belongs to $R'_X(w)$ (see Figure 5.1.2).

![Figure 5.1.2: Two occurrences of $w$ in $x$](image)
5.1. PARTITIONS IN TOWERS

We set \( v = x_{[i,j-1]} \in R'_X(w) \) and observe that we have \( 0 \leq j < |v| \). Thus \( x \) belongs to \( S^j[w] \). Since \( v \) and \( j \) are unique, this shows that \( \Psi \) is a partition.

Example 5.1.2 Let \((X, S)\) be the two-sided Fibonacci shift on the alphabet \( A = \{a, b\} \). We have \( ab \in L(X) \) and \( R'_X(ab) = \{ab, aba\} \). The corresponding partition in towers is represented in Figure 5.1.3 with \( u \cdot v \) representing the set \( S^j[uv] \) for \( j = |u| \).

![Figure 5.1.3: A partition in towers of the Fibonacci shift](image)

5.1.2 Partitions of substitution shifts

The second example is a partition of a substitution shift.

Proposition 5.1.3 Let \( \varphi : A^* \to A^* \) be a primitive substitution and let \( X \) be the associated shift space. If \( X \) is infinite, the family

\[
\Psi(n) = \{ S^j \varphi^n([a]) \mid a \in A, 0 \leq j < |\varphi^n(a)| \}
\]

is, for every \( n \geq 1 \), a partition in towers with basis \( \varphi^n(X) \).

Proof. Set \( \psi = \varphi^n \). Since \( \varphi \) is primitive and \( X \) is infinite, \( \varphi \) is recognizable on \( X \) and thus \( \psi \) is recognizable on \( X \). This implies clearly that \( \Psi(n) \) is a partition.

The partition \( \Psi(n) \) is called the partition associated to the substitution \( \varphi^n \). We illustrate Proposition 5.1.3 with the following example.

Example 5.1.4 Let \( \varphi \) be the Fibonacci substitution and let \((X, S)\) be the Fibonacci shift. The partition \( \Psi(n) \) corresponding to Proposition 5.1.3 has two towers with basis \( \varphi^n([a]) \) and \( \varphi^n([b]) \) respectively. One has \( \varphi^n([a]) = [u_n] \) with \( u_0 = a \) and where \( u_n \) is, for \( n \geq 1 \), the shortest right-special factor which has \( \varphi(u_{n-1}) \) as a prefix. A similar rule holds for \( \varphi^n([b]) \). The partition in towers for \( n = 2 \) is identical with the partition represented in Figure 5.1.3.
CHAPTER 5. PARTITIONS IN TOWERS

Proposition 5.1.5 Let \((X, T)\) be a minimal Cantor system. Let \(\mathcal{Q}\) be a clopen partition of \(X\) and \(C\) be a clopen set. Then there exists a clopen partition \(C_1, \ldots, C_t\) of \(C\) and integers \((h_i)_{1 \leq i \leq t}\) such that

\[
\mathcal{P} = \{T^j C_i \mid 0 \leq j < h_i, \ 1 \leq i \leq t\}
\]

is finer than \(\mathcal{Q}\).

Proof. Let \(r_C : X \to \mathbb{Z}\) be the first return map to \(C\) defined by \(r_C(x) = \inf\{n > 0 \mid T^n x \in C\}\). Since \((X, T)\) is minimal, \(r_C\) is well-defined and continuous. Let \(r_1, r_2, \ldots, r_{t'}\) be the set of values taken by \(r_C\). For every \(i\) with \(1 \leq i \leq t'\), we define \(C'_i = \{x \in C \mid r_C(x) = r_i\}\). Then

\[
\mathcal{P}' = \{T^j C'_i \mid 0 \leq j < r_i, \ 1 \leq i \leq t'\}
\]

is a clopen partition of \(X\). Indeed, since \((C'_i)_{1 \leq i \leq t'}\) is a partition of \(C\), the family \(\mathcal{P}'\) is formed of disjoint sets. Their union is a nonempty closed invariant subset of \(X\) and since \((X, T)\) is minimal, it is equal to \(X\). It is however not necessarily finer than \(\mathcal{Q}\). Let \(\mathcal{Q}' = \{P' \cap Q \mid P' \in \mathcal{P}', Q \in \mathcal{Q}\}\). It suffices to find \(\mathcal{P}\) finer than \(\mathcal{Q}'\). Let \(Q'\) be an atom of \(\mathcal{Q}'\). There exists a unique pair \((i_0, j_0)\) with \(1 \leq i_0 \leq t'\) and \(0 \leq j_0 < r_{i_0}\) such that \(Q' \subset T^{j_0} C'_i\). Set \(Q'' = T^{j_0} C'_i \setminus Q'\). We divide the tower \(i_0\) into two new towers and obtain a new KR-partition \(\mathcal{P}''\) with \(t' + 1\) towers

\[
\mathcal{P}'' = \{T^j C'_i \mid 0 \leq j < r_i, 1 \leq i \leq t', i \neq i_0\} \\
\cup \{T^j Q' \mid -j_0 \leq j < r_{i_0} - j_0\} \\
\cup \{T^j Q'' \mid -j_0 \leq j < r_{i_0} - j_0\}
\]

with a split of the \(i_0\)-tower propagating up and down the split of \(T^{j_0} C'_i\) in two parts, namely \(Q'\) and \(Q''\). We repeat this procedure for every atom of \(\mathcal{Q}'\). The result is the desired KR-partition. QED

5.1.3 Sequences of partitions

A partition \(\alpha\) is a refinement of a partition \(\rho\) if every element of \(\alpha\) is a subset of an element of \(\rho\) or, equivalently, if every element of \(\rho\) is a union of elements of \(\alpha\). We also say that \(\rho\) is coarser than \(\alpha\). We denote \(\alpha \geq \rho\).

Let \(A\) and \(B\) be two families of subsets of some set \(X\). We set

\[
A \lor B = \{A \cap B \mid A \in A, \ B \in B\}.
\]

For finitely many families \(A_1, A_2, \ldots, A_n\) of subsets of \(X\) we set

\[
\bigvee_{i=1}^n A_i = A_1 \lor A_2 \lor \cdots \lor A_n
\]

while \(A \cup B\) is the union of the families of \(A \in A\) and \(B \in B\).
We also say that a partition in towers $\mathfrak{P}$ with basis $B$ is nested in a partition in towers $\mathfrak{P}'$ with basis $B'$ if $B \subseteq B'$ and $\mathfrak{P}$ is a refinement, as a partition, of $\mathfrak{P}'$.

A sequence $(\mathfrak{P}(n))$ of partitions in towers is nested if $\mathfrak{P}(n + 1)$ is nested in $\mathfrak{P}(n)$ for all $n \geq 1$.

The following statement shows that the sequence of partitions of a primitive substitution shift defined in Proposition 5.1.3 is nested.

**Proposition 5.1.6** Let $\varphi : A^* \to A^*$ be a primitive substitution and let $X$ be the associated shift space. If $X$ is infinite, the sequence of partitions

$$\mathfrak{P}(n) = \{ S^j \varphi^n([a]) \mid a \in A, 0 \leq j < |\varphi^n(a)| \}$$

is nested.

**Proof.** If $\varphi(a)$ begins with $b$, we have $\varphi^{n+1}([a]) \subset \varphi^n([b])$. Thus, the base of $\mathfrak{P}(n + 1)$ is contained in the base of $\mathfrak{P}(n)$. Next, if $0 \leq j < |\varphi^{n+1}(a)|$, there is a factorization $\varphi(a) = xcy$ with $c \in A$ such that $|\varphi^n(x)| \leq j < |\varphi^n(xc)|$. Set $k = j - |\varphi^n(x)|$. Then

$$S^j \varphi^{n+1}([a]) \subset S^k \varphi^n([c])$$

with $0 \leq k < |\varphi^n(c)|$. This shows that $\mathfrak{P}(n + 1)$ refines $\mathfrak{P}(n)$. $\blacksquare$

We say that a sequence $(\mathfrak{P}(n))$ of KR-partitions of $X$ with bases $B(n)$ is a refining sequence if it satisfies the three following conditions.

(KR1) $\cap_n B(n) = \{ x \}$ for some $x \in X$, that is, the intersection of the bases consists in one point $x \in X$,

(KR2) the sequence $(\mathfrak{P}(n))$ is nested,

(KR3) $\cup_{n \geq 1} \mathfrak{P}(n)$ generates the topology of $X$ (that is every open set is a union of elements of the partitions $\mathfrak{P}(n)$).

Condition (KR3) can be expressed equivalently by the condition that the sequence of partitions $(\mathfrak{P}(n))$ tends to the point partition (that is, for every $\varepsilon > 0$, there is an $n$ such that all elements of $\mathfrak{P}(n)$ are contained in a ball of radius $\varepsilon$).

None of the conditions (KR1) or (KR3) implies the other one (see Exercises 5.2, 5.3).

We deduce from Proposition 5.1.5 the following statement.

**Theorem 5.1.7** Let $(X, T)$ be a minimal Cantor system. There exists a refining sequence of KR-partitions of $X$.

**Proof.** Let $x \in X$. We start choosing a decreasing sequence of clopen sets $(C_n)_{n \geq 1}$ whose intersection is $\{ x \}$ and an increasing sequence of partitions $(\mathfrak{P}'(n))_n$ generating the topology. We apply Proposition 5.1.5 to $\Omega = \mathfrak{P}'(1)$ and $C = C_1$ to obtain $\mathfrak{P}'(1)$.
Applying Proposition 5.1.5 iteratively for \( n \geq 2 \) to \( C = C_n \) and by setting now
\[
Ω = \Psi'(n) \vee \Psi(n - 1),
\]
we obtain a partition \( \Psi(n) \) with basis \( C_n \) which is finer than \( \Psi'(n) \) and \( \Psi(n - 1) \).

Condition (KR1) holds because for each \( n \), the basis of \( \Psi(n) \) is \( C_n \) and \( \cap_{n} C_n = \{ x \} \) by hypothesis.

Condition (KR2) holds because, by construction, \( \Psi(n) \) is nested in \( Ω \), which is nested in \( \Psi(n - 1) \).

Finally, condition (KR3) holds because \( \cup_{n \geq 1} \Psi(n) \) is finer than \( \cup_{n \geq 1} \Psi'(n) \), which generates the topology.

Note that, by definition, in a nested sequence \( (\Psi(n)) \) of partitions in towers, the sequence \( B(\Psi(n)) \) is decreasing.

We give two simple examples illustrating Theorem 5.1.7. The first one is the ring of \( p \)-adic integers (see Section 2.1.4).

**Example 5.1.8** We show that the odometer on the ring of \( p \)-adic integers can be represented by a sequence of partitions in towers with one tower. For \( n \geq 1 \), let \( B(n) = p^n \mathbb{Z}_p \), that is the ball of \( \mathbb{Z}_p \) centered in 0 of radius \( p^n \). Then the family \( \Psi(n) = \{ T^j B(n) \mid 0 \leq j < p^n \} \) is, for each \( n \), a partition formed of one tower. It is easy to verify that the sequence \( (\Psi(n)) \) satisfies the conditions of Theorem 5.1.7.

In the second example, we show how a nested sequence of partitions can modified to become a refining sequence.

**Example 5.1.9** Let \( X \) be the two-sided Fibonacci shift. For \( n \geq 1 \), let \( \Psi(n) \) be the partition \( \Psi(n) = \{ S^j \varphi^n([a] \mid a \in A, 0 \leq j < |\varphi^n(a)|) \} \) (see Proposition 5.1.3). Properties (KR1) and (KR3) do not hold for this sequence of partitions. Indeed, \( \varphi^2 : a \mapsto aba, b \mapsto ab \) has one right infinite fixed point \( x \) (the Fibonacci word) and two left infinite fixed points \( y, z \). The two points (actually fixed points) \( y \cdot x \) and \( z \cdot x \) belong to all \( \varphi^n[a] \).

Using instead the substitution \( \psi : a \mapsto baa, b \mapsto ba \) (related to \( \varphi^2 \) by \( a\psi(u) = \varphi^2(u)a \) for every word \( u \)), the shift defined remains the same. Indeed, set \( u_n = \varphi^{2n}(a) \) and \( v_n = \psi^n(b) \). Then \( u_{n+1} = a\psi(u_n) \) and \( v_{n+1} = bu_nw_n \) with \( w_n = \psi(v_n) \) (because \( v_{n+1} = \psi(v_n) = \psi(bu_{n-1}w_{n-1}) = bu_{n}w_{n} \)). Thus \( \psi^n(b) = b\varphi^n(a) \). The sequence of partitions \( (\Psi(n)) \) associated to the substitutions \( \psi^n \) satisfies the three conditions because there is a unique fixed point (we shall see in Chapter 7 that this example describes a general situation).

### 5.2 Ordered group associated with a partition

Let \((X,T)\) be a minimal Cantor system. Let
\[
Ψ = \{ T^j B_i \mid 1 \leq i \leq m, 0 \leq j < h_i \}
\]
be a KR-partition of \((X,T)\) built on \(B_1,\ldots,B_m\) and with heights \(h_1,\ldots,h_m\). Denote \(C(\mathfrak{P})\) the subgroup of \(C(X,Z)\) formed of the functions which are constant on every element of the partition \(\mathfrak{P}\) and \(C^+(\mathfrak{P}) = C(\mathfrak{P}) \cap C(X,Z^+).\) The triple \((C(\mathfrak{P}), C^+(\mathfrak{P}), \chi_X)\) is a unital ordered group.

Next, denote \(G(\mathfrak{P})\) the subgroup of \(C(B(\mathfrak{P}),Z)\) formed of the functions constant on the basis \(B_i\) of each tower, denote \(G^+(\mathfrak{P}) = G(\mathfrak{P}) \cap C(B(\mathfrak{P}),Z^+)\) and \(1_\mathfrak{P}\) the function with value \(h_i\) on \(B_i\). We define the unital ordered group associated to \(\mathfrak{P}\) as the triple \((G(\mathfrak{P}), G^+(\mathfrak{P}), 1_\mathfrak{P})\).

Let \(I(\mathfrak{P}) : C(\mathfrak{P}) \to G(\mathfrak{P})\) be the group morphism defined by
\[
(I(\mathfrak{P})f)(x) = f^{(h_i)}(x)
\]
for every \(f \in C(\mathfrak{P})\) and \(x \in B_i\). It is the restriction to \(C(\mathfrak{P})\) of the morphism \(I_{B(\mathfrak{P})}\) introduced in Section 4.7.

By Proposition 4.7.2, the kernel of \(I(\mathfrak{P})\) consists in coboundaries and there exists a morphism \(\pi(\mathfrak{P}) : G(\mathfrak{P}) \to H(X,T,Z)\) which makes the diagram of Figure 5.2.1 commutative (we denote by \(\pi\) the canonical morphism from \(C(X,Z)\) onto \(H(X,T,Z) = C(X,Z)/\partial_T C(X,Z)\)).

\[
\begin{tikzpicture}
  \node (1) at (0,0) {\(I(\mathfrak{P})\)};
  \node (2) at (0,-1) {\(G(\mathfrak{P})\)};
  \node (3) at (1,0) {\(C(\mathfrak{P})\)};
  \node (4) at (1,-1) {\(H(X,T,Z)\)};
  \draw[->] (1) to node {\(\pi\)} (3);
  \draw[->] (2) to node [swap] {\(\pi(\mathfrak{P})\)} (4);
  \draw[->] (3) to node {\(\chi_X\)} (4);
\end{tikzpicture}
\]

Figure 5.2.1: The morphism \(\pi(\mathfrak{P})\).

**Proposition 5.2.1** The morphism \(\pi(\mathfrak{P})\) defines a morphism of unital ordered groups from \((G(\mathfrak{P}), G^+(\mathfrak{P}), 1_\mathfrak{P})\) to \(K^0(X,T)\).

**Proof.** It is clear that \(\pi(\mathfrak{P})(G^+(\mathfrak{P}))\) is included in \(H^+(X,T,Z)\). Next, for any \(x \in B_i\)
\[
(I(\mathfrak{P})\chi_X)(x) = h_i.
\]
Thus \(\pi(\mathfrak{P})(1_\mathfrak{P}) = \pi(\chi_X) = 1_X\).

Assume now that
\[
\mathfrak{P}' = \{T^\ell B'_k \mid 1 \leq k \leq m',\ 0 \leq \ell < h'_k\}
\]
is an other KR-partition in tower (with basis \(B'_1,\ldots,B'_m\) and heights \(h'_1,\ldots,h'_m\)) which is nested in \(\mathfrak{P}\). The homomorphism \(R_{B(\mathfrak{P})} : C(B(\mathfrak{P}),Z) \to C(X,Z)\) defined in Section 4.7 maps \(G(\mathfrak{P})\) to \(C(\mathfrak{P})\), and thus in \(C(\mathfrak{P}')\). The homomorphism
\[
I(\mathfrak{P}', \mathfrak{P}) = I(\mathfrak{P}') \circ R_{B(\mathfrak{P})}
\]
maps \( G(\mathfrak{P}) \) to \( G(\mathfrak{P}') \) (see the commutative diagram in Figure 5.2.2). It is clearly a morphism of ordered groups and

\[
I(\mathfrak{P}', \mathfrak{P}) \circ I(\mathfrak{P}) = I(\mathfrak{P}') \quad \text{and} \quad \pi(\mathfrak{P}') \circ I(\mathfrak{P}', \mathfrak{P}) = \pi(\mathfrak{P}).
\]

Note that in the first equality, the right-hand side is actually the restriction of \( I(\mathfrak{P}') \) to \( C(\mathfrak{P}) \).

It can be useful to write the morphism \( I(\mathfrak{P}', \mathfrak{P}) \) in matrix form. We can make the following identification

\[
G(\mathfrak{P}) = \mathbb{Z}^m, \quad G(\mathfrak{P}') = \mathbb{Z}^{m'}, \quad G(\mathfrak{P}^+), G(\mathfrak{P}')^+ = \mathbb{Z}^{m'}_+.
\]

The units \( 1_{\mathfrak{P}} \) and \( 1_{\mathfrak{P}'} \) are identified with the vectors of heights. The \( m' \times m \)-matrix \( M(\mathfrak{P}', \mathfrak{P}) \) of the morphism \( I(\mathfrak{P}', \mathfrak{P}) \) is given by

\[
M(\mathfrak{P}', \mathfrak{P})_{k,i} = \text{Card}\{ \ell \mid 0 \leq \ell < h'_k, T^\ell B'_k \subset B_i \} \quad (5.2.1)
\]

for \( 1 \leq k \leq m' \) and \( 1 \leq i \leq m \). Indeed, let \( f \in G(\mathfrak{P}) \) be the function with value \( \alpha_i \) on \( B_i \) for \( 1 \leq i \leq m \). Then, for \( x \in B'_k \), we have

\[
I(\mathfrak{P}', \mathfrak{P})f(x) = f^{(h'_k)}(x) = \sum_{i=1}^{m} M(\mathfrak{P}', \mathfrak{P})_{k,i} \alpha_i.
\]

**Example 5.2.2** Let \((X, S)\) be the two-sided Fibonacci shift. Let \( \mathfrak{P} \) be the partition corresponding to the return words on \( a \). We have \( R'_X(a) = \{a, ab\} \), and the partition \( \mathfrak{P} \) is represented in Figure 5.2.3 on the left. Let \( \mathfrak{P}' \) be the partition corresponding to the return words on \( ab \) (see Example 5.1.2) represented in Figure 5.2.3 on the right.

Since \( a \) is a prefix of \( ab \), the partition \( \mathfrak{P}' \) is nested in \( \mathfrak{P} \).

The matrix \( M(\mathfrak{P}', \mathfrak{P}) \) is

\[
M(\mathfrak{P}', \mathfrak{P}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

For example, the second row is \([1 1]\) because, in the second tower of \( \mathfrak{P}' \), the lower element is contained in the basis of the second tower of \( \mathfrak{P} \) and the upper element in the basis of the first one.
Figure 5.2.3: The partitions \( \Psi \) and \( \Psi' \).

5.3 Ordered groups of sequences of partitions

We now give a description of the ordered group of the minimal Cantor system \((X, T)\) in terms of a refining sequence of KR-partitions. Denote for simplicity \( B(n) \) the base of the partition \( \Psi(n) \) and 
\[
G(n) = G(\Psi(n)), \quad G^+(n) = G^+(\Psi(n)), \quad 1_n = 1_{\Psi(n)}. 
\]

**Proposition 5.3.1** Let \((X, T)\) be an invertible minimal Cantor system. Let \((\Psi(n))\) be a refining sequence of KR-partitions. The ordered group \( K^0(X, T) \) is the inductive limit of the unital ordered groups \((G(n), G^+(n), 1_n)\).

The proof is given below. Before we need some lemmas.

Let \((G, G^+, 1)\) be the inductive limit of the sequence of ordered groups \((G(n), G^+(n), 1_n)\) with the morphisms \( I(n + 1, n) = I(\Psi(n + 1), \Psi(n)) \). Let \( i(n) : G(n) \to G \) be the natural morphism. Note that for \( m > n \), we have \( i(n) = i(m) \circ I(m, n) \).

**Lemma 5.3.2** Let \((X, T)\) be a minimal Cantor system with \( T \) a homeomorphism. Let \((\Psi(n))\) be a nested sequence of KR-partitions. There is a unique morphism \( \sigma : (G, G^+, 1) \to K^0(X, T) \) such that \( \sigma \circ i(n) = \pi(n) \) for every \( n \geq 0 \).

**Proof.** By Proposition 5.2.1, the morphism \( \pi(n) : G(n) \to H(X, T, Z) \) is for every \( n \geq 0 \) a morphism of unital ordered groups. By Proposition 3.3.8 there is a unique morphism \( \sigma : (G, G^+, 1) \to K^0(X, T) \) such that \( \sigma \circ I(n) = \pi(n) \).

We illustrate the proof in Figure 5.3.1, where the upper part reproduces Figure 5.2.1 for \( \Psi = \Psi(n) \) with \( C(n) = C(\Psi(n)) \), \( I(n) = I(\Psi(n)) \) and \( \pi(n) = \pi(\Psi(n)) \).

The proof uses the following second lemma (in which we do not yet assume that \( Y \) is reduced to one point, as in Proposition 5.3.1). A sequence 
\[
G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} G_2 \to \cdots \xrightarrow{\varphi_n} G_n
\]
of group morphisms is exact if the image of each morphism is the kernel of the next one.

**Lemma 5.3.3** Let \((X,T)\) be a minimal Cantor system with \(T\) a homeomorphism. Let \((\mathcal{P}(n))\) be a nested sequence of partitions such that \(\mathcal{P}(n)\) converges to the point partition. Let \(Y = \cap_{n \geq 0} B(n)\). There is a group morphism \(r : C(Y,Z) \rightarrow \ker(\sigma)\) such that the sequence of group morphisms

\[
0 \rightarrow Z \rightarrow C(Y,Z) \xrightarrow{r} G \xrightarrow{\sigma} H(X,T,Z) \rightarrow 0
\]  

(5.3.1)

is exact, with \(Z\) identified with the group of constant functions on \(Y\) where \(G\) is the inductive limit of the groups \(G(n)\) and \(\sigma\) is defined in Lemma 5.3.2.

**Proof.** Recall that \(C(n)\) is the set of functions in \(C(X,Z)\) which are constant on every element of \(\mathcal{P}(n)\). Since the sequence \((\mathcal{P}(n))\) tends to a partition in points, and since a continuous function is locally constant, we have \(C(X,Z) = \bigcup_n C(n)\).

This implies, as a first step, that \(\sigma\) is surjective by Proposition 3.3.8.

As a second step, let us define the morphism \(r\). For this, let \(u \in C(Y,Z)\), let \(h \in C(X,Z)\) having \(u\) as restriction to \(Y\) and let \(g = \partial T h\). For \(n\) large enough, since \(C(X,Z) = \bigcup_n C(n)\), we have \(g \in C(n)\). Let \(f = I(n)(g)\) and let \(\alpha = i(n)(f)\).

Since \(\alpha\) is the image in \(G\) of a coboundary, it belongs to the kernel of \(\sigma\). Indeed, we have (see Figure 5.3.1)

\[
\sigma(\alpha) = \sigma(i(n)f) = \pi(n)(f) = \pi(n) \circ I(n)(g) = \pi(g) = 0.
\]

Thus the map \(r : u \mapsto \alpha\) is a group morphism from \(C(Y,Z)\) onto \(\ker(\sigma)\).

Let us finally prove that \(\alpha = 0\) if and only if \(u\) is constant.

If \(u\) is constant, \(h\) is constant on \(Y\) and thus on some neighborhood of \(Y\). For \(m \geq n\) large enough, since the sequence \(B(n)\) is decreasing, \(h\) is constant on \(B(m)\). We may assume that \(m\) is large enough to have also \(g = \partial T h \in C(m)\). Then, by Equation (4.1.2), for any \(x \in B_i(m)\), since \(T^{h_i(m)} x\) belongs to \(B(m)\), we have

\[
I(m)g(x) = g^{(h_i(m))}(x) = h(T^{h_i(m)} x) - h(x) = 0.
\]  

(5.3.2)
5.3. ORDERED GROUPS OF SEQUENCES OF PARTITIONS

Since \( I(m)g = I(m, n) \circ I(n)g = I(m, n)f \), we have also \( I(m, n)f = 0 \) and thus \( \alpha = i(n)f = i(m) \circ I(m, n)f = 0 \).

Conversely, if \( \alpha = 0 \), there exists \( m \geq n \) such that \( 0 = I(m, n)f = I(m)g \). Thus, by Equation 5.3.2, \( h \) is constant on a set which is dense in \( B(m) \), which implies that it is constant on \( B(m) \) and therefore that \( u \) is constant.

It follows that the kernel of \( r \) is the group of constant functions on \( Y \).

This completes the proof that the sequence of Equation 5.3.1 is an exact sequence.

Proof of Proposition 5.3.4. Since the intersection of the bases \( B(n) \) consists in one point, we have \( \ker(r) = C(Y, \mathbb{Z}) \) and \( \text{Im}(r) = 0 \). Thus \( \sigma \) is an isomorphism and the exact sequence displayed in (5.3.1) reduces to the isomorphism of \( G \) with \( H(X, T, \mathbb{Z}) \).

We deduce from Proposition 5.3.4 the following important result.

**Theorem 5.3.4 (Herman, Putnam, Skau)** For any minimal Cantor system \((X, T)\), with \( T \) a homeomorphism, the ordered group \( K^0(X, T) \) is a simple dimension group.

**Proof.** By Theorem 5.1.7 there exists a sequence \((\Psi(n))\) of partitions in towers satisfying the hypotheses of Proposition 5.3.1. Thus \( K^0(X, T) \) is the direct limit \( G \) of the ordered groups \((G(n))\). Since each \( G(n) \) is isomorphic to some \( \mathbb{Z}^m \) with the natural order, it follows that \( K^0(X, T) \) is a dimension group.

For \( n \geq 1 \), let \( [i, n] \) be the class of the characteristic function \( \chi_{B_i(n)} \) of an element \( B_i(n) \) of the basis of \( \Psi(n) \), let \( I(i, n) \) be the ideal of \( G \) formed of the classes of the functions \( f \in G \) such that \( -k[i, n] \leq f \leq k[i, n] \) for some positive integer \( k \). It is easy to see that \( I(i, n) \) is an order ideal. Next, suppose that \( I \) is any nonzero order ideal in \( G \). Let \( x \) be a nonzero positive element of \( I \). It must be represented by some strictly positive element in some \( G(n) \) and, if \( B_i(n) \) is an element of \( B(n) \) on which \( x \) is positive, we have \( 0 \leq [i, n] \leq x \) and hence \( [i, n] \) is also in \( I \). It follows that \( I(i, n) \subset I \).

We have shown that if \( G \) contains a nonzero order ideal, then it contains one of the form \( I(i, n) \). On the other hand, since \((X, T)\) is minimal, one has \( I(i, n) = G \) for every element \( B_i(n) \) of \( B(n) \). Indeed, denote by \( M(n, m) \) the matrix \( M(\Psi(m), \Psi(n)) \) defined by Equation 5.2.1. Since \((X, T)\) is minimal, there is an \( m > n \) such that the matrix \( M(n, m) \) has all its elements positive. Thus \( k[i, n] \) can exceed any positive element \( x \) of \( G(m) \), which implies \( x \in I(i, n) \). Thus we conclude that \( G \) is simple.

In agreement with Theorem 5.3.4 the group \( K^0(X, T) \) for a minimal Cantor system \((X, T)\), with \( T \) a homeomorphism, is called the dimension group of \((X, T)\).

We give two examples illustrating Theorem 5.3.4. Many more examples will appear later.
Example 5.3.5 The dimension group of the odometer on the ring of 2-adic integers is the group of dyadic rationals. Indeed, we have seen in Example 5.1.8 that it can be represented by a sequence $\mathcal{P}(n)$ of partitions with one tower and the map $i_{n+1,n}$ is easily seen to be the multiplication by 2. Thus the dimension group $K^0(X,T)$ is the group $\mathbb{Z}[1/2]$ (see Example 3.3.1).

Example 5.3.6 We have seen before (Proposition 4.9.4) that the dimension group of a Sturmian shift of slope $\alpha$ is $\mathbb{Z} + \mathbb{Z}\alpha$, which is a simple dimension group.

5.4 Dimension groups and return words

In this section, we show how to use return words to compute the dimension group of a minimal shift space. This method has the advantage of using in general abelian groups of smaller dimension than with the cylinder functions (as seen in Section 1.5.1). For example, in a Sturmian shift space, the number of return words is constant while the word complexity is linear.

In the first part, we show that the group $K^0(X,T)$ is, for every minimal shift space, the direct limit of a sequence of groups associated with return words. In the second part we illustrate the use return words to compute the dimension groups of episturmian shifts.

5.4.1 Sequences of return words

Let $X$ be a minimal shift space. We have already introduced in Section 5.4.1 the sets $\mathcal{R}_X(w)$ and $\mathcal{R}_X^r(w)$ of right and left return words to $w \in \mathcal{L}(X)$.

We fix a point $x \in X$ and we denote $W_n(x) = \mathcal{R}_X(x_{[n-1]})$. Let $G_n(x)$ be the group of maps from $W_n(x)$ to $\mathbb{Z}$, $G^+_n(x)$ the subset of nonnegative ones and $1_n(x)$ the map which associates to $v \in W_n(x)$ its length.

Since $x_{[n-1]}$ is a prefix of $x_{[n]}$, the set $W_{n+1}(x)$ is contained in the submonoid generated by the set $W_n(x)$. This means that for every $v \in W_{n+1}(x)$ there is a decomposition $v = w_1(v)w_2(v)\cdots w_k(v)$ as a concatenation of elements of $W_n(x)$. The decomposition is moreover easily seen to be unique. For every $\phi \in G_n(x)$, let $i_{n+1,n}\phi \in G_{n+1}(x)$ be defined by

$$i_{n+1,n}\phi(v) = \sum_{i=1}^{k(v)} \phi(w_i(v)).$$

It is clear that $i_{n+1,n} : (G_n(x), G^+_n(x), 1_n(x)) \to (G_{n+1}(x), G^+_{n+1}(x), 1_{n+1}(x))$ is a morphism of unital ordered groups. Indeed, in particular, for $v \in W_{n+1}(x)$,

$$i_{n+1,n}(1_n(x))(v) = \sum_{i=1}^{k(v)} 1_n(x)(w_i(v)) = \sum_{i=1}^{k(v)} |w_i(v)| = |v| = 1_{n+1}(x)(v).$$
We will prove the following result using the partition in towers associated with return words (see Proposition 5.1.1). However, we will not be able to use Proposition 5.3.1 because the sequence of partitions in towers associated with the sets \(W_n(x)\) do not satisfy the hypotheses of Proposition 5.3.1 (the intersection of the bases does not consist in one point).

**Proposition 5.4.1** Let \((X, S)\) be a minimal shift space. The group \(K^0(X,S)\) is the inductive limit of the family \((G_n(x), G^+_n(x), 1_n(x))\) with the morphisms \(i_{n+1,n}\).

**Proof.** By Proposition 5.1.1 for every \(n > 0\), the family \(\mathcal{P}_n = \{S^j[vx|0,n-1]\} \mid v \in W_n(x), 0 \leq j < |v|\} is a partition in towers with basis \([x|0,n-1]\]. We can identify \(G(n) = G(\mathcal{P}_n)\) with \(G_n(x)\) so that

\[
(G_n(x), G^+_n(x), 1_n(x)) = (G(n), G^+(n), 1_n)
\]

The morphisms \(i_{n+1,n}\) are then identified to the morphisms \(I(n+1,n)\). Let \((G, G^+, 1)\) be the direct limit of the sequence of ordered groups \((G(n), G^+(n), 1_n)\) with the morphisms \(I(n+1,n)\). By Lemma 5.3.2 there is a unique morphism \(\sigma : G \to H(X,T,Z)\) such that \(\sigma \circ I(n) = \pi(n)\) for every \(n \geq 1\). We show that \(\sigma\) is an isomorphism.

Let \(y, z \in S^j[vx|0,n-1]\) for some \(n \geq 1\), some \(v \in W_n(x)\) and some \(j\) with \(0 \leq j < |v|\). Then \(y, z\) have the same prefix of length \(|v| - j + n\) and thus the same prefix of length \(n\). Thus \(y|0,n-1| = z|0,n-1|\). It follows that the partition \(\mathcal{P}_n\) is finer than the partition in \(n\)-cylinders, and that \(C(\mathcal{P}_n)\) contains every cylinder function corresponding to some \(\phi \in Z_n(X)\). Consequently \(\cup_{n \geq 1} C(\mathcal{P}_n)\) contains every cylinder function. Since every \(f \in C(X,Z)\) is cohomologous to some cylinder function, the morphism \(\sigma\) is onto and maps \(G^+\) onto \(H^+(X,T,Z)\).

Assume now that \(\alpha \in G\) is in the kernel of \(\sigma\). For some \(n\), \(\alpha\) is the image in \(G\) of some \(f \in C(\mathcal{P}_n)\) and \(f\) is a coboundary. The function \(g = R_{\beta(\mathcal{P}_n)} \circ I(n)f\) is, by Proposition 4.7.2 cohomologous to \(f\) and thus is also a coboundary. It is a cylinder function because it is constant on \([vx|0,n-1]\) for each \(v \in W_n(x)\) and null outside \([x|0,n-1]\). Thus, by Lemma 5.4.2 it is the coboundary of some cylinder function \(h\). For \(m\) large enough, \(h\) is constant on the basis \(x|0,m-1\) of \(\mathcal{P}_m\). This implies that \(I(m)f = I(m)g = 0\) and the image \(\alpha\) of \(f\) in \(G\) is 0. ■

**Example 5.4.2** Let \(X\) be the two-sided Fibonacci shift. Let \(x \in X\) be such that \(x_0x_1\cdots\) is the Fibonacci word. We have

\[
\begin{align*}
W_1(x) & = \{a, ab\} \\
W_2(x) & = \{ab, aba\} \\
W_3(x) & = \{ab, aba\} \\
W_4(x) & = \{aba, abaab\}
\end{align*}
\]

In general, one has \(W_{n+1}(x) = W_n(x)\) if \(x|0,n-1\) is not right special and otherwise \(W_{n+1}(x) = \{v, vu\}\) if \(W_n(x) = \{u, v\}\) with \(|u| < |v|\). Indeed, the right-special prefixes of \(x\) are its palindrome prefixes \(u_n\) and \(R'_n(u_n) = \{\varphi^n(a), \varphi^n(b)\}\)
Proposition 5.4.3. The set $W_n(x)$ has two elements for every $n \geq 1$ and thus $G_n(x) = \mathbb{Z}^2$ for every $n \geq 1$.

The fact that $W_n(x)$ has always two elements is true for every Sturmian shift, and thus also that $G_n(x) = \mathbb{Z}^2$ for all $n \geq 1$, as we have already seen in Example 4.3.3.

5.4.2 Dimension groups of episturmian shifts

We now use return words to describe the dimension group of an arbitrary strict episturmian shift. Recall from Section 2.5 that if $s$ is a standard episturmian shift there is a word $x = a_0a_1 \cdots$ called its directive word such that $s = \text{Pal}(x)$. Moreover, the words $u_n = \text{Pal}(a_0 \cdots a_{n-1})$ are the palindrome prefixes of $s$ and the set of left return words to $u_n$ is, by Equation (2.7.3),

$$R_X(u_n) = \{L_{a_0 \cdots a_{n-1}}(a) \mid a \in A\}$$

Denote by $M_a$ the incidence matrix of the morphism $L_a$. Thus, if $A = \{a, b\}$, we have

$$M_a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad M_b = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$  

The following result allows us to compute the dimension group of a strict episturmian shift (also called Arnoux-Rauzy shift).

**Proposition 5.4.3** Let $s$ be a strict standard episturmian word on the alphabet $A$, let $x = a_0a_1 \cdots$ be its directive sequence and let $X$ be the shift generated by $s$. The dimension group of $X$ is the direct limit of the sequence

$$\mathbb{Z}^k \overset{M_{a_0}}{\to} \mathbb{Z}^k \overset{M_{a_1}}{\to} \mathbb{Z}^k \overset{M_{a_2}}{\to} \cdots$$

with $k = \text{Card}(A)$.

**Proof.** Set $u_n = \text{Pal}(a_0 \cdots a_{n-1})$ and $\alpha_n = |u_n|$. We set $W_n(s) = R_X(s_{[0, n-1]})$ and we use the notation of the previous section with $s$ in place of $x$. We identify $G_n(s)$ with $\mathbb{Z}^k$ via the bijection $a \mapsto L_{a_0 \cdots a_{n-1}}(a)$ from $A$ onto $W_n(s)$. It is enough to prove that the matrix of the map $i_{\alpha_{n+1}, \alpha_n}$ is the matrix $M_{a_n}$. Note that for every $a \in A$, since $L_{a_0 \cdots a_n}(a) = L_{a_0 \cdots a_{n-1}}(L_{a_n}(a))$, we have

$$L_{a_0 \cdots a_n}(a) = \begin{cases} L_{a_0 \cdots a_{n-1}}(a_n)L_{a_0 \cdots a_{n-1}}(a) & \text{if } a \neq a_n \\ L_{a_0 \cdots a_{n-1}}(a) & \text{otherwise.} \end{cases} \quad (5.4.1)$$

This gives the decomposition of an element of $W_{\alpha_{n+1}}(s)$ as a product of elements of $W_{\alpha_n}(s)$. Consider now $\phi \in G_{\alpha_n}(s)$. We consider $\phi$ as a column vector with components $\phi_a$ for $a \in A$ (via the identification above). Then, by (5.4.1), we have

$$(i_{\alpha_{n+1}, \alpha_n}\phi)(b) = \begin{cases} \phi_{a_n} + \phi_b & \text{if } b \neq a_n \\ \phi_b & \text{otherwise.} \end{cases}$$
This shows that \( i_{\alpha_{n+1}, \alpha_n} \phi = M_{\alpha_n} \phi \) and completes the proof.

We give two examples with the Fibonacci shift (which is Sturmian and thus we already know its dimension group by Theorem 4.9.4) and the Tribonacci shift.

**Example 5.4.4** Let \( s \) be the Fibonacci word, which generates the Fibonacci shift \( X \) (see Example 2.4.1). The directive word of \( s \) is \( x = (ab)\omega \). Indeed, one has by Justin’s Formula \( x = L_{ab}(x) \) whence the result since \( L_{ab} = \varphi^2 \) where \( \varphi \) is the Fibonacci morphism. It follows from Proposition 5.4.3 that the dimension group of \( X \) is the ordered group \( \Delta_M \) of the matrix \( M = M_b M_a = M(\varphi)^2 \) where \( \varphi \) is the Fibonacci morphism. Thus we prove again that the dimension group of \( X \) is the group of algebraic integers \( \mathbb{Z} + \frac{1+\sqrt{5}}{2} \mathbb{Z} \), in agreement with Theorem 4.9.4.

**Example 5.4.5** Let now \( s \) be the Tribonacci word which is the fixed point of the morphism \( \varphi : a \mapsto ab, b \mapsto ac, c \mapsto a \) (see Example 2.5.4). Its directive word is, as we have seen, \( x = (abc)\omega \). Thus the dimension group of the Tribonacci shift \( X \) generated by \( s \) is the group \( \Delta_M \) of the incidence matrix \( M \) of the morphism \( \varphi \). We have

\[
M = \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\]

The dominant eigenvalue is the positive real number \( \lambda \) such that \( \lambda^3 = \lambda^2 + \lambda + 1 \). A corresponding row eigenvector is \( [\lambda^2, \lambda, 1] \). Thus \( K^0(X, S) \) is isomorphic to \( \mathbb{Z}[\lambda] \).

### 5.5 Dimension groups and Rauzy graphs

We now show how to use Rauzy graphs to compute the dimension group of a minimal shift space. We begin with considerations valid for all graphs. We use the fundamental group \( G(\Gamma) \) of a connected graph \( \Gamma \) to define the ordered cohomology group of a connected graph. We then define the notion of Rauzy graphs associated to a recurrent shift space \((X, S)\). We show that the direct limit sequence of ordered cohomology groups of the Rauzy graphs is the group \( K^0(X, S) \) (Proposition 5.5.6).

#### 5.5.1 Group of a graph

Let \( \Gamma = (V, E) \) be a finite oriented graph with \( V \) as set of vertices and \( E \) as set of edges. We have already met basic notions concerning graphs such as paths or cycles, but we recall them now for convenience (see also Appendix D where more details are given). For an edge \( e \in E \), we denote by \( s(e) \) its source (also called its origin) and by \( r(e) \) its range (also called its end). There may be several edges with the same source and range (thus, \( G \) is actually a multigraph).
Two edges \( e, f \) are \textit{consecutive} if the range of \( e \) is the source of \( f \). A \textit{path} in \( \Gamma \) is a sequence of consecutive edges. A \textit{cycle} is a path \( (e_1, \ldots, e_n) \) such that the source of \( e_1 \) is the range of \( e_n \). To every path \( p = (e_1, \ldots, e_n) \) in \( \Gamma \), we associate its \textit{composition} \( \kappa(p) = e_1 + \ldots + e_n \), which is an element of the free abelian group \( \mathbb{Z}(E) \) on the set \( E \). The \textit{group of cycles} of \( \Gamma \), denoted \( \Sigma(\Gamma) \), is the subgroup of \( \mathbb{Z}(E) \) spanned by the compositions of the cycles of \( \Gamma \).

The elements of \( \mathbb{Z}(E) \) can be represented by row vectors indexed by \( E \). Thus, the elements of \( \Sigma(\Gamma) \) are also represented by row vectors indexed by \( E \).

We consider, for \( v \in V \), the \textit{fundamental group} \( G(\Gamma, v) \) of \( \Gamma \) (see Appendix D). When \( \Gamma \) is strongly connected, the group \( \Sigma(\Gamma) \) is the abelianization of any of the groups \( G(\Gamma, v) \). Indeed, any cycle \( p = p_s(e)q_{r(e)}^{-1} \) can be written \( (p_s(e)q)_{r(e)}^{-1} \) where \( q \) is a path from \( r(e) \) to \( v \) and then \( \kappa(p) = \kappa(p_s(e)q) - \kappa(p_{r(e)}q) \). Moreover, if \( p \) is a cycle around \( v' \), let \( q \) be a path from \( v \) to \( v' \). Then \( r = qpq^{-1} \) is a cycle around \( v \) and \( \kappa(r) = \kappa(p) \).

Let \( C(\Gamma) = \text{Hom}(\Sigma(\Gamma), \mathbb{Z}) \) and let \( C^+(\Gamma) \) be the submonoid of \( C(\Gamma) \) formed by the morphisms giving a nonnegative value to every cycle of \( \Gamma \). It will be convenient, given a basis \( B \) of \( \Sigma(\Gamma) \), to represent an element of \( C(\Gamma) \) as a column vector indexed by \( B \).

The coboundary homomorphism \( \partial : \mathbb{Z}^V \to \mathbb{Z}^E \) is defined by
\[
(\partial \phi)(e) = \phi(r(e)) - \phi(s(e))
\]
for every \( \phi \in \mathbb{Z}^V \) and \( e \in E \). We denote \( H(\Gamma) = \mathbb{Z}^E / \partial \mathbb{Z}^V \) and \( H^+(\Gamma) = \mathbb{Z}^E_+ / \partial \mathbb{Z}^V \).

We identify \( \mathbb{Z}^E \) with \( \text{Hom}(\mathbb{Z}(E), \mathbb{Z}) \) by duality. Thus, it will be convenient to consider the elements of \( \mathbb{Z}^E \) as column vectors indexed by \( E \).

**Example 5.5.1** Consider the graph \( \Gamma \) of Figure 5.5.1.

![Figure 5.5.1: A connected graph.](image)

We use the basis \( \{e, fg, fhf^{-1}\} \) of \( G(\Gamma, 1) \) and the corresponding basis \( B = \{e, f + g, h\} \) of \( \Sigma(\Gamma) \). Thus \( \Sigma(\Gamma) \) is formed of integer row vectors of size 3 and is isomorphic to \( \mathbb{Z}^3 \), while \( C(\Gamma) \) is formed of integer column vectors of the same size. The matrix of the coboundary map is the \textit{incidence matrix} of the graph \( \Gamma \), which is the \( E \times V \) matrix defined by
\[
D_{e,v} = \begin{cases} 
1 & \text{if } v = r(e) \text{ and } v \neq s(e) \\
-1 & \text{if } v = s(e) \text{ and } v \neq r(e) \\
0 & \text{otherwise.}
\end{cases}
\]
In our example, we find
\[
D = \begin{pmatrix}
1 & e & f & g & h \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0
\end{pmatrix}
\]

The group \( \partial \mathbb{Z}^V \) is the group generated by the rows of the matrix \( D \) and thus it is, in this example, isomorphic to \( \mathbb{Z}^3 \). Accordingly, the group \( H(\Gamma) \) is isomorphic to \( \mathbb{Z}^3 \).

Let \( \Gamma = (V, E) \) be a strongly connected graph and let \( \rho : \mathbb{Z}^E \to C(\Gamma) \) be the morphism assigning to an element of \( \text{Hom}(\mathbb{Z}(E), \mathbb{Z}) \) its restriction to \( \Sigma(\Gamma) \). Then \( \rho \) is a positive morphism such that \( \rho(\mathbb{Z}_+^E) = C_+(\Gamma) \). Given a basis \( B \) of \( \Sigma(\Gamma) \), the matrix of the morphism \( \rho \) is the matrix having as rows the elements of \( B \) (considered as row vectors indexed by \( E \)).

The following statement is just the dual of the classical statement that the sequence \( 0 \to \mathbb{Z}(\Gamma) \xrightarrow{\kappa} \mathbb{Z}(E) \xrightarrow{\beta} \mathbb{Z}(V) \xrightarrow{\gamma} \mathbb{Z} \to 0 \), where \( \kappa \) is the composition map, \( \beta(e) = r(e) - s(e) \) and \( \gamma(v) = 1 \) identically, is exact (Exercise 5.4). We give, however, a direct proof for the sake of clarity.

**Proposition 5.5.2** For every strongly connected graph \( \Gamma \), the sequence
\[
0 \to \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z}^V \xrightarrow{\partial} \mathbb{Z}^E \xrightarrow{\rho} C(\Gamma) \to 0
\]
where \( \gamma(i) \) is the constant map equal to \( i \), is exact and there is an isomorphism from \( C(\Gamma) \) onto \( H(\Gamma) \) sending \( \mathbb{Z}_+^E \) onto \( H_+(\Gamma) \).

**Proof.** The equality \( \text{Im}(\gamma) = \ker(\partial) \) results from the hypothesis that \( \Gamma \) is strongly connected.

Let us now show that \( \text{Im}(\partial) = \ker(\rho) \). If \( \theta \) belongs to \( \mathbb{Z}^V \) and \( \psi = \partial \theta \), then \( \psi(\kappa(p)) = \theta(v') - \theta(v) \) for every path \( p \) from \( v \) to \( v' \) and thus \( \rho(\psi) = 0 \). Conversely, suppose that \( \rho(\psi) = 0 \). Let \( T \) be a spanning tree of \( \Gamma \) rooted at \( v \) and let \( p_w \) be the path in \( T \) from \( v \) to \( w \), for every \( w \in V \). Let \( \theta \in \mathbb{Z}^E \) be defined by \( \theta(w) = \psi(\kappa(p_w)) \). Then it is easy to verify that \( \psi = \partial \theta \) and thus that \( \psi \) belongs to \( \text{Im}(\partial) \).

We now verify that \( \rho \) is surjective. Let \( T \) be a spanning tree of \( \Gamma \) rooted at \( v \in V \) and let \( B' \) be the corresponding basis of \( \mathbb{G}(\Gamma, v) \) given by (D.3.1). Then the set \( B = \{ \kappa(p) \mid p \in B' \} \) is a basis of \( C(\Gamma) \). It is enough to show that for each \( \pi = \kappa(p) \in B \) there is some \( \psi \in \mathbb{Z}^E \) such that \( \rho(\psi) \) is the unit vector \( u_\pi \) (recall that the elements of \( C(\Gamma) \) are column vectors indexed by \( B \)). Let \( e \in E \setminus T \) be the unique edge such that \( p = p_\pi e p_{r(e)}^{-1} \). Let \( \psi \in \mathbb{Z}^E \) be the characteristic function of \( e \). Then clearly \( \rho(\psi) = u_\pi \). Since \( \psi \in \mathbb{Z}_+^E \), this shows also that \( \rho(\mathbb{Z}_+^E) = C_+(\Gamma) \).

Let \( 1_\Gamma \) be the function which associates to each cycle its length. The triple \( (C(\Gamma), C_+(\Gamma), 1_\Gamma) \) is an ordered group called the ordered cohomology group associated to the graph \( \Gamma = (V, E) \).
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Example 5.5.3 Consider again the graph $\Gamma$ of Figure 5.5.1. As in Example 5.5.1 we use the basis $\{e, fg, fh^{-1}\}$ of $G(\Gamma, 1)$ and the corresponding basis $B = \{e, f + g, h\}$ of $\Sigma(\Gamma)$. Thus $\Sigma(\Gamma)$ is isomorphic to $\mathbb{Z}^3$. The corresponding matrix of the morphism $\rho$ is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The rows of $P$ are the coefficients of the elements of $B$ in $\mathbb{Z}(E)$. The group $H(\Gamma)$ is the group of linear maps from $\Sigma(\Gamma)$ to $\mathbb{Z}$ and thus it is isomorphic to $\mathbb{Z}^3$. The submonoid $H_+(\Gamma)$ is formed by the non-negative linear maps on the space generated by $B$. Thus it is isomorphic to $\mathbb{N}^3$ and the ordered cohomology group of $\Gamma$ is $\mathbb{Z}^3$ with the natural ordering. The order unit is the vector $[1 \ 2 \ 1]^t$.

We give below an example of a graph with an ordered cohomology group which is not isomorphic to $\mathbb{Z}^n$ with the natural ordering.

Example 5.5.4 Let $\Gamma$ be the graph represented in Figure 5.5.2.

![Figure 5.5.2: The graph $\Gamma$.](image)

We take this time $\{e + g, f + g, f + h\}$ as basis of $\Sigma(\Gamma)$. Thus the matrix $P$ is

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The ordered cohomology group is again $\mathbb{Z}^3$ but this time $H_+(\Gamma) = \{(\alpha, \beta, \gamma) \in \mathbb{Z}_+^3 \mid \alpha + \gamma \geq \beta\}$. Indeed, let $\theta = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$ be an element of $H(\Gamma)$. Then $\theta \in H_+(\Gamma)$ if and only if its value on any cycle of $\Gamma$ is nonnegative, that is, if and only if $\alpha, \beta, \gamma \geq 0$ and

$$\theta(e + h) = \theta(e + g) - \theta(f + g) + \theta(f + h) = \alpha - \beta + \gamma$$

is non-negative, that is, if $\alpha + \gamma \geq \beta$. The order unit is $[2 \ 2 \ 2]^t$. The ordered group is not a Riesz group because, in $H(\Gamma)$, the sum of the first two columns of $P$ is equal to the sum of the two last ones, although none of these vectors can be written as a sum of positive vectors.
5.5.2 Rauzy graphs

Let $X$ be a shift space on the alphabet $A$. Recall from Section 2.2.5 that the Rauzy graph of $X$ of order $n$ is the graph $\Gamma_n(X)$ with $L_{n-1}(X)$ as set of vertices and $L_n(X)$ as set of edges. The edge $w$ goes from $u$ to $v$ if $w = ua = bv$ with $a, b \in A$.

Example 5.5.5 Let $X$ be the Fibonacci shift. The Rauzy graphs of order $n = 1, 2, 3$ are represented in Figure 5.5.3 (with the edge from $w = ua$ labeled $a$).

![Figure 5.5.3: The Rauzy graphs of order $n = 1, 2, 3$ of the Fibonacci shift.](image)

There is a positive morphism from the group $C(\Gamma_n(X))$ to the group $C(\Gamma_{n+1}(X))$. Indeed, the prefix map defines a projection from the graph $\Gamma_{n+1}(X)$ onto the graph $\Gamma_n(X)$. The set of edges of the first one is mapped onto the set of edges of the second one and the set of cycles onto the set of cycles. It follows that this projection defines a positive morphism from $\Sigma(\Gamma_{n+1}(X))$ to $\Sigma(\Gamma_n(X))$ and by duality from $C(\Gamma_{n+1}(X))$ to $C(\Gamma_n(X))$.

Proposition 5.5.6 For any recurrent shift space $X$, the unital ordered group $K^0(X,S)$ is the direct limit of the ordered groups associated with its Rauzy graphs.

Proof. Since the set of vertices of $\Gamma_n(X)$ is $L_{n-1}(X)$ and its set of edges is $L_n(X)$, we can identify $H(\Gamma_n(X))$ to $G_n(X)$ (defined by (4.5.3)) and $H^+(\Gamma_n(X))$ to $G^+_n(X)$. The unital ordered group $(C(\Gamma_n(X)), C_+(\Gamma_n(X)), 1_{\Gamma_n(X)})$ can be identified, by Proposition 5.5.2, with $(G_n(X), G^+_n(X), 1_n(X))$. Since the morphism from $C(\Gamma_n(X))$ to $C(\Gamma_{n+1}(X))$ induced by taking the prefixes is the same as the morphism $i_{n+1,n}$ from $G_n(X)$ to $G_{n+1}(X)$, the result follows from Proposition 5.4.1.

Example 5.5.7 We consider again the Fibonacci shift $X$. We already know that its dimension group is $\mathbb{Z}[\alpha]$ with $\alpha = (1 + \sqrt{5})/2$ (see Examples 5.4.2 and 5.4.4). We will not prove it again but our point is to put in evidence the isomorphisms used in the proof of Proposition 5.5.6.

The prefix $p_n$ of length $n$ of the Fibonacci word $x$ is the vertex labeled 1 in $\Gamma_{n+1}(X)$ in Figures 5.5.3 and 5.5.4. Thus, the elementary cycles around 1 in
Figure 5.5.4: The Rauzy graphs of order 4, 5, 6 of the Fibonacci shift.

\[ \Gamma_{n+1}(X) \] are labeled by the right return words to \( p_n \) (conjugate to the left return words forming the set denoted \( W_n(x) \) in Example 5.4.2). For example, the vertex 1 of \( \Gamma_5(X) \) is the word \( p_4 = abaa \). The elementary cycles around 1 are labeled by \( baa \) and \( babaa \), which are conjugate to the elements of \( W_4(x) = \{aba, abaaab\} \) by the conjugacy \( u \to (abaa)u(abaa)^{-1} \).

The prefix map from \( \Gamma_4(X) \) to \( \Gamma_3(X) \) sends the vertices 2 and 4 to the vertex \( ba \). Since \( W_2(x) = W_3(x) \), the morphism from \( \Sigma(\Gamma_3(X)) \) to \( \Sigma(\Gamma_4(X)) \) induced by the prefix map is the identity. In contrast, the morphism from \( \Sigma(\Gamma_4(X)) \) to \( \sigma(\Gamma_5(X)) \) is given by the matrix

\[
\begin{bmatrix}
aba & ab & aba \\
aba & 0 & 1 \\
abaab & 1 & 1
\end{bmatrix}
\]

where the rows are indexed by the words of \( W_4(X) \), in bijection with the (composition of the) cycles around 1 in \( \Gamma_5(X) \). The columns are indexed by the words of \( W_3(X) \). Finally, the morphism from \( C(\Gamma_4(X)) \) to \( C(\Gamma_5(X)) \) is obtained by transposition of this matrix.

5.6 Dimension groups of substitution shifts

We show in this section how to compute the dimension group of a substitutive shift.

We begin by establishing a connection between the two block presentation of the shift space and its Rauzy graph of order two.

5.6.1 Two-block presentation and Rauzy graph

Let \( \varphi : A \to A^* \) be a substitution and let \( X \) be the shift space associated to \( \varphi \).

We have seen that there is a matrix \( M \) associated to \( \varphi \), called its composition matrix. Unfortunately, the composition (or incidence) matrix of the substitution does not determine the ordered group of the substitution shift associated to \( \varphi \), as shown by the following example. We show below that it is however determined
by the incidence matrix $M_2$ of the 2-block presentation $\varphi_2$ of $\varphi$ (see Section 2.4 for the definition of the matrix $M_2$).

**Example 5.6.1** Let $(X,S)$ be the Morse shift. The matrix $M$ associated to the Morse substitution is

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

and it is the same as for the substitution $a \mapsto ab, b \mapsto ab$. The shift space corresponding to the second substitution has two elements and its dimension group is $\mathbb{Z}$. The dimension group of the Morse shift is not isomorphic to $\mathbb{Z}$. Indeed (see Example 4.8.16), one has $\mu([aa]) \neq \mu([ab])$ for the unique invariant measure on $(X,S)$ and thus, by Proposition 4.9.1, the difference $\chi_{[aa]} - \chi_{[ab]}$ of the characteristic functions of the cylinders $[ab]$ and $[ba]$ is not a coboundary (we will see shortly that actually the group $H(X,S,\mathbb{Z})$ is isomorphic to $\mathbb{Z}[1/2] \times \mathbb{Z}$). Thus the dimension groups are not the same for the two shift spaces.

We prove a statement connecting the endomorphism of $R_2(X)$ defined by the matrix $M_2$ with the fundamental group $\Sigma(\Gamma_2(X))$ of the Rauzy graph $\Gamma_2(X)$. Recall that $R_n(X)$ denotes the group of maps from $L_n(X)$ to $\mathbb{R}$ and that $M_2(X)$ operates on the left on the elements of $R_2(X)$ considered as column vectors.

Recall from Section 4.5 that the map $\partial_1 : Z_1(X) \to Z_2(X)$ is the morphism defined by $\partial_1 \varphi(ab) = \varphi(b) - \varphi(a)$. Since $L_1(X)$ (resp. $L_2(X)$) is the set of vertices (resp. of edges) of the graph $\Gamma_2(X)$, it coincides with the map $\partial$ of Section 5.5.1 and we shall denote it simply $\partial$.

Let $\tau : A \to A$ be the map which sends each letter $a \in A$ to the first letter of $\varphi(a)$. We define an endomorphism $I$ of $R_1(X)$ by

$$(I\varphi)(a) = \varphi(\tau(a))$$

for every $\varphi \in R_1(X)$ and $a \in A$.

Let $P$ be a matrix whose rows form a basis of the group $\Sigma(\Gamma_2(X))$. The matrix $P$ is the matrix of the morphism $\rho$ in Proposition 5.5.2.

We may choose $P$ as a nonnegative matrix by selecting a set of cycles whose composition form the rows of $P$.

Moreover these cycles may be chosen to correspond to return words to a vertex $a \in A$, where $a$ is such that $\varphi(a)$ begins with $a$ (we may always find such $a$, up to replacing $\varphi$ by some power). Indeed, by Proposition 8.1.21 applied with $u = a$, there is an integer $n$ such that the set labels of paths in $\Gamma_{n+1}(X)$ to a word $x \in L_n(X)$ ending with $a$ is generated by $R_X(a)$. This implies, by projection from $\Gamma_{n+1}(X)$ to $\Gamma_2(X)$, that every cycle from $a$ to $a$ in $\Gamma_2(X)$, is a product of return words to $a$. Thus the compositions of return words to $a$ generate the group $\Sigma(\Gamma_2(X))$.

We illustrate this on an example.

**Example 5.6.2** Let $A = \{a, b, c, d\}$ and let $\varphi : a \to ab, b \to cda, c \to cd, d \to abc$ (we shall consider again this substitution below and later in Example 8.1.4).
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Figure 5.6.1: The graph $\Gamma_2(X)$.

We have $L_2(X) = \{ab, ac, bc, ca, cd, da\}$ and the graph $\Gamma_2(X)$ is represented in Figure 5.6.1.

We have $R_X(a) = \{cda, bcda, bca\}$. The choice for the matrix $P$ is then

$$P = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}$$

For example, the first row corresponds to the return word $bca$ since it gives the sequence $(ab)(bc)(ca)$ of 2-blocks.

**Proposition 5.6.3** We have $M_2 \circ \partial = \partial \circ I$, so that $\partial(R_1(X))$ and $\partial(Z_1(X))$ are invariant by $M_2$. Moreover, assuming that the rows of $P$ correspond to return words to a letter $a \in A$ such that $\varphi(a)$ begins with $a$, there is a unique nonnegative matrix $N_2$ such that $PM_2 = N_2P$.

**Proof.** We first prove that $M_2 \circ \partial = \partial \circ I$. Let $\phi \in R_1(X)$. Then

$$(\partial \circ I \circ \phi)(ab) = (\partial \circ \phi \circ \tau)(ab) = \phi \circ \tau(b) - \phi \circ \tau(a) = \phi(\tau(b)) - \phi(\tau(a)).$$

To evaluate $(M_2 \circ \partial \phi)(ab)$, set $\varphi(a) = a_1a_2\cdots a_k$ and $\varphi(b) = b_1b_2\cdots b_{\ell}$. Then, identifying $A_2$ and $L_2(X)$, we have

$$\varphi_2(ab) = (a_1a_2)(a_2a_3)\cdots(a_kb_1).$$

Next, considering accordingly $M_2$ as an endomorphism of $R_2(X)$, we have for any $\psi \in R_2(X)$, using Equation (5.6.1),

$$(M_2\psi)(ab) = \psi(a_1a_2) + \psi(a_2a_3) + \cdots + \psi(a_kb_1).$$

We obtain

$$(M_2\partial \phi)(ab) = \phi(a_2) - \phi(a_1) + \phi(a_3) - \phi(a_2) + \cdots + \phi(b_1) - \phi(a_k) = \phi(b_1) - \phi(a_1) = \phi(\tau(b)) - \phi(\tau(a))$$

and thus the conclusion.
Since $M_2 \circ \partial = \partial \circ I$, the subgroups $\partial(R_1(X))$ and $\partial(Z_1(X))$ are invariant by $M_2$.

Let $p = (a_1a_2)(a_2a_3) \cdots (a_{n-1}a_n)(a_na_1)$ be a cycle in $\Gamma_2(X)$ which is a row of $P$ and thus with $a_1 = a$. We can consider $p$ as a word on the alphabet $L_2(X)$ and compute its image by the morphism $\varphi_2$. Since the rows of $P$ correspond to return words to $a$, the first letter of $\varphi(a_1)$ is $a$. Then the first element of $\varphi_2(a_1a_2)$ is equal to $ab$ for some $b \in A$, and the last element of $\varphi_2(a_na_1)$ is equal to $ca$ for some $c \in A$ (as we have seen above computing $\varphi_2(ab)$). It follows that $\varphi_2(p)$ is a cycle around $a$ in $\Gamma_2(X)$ and thus a composition of return words to $a$. By the choice of $P$, it is a nonnegative combination of rows of $P$.

This implies that $PM_2 = N_2P$ where $N_2$ is the matrix of the map sending the composition of a cycle $p$ on the composition of $\varphi_2(p)$.

Note that the two assertions of Proposition 5.6.3 are related. Indeed, the matrix of the map $\partial$ is the transpose of the incidence matrix $D$ of the graph $\Gamma_2(X)$. The rows of $D$ generate the orthogonal of the space generated by the rows of $P$. It is thus equivalent to say the space generated by the rows of $P$ is invariant by $M_2$ and that the space $\partial(R_1(X))$ is invariant by $M_2$. This duality is described in more detail in the following example.

**Example 5.6.4** Consider again the substitution shift of Example 5.6.2. Using the bijection $\{ab, ac, bc, ca, cd, da\} \to \{x, y, z, t, u, v\}$, we find $\varphi_2 : x \to xz, y \to xz, z \to wvy, t \to uv, u \to uvx, v \to xzt$. The matrices $M, M_2$ are

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The matrices $P$ and $N_2$ are then

$$P = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

so that $N_2$ has dimension less than $M$. The matrix $D$ is

$$D = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} ab & ac & bc & ca & cd & da \\ -1 & -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

One may easily verify that the rows of $D$ generate a vector space of dimension 3 orthogonal to the space generated by the rows of $P$. 
Example 5.6.5 Let $\varphi : a \mapsto ab, b \mapsto a$ be the Fibonacci substitution and let $X$ be the Fibonacci shift. Let $f : x \mapsto aa, y \mapsto ab, z \mapsto ba$ be a bijection from $A_2 = \{x, y, z\}$ onto $L_2(X) = \{aa, ab, ba\}$. The morphism $\varphi_2$ is $x \mapsto yz, y \mapsto yz, z \mapsto x$. The associated matrices $M$ and $M_2$ and the vector $v$ spanning the group $\partial_1(R_1(X))$ are (see Example 5.4.4)

$$
M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
$$

The Rauzy graph $\Gamma_2(X)$ is represented in Figure 5.6.2 with the edges labeled by the corresponding element of $A_2$.

![Figure 5.6.2: The Rauzy graph $\Gamma_2(X)$.](image)

The matrices $P$ corresponding to the basis $\{x, yz\}$ and the matrix $N_2$ associated with the action of $\varphi_2$ on the rows of $P$ are

$$
P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.
$$

Thus we conclude that $N_2 = M$ in this case.

### 5.6.2 Dimension group of a substitution shift

Recall first from Section 3.3 that, for a $d \times d$ matrix $M$, $R_M$ denotes the eventual range of $M$ and $K_M$ its eventual kernel.

For $M$ nonnegative, we have defined a unital ordered group $(\Delta_M, \Delta_M^+, 1_M)$. By Equation (3.3.2), the group $\Delta_M$ is given by

$$
\Delta_M = \{v \in R_M \mid \text{for some } k \geq 1, M^k v \in \mathbb{Z}^d\},
$$

with positive cone

$$
\Delta_M^+ = \{v \in R_M \mid \text{for some } k \geq 1, M^k v \in \mathbb{Z}^d_+\}
$$

and order unit $1_M$ equal to the projection on $R_M$ along $K_M$ of the vector with all its components equal to 1.

The following statement gives a surprisingly simple way to compute the dimension group of a substitution shift. Indeed, it shows that the dimension group of a primitive substitution shift is defined by the matrix $N_2$ of Proposition 5.6.3 and the nonnegative matrix $P$ such that $PM_2 = N_2 P$. 
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**Proposition 5.6.6** Assume that $\varphi$ is a primitive substitution and that the shift $X$ associated to $\varphi$ is infinite. The dimension group $K^0(X, S)$ is isomorphic to $(\Delta_{N_2}, \Delta^+_N, P_{M_2})$.

Note that the order unit is not the vector $1_{N_2}$ but the projection on $G_2(X)$ of the vector $1_{M_2}$ (see Example [5.6.11]). The proof relies on several lemmas.

We first introduce a sequence of partitions in towers associated with the primitive morphism $\varphi$.

**Lemma 5.6.7** The sequence $(\mathfrak{P}(n))_{n \geq 0}$ with

$$\mathfrak{P}(n) = \{S^j \varphi^n([ab]) \mid ab \in \mathcal{L}_2(X), 0 \leq j < |\varphi^n(a)|\}. \quad (5.6.2)$$

is a nested sequence of partition in towers of $X$.

**Proof.** Let $f : \mathcal{L}_2(X) \to A_2$ be a bijection and let $X^{(2)}$ be the second higher block presentation of $X$. We consider the sequence $(\Omega(n))$ of partitions in towers of the second higher block presentation $X^{(2)}$ of $X$ associated with the 2-block presentation $\varphi^n_2$ of $\varphi^n$ as in Proposition ???. Thus

$$\Omega(n) = \{S^j \varphi^n_2([u]) \mid u \in A_2, 0 \leq j < |\varphi^n_2(u)|\}.$$ 

Let $\pi : A_2 \to A$ be the morphism assigning to $u \in A_2$ the first letter of $f(u)$. The extension of $\pi$ to $A^*_2$ defines an isomorphism from $X^{(2)}$ onto $X$. Let $(\mathfrak{P}(n))$ be the image of the sequence $(\Omega(n))$ by the isomorphism $\pi$. Since $\pi([u]) = [ab]$ when $f(u) = ab$, the partition $\mathfrak{P}(n)$ is given by Equation (5.6.2). By Proposition 5.1.6 the sequence $\mathfrak{P}(n)$ is nested.

Denote $G(n) = G(\mathfrak{P}(n))$, $G^+(n) = G^+(\mathfrak{P}(n))$ and $1_n = 1_{\mathfrak{P}(n)}$. Let $(G, G^+, 1)$ be the inductive limit of the sequence $(G(n), G^+(n), 1_n)$ with the morphisms $I(n+1, n) = I(\mathfrak{P}(n+1), \mathfrak{P}(n))$.

**Lemma 5.6.8** The map from $Z_2(X)$ to $C(X, \mathbb{Z})$ sending $\phi \in Z_2(X)$ to the map equal to $\phi(ab)$ on the cylinder $[ab]$ defines an isomorphism of unital ordered groups from $(\Delta_{M_2}, \Delta^+_{M_2}, 1_{M_2})$ onto $(G, G^+, 1)$.

**Proof.** By associating to each $\phi \in Z_2(X)$ the function equal to $\phi(ab)$ on $\varphi^n([ab])$, we can identify $G(n)$ to $Z_2(X)$, $G^+(n)$ to $Z^*_2(X)$ and $1_n$ to the map $ab \mapsto |\varphi^n(a)|$. Given $k$ with $0 \leq k < |\varphi^n(a)|$, we have (see Figure 5.6.3)

$$S^k(\varphi^{n+1}([ab]) \subset \varphi^n([cd]) \iff \left\{ \begin{array}{ll} \text{there exists } \ell \text{ with } 0 \leq \ell < |\varphi(a)| \text{ such that } \\ S^\ell([ab]) \subset [cd] \text{ and } k = |\varphi^n(\varphi([ab])**{[0, \ell-1]})|. \end{array} \right.$$ 

Therefore, for every $u, v \in A_2$ with $f(u) = ab$ and $f(v) = cd$, the number

$$\text{Card}\{k \mid 0 \leq k < |\varphi^n(a)|, S^k \varphi^{n+1}([ab]) \subset \varphi^n([cd])\}$$

is the number of occurrences of $v$ in $\varphi_2(u) = (M_2)_u$. Consequently, we may identify the morphism $I(n+1, n)$ to the morphism $M_2 : Z_2(X) \to Z_2(X)$. 


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Figure 5.6.3: Representing \( S^k(\varphi^{n+1}([ab])) \subset \varphi^n([cd]) \)

Thus, the inductive limit \((G, G^+, 1)\) associated to the sequence \((\mathcal{P}(n))\) of partitions in towers can be identified to \((\Delta M_2, \Delta M_2^+, 1 M_2)\).

Recall from Proposition 5.2.1 that there is a morphism \(\pi(n) : G(n) \to H(X, S, Z)\) making the diagram of Figure 5.6.4 commutative and from Lemma 5.3.2 that there is a unique morphism \(\sigma : (G, G^+, 1) \to K_0(X, S)\) such that \(\sigma \circ I(n) = \pi(n)\) for every \(n \geq 0\).

Recall also from Section 4.5.1 that \(\partial_1 : R_1(X) \to R_2(X)\) is the morphism defined by \((\partial_1 \phi)(ab) = \phi(b) - \phi(a)\).

**Lemma 5.6.9** The morphism \(\sigma : G \to H(X, S, Z)\) is onto and its kernel is \(\Delta M_2 \cap \partial_1(Z_1(X))\).

**Proof.** We first show that the morphism \(\sigma\) is surjective. Since every function in \(C(X, Z)\) is cohomologous to a cylinder function by Proposition 4.4.1, it is enough to consider a cylinder function \(\phi\) associated to \(\phi \in Z_2(X)\). Choose \(n\) so large that \(|\varphi^n(a)| > k\) for every letter \(a \in A\). Since all elements of the atoms of the partition \(\mathcal{P}_n\) have the same prefix of length \(k\), the cylinder function \(\phi\) is constant on every element of the partition \(\mathcal{P}_n\) and thus belongs to \(C(n)\). Its image by \(I(n)\) is sent by \(\pi(n)\) to the class of \(\phi\) modulo the coboundaries (see Figure 5.6.4).

![Diagram](image)

Figure 5.6.4: The morphism \(\pi(n)\).

We now prove that the kernel of \(\sigma\) is \(\Delta M_2 \cap \partial_1(Z_1(X))\). Let \(\phi\) be in \(G\), or equivalently in \(\Delta M_2\). In particular, \(\phi\) belongs to \(R_2(X)\) and there exists \(k \geq 0\) such that \(M_2^k \phi \in Z_2(X)\). Assume that \(\sigma(\phi) = 0\). By definition this means
that \( \pi(\tilde{\phi}) = 0 \) where \( \pi : C(n) \to H(X, S; \mathbb{Z}) \) is the natural projection and \( \tilde{\phi} \) is defined by

\[
\tilde{\phi}(x) = \begin{cases} 
(M^k_2 \phi)(ab) & \text{if } x \in \varphi^k([ab]) \text{ for some } ab \in \mathcal{L}_2(X) \\
0 & \text{otherwise.}
\end{cases}
\]

Note that, as in the proof of Proposition 5.6.3, we identify \( A_2 \) with \( \mathcal{L}_2(X) \) and consequently consider \( M_2 \) as an endomorphism of \( R_2(X) \).

Let \( g \in C(X, \mathbb{Z}) \) be such that \( \tilde{\phi} = g \circ S - g \). We claim that there exists \( n \geq 0 \) such that \( g \) is constant on the set \( \varphi^n([ab]) \) for every \( ab \in \mathcal{L}_2(X) \). Since \( g \) is continuous, it is locally constant and there is an \( m \geq 1 \) such that \( g(x) \) depends only on \( x_{[-m,m]} \). Choose \( n > k \) so large that \( |\varphi^n(a)| > m \) for every letter \( a \). Let \( ab \in \mathcal{L}_2(X) \) and \( y, z \in \varphi^n([ab]) \). Since \( g \) depends only on \( x_{[0,2m]} \), \( g(S^m x) \) depends only on \( x_{[0,2m]} \). Since \( y, z \in \varphi^n([ab]) \) and \( |\varphi^n(a)|, |\varphi^n(b)| > m \), \( y \) and \( z \) share the same \( 2m \) first coordinates and therefore \( g(S^m x) = g(S^m y) \).

On the other hand, for all \( 0 \leq j < |\varphi^n(a)| \), \( S^j y \) and \( S^j z \) are in the same atom of the partition \( \mathfrak{P}(n) \). Since \( m < |\varphi^n(a)| \) and since \( \tilde{\phi} \) is constant on the atoms of \( \mathfrak{P}(n) \), we obtain \( \tilde{\phi}^j(y) = \tilde{\phi}^j(z) \). Since finally \( g = g \circ S^m - \tilde{\phi}^j \) by Equation (4.1.2), we conclude that \( g(y) = g(z) \).

Let \( \psi \in Z_2(X) \) be such that \( \psi(ab) = g(x) \) for \( x \in \varphi^n([ab]) \). Then if \( x \in \varphi^n([ab]) \) and \( S^i \varphi^n(x) \in \varphi^n([bc]) \), we have with \( \ell = |\varphi^n(a)| \),

\[
\psi(bc) - \psi(ab) = g(S^\ell x) - g(x) = \tilde{\phi}^\ell(x).
\]

Recall that

\[
\tilde{\phi}^\ell(x) = \tilde{\phi}(x) + \tilde{\phi} \circ S(x) + \ldots + \tilde{\phi}(S^{\ell-1}(x))
\]

and note that the term \( \tilde{\phi} \circ S^j(x) \) of this sum is equal to \( (M^k \phi)(cd) \) if there is \( cd \in \mathcal{L}_2(X) \) such that \( S^j \varphi^n([ab]) \subset \varphi^n([cd]) \) and equal to \( 0 \) otherwise. Thus

\[
\tilde{\phi}^\ell(x) = \sum_{cd \in \mathcal{L}_2(X)} \text{Card}\{0 \leq j < \ell \mid S^j \varphi^n([ab]) \subset \varphi^n([cd])\} (M^k_2 \phi)(cd)
\]

\[
= \sum_{cd \in \mathcal{L}_2(X)} M_2^{-k(ab,cd)} (M^k_2 \phi)(cd) = (M^k_2 \phi)(ab).
\]

Consequently, we have for every \( abc \in \mathcal{L}_3(X) \),

\[
\psi(bc) - \psi(ab) = (M^k_2 \phi)(ab).
\]

Choose \( m \) so large that \( |\varphi^m(a)| \geq 2 \) for every \( a \in A \) and define \( \theta \in Z_1(X) \) by

\[
\theta(a) = \psi(a_1 a_2)
\]

for \( a \in A \) if \( \varphi^m(a) = a_1 \cdots a_\ell \).
If \( ab \) belongs to \( L_2(X) \) with \( \varphi^m(a) = a_1 \cdots a_r \) and \( \varphi^m(b) = b_1 \cdots b_s \), we obtain
\[
(M_2^{n+m} \phi)(ab) = (M_2^2 \phi)(a_1a_2) + \cdots + (M_2^2 \phi)(a_r b_1)
\]
\[
= \psi(b_1 b_2) - \psi(a_1 a_2) = \theta(b) - \theta(a)
\]
\[
= (\partial_1 \theta)(ab).
\]

If follows that \( M_2^{n+m} \phi \) belongs to \( \partial_1(Z_1(X)) \). Choosing \( m \) large enough, we may assume that \( M_2^{n+m} \phi \) is in \( R_{M_2} \). Since \( M_2 \) defines an automorphism of \( R_{M_2} \), and since, by Proposition 5.6.3, the subspace \( \partial_1(R_1(X)) \) is invariant by \( M_2 \), we conclude that \( \phi \) is an element of \( \Delta_{M_2} \cap \partial_1(R_1(X)) \). Thus the kernel of \( \sigma \) is included in \( \Delta_{M_2} \cap \partial_1(R_1(X)) \). Since the converse inclusion is obvious, the conclusion follows.

Proof of Proposition 5.6.6. By Lemma 5.6.8, the ordered groups \( (\Delta_{M_2}, \Delta_{M_2}^+, 1_{M_2}) \) and \( (G, G^+, 1) \) can be identified. By Lemma 5.6.9 the morphism \( \sigma \) defines an isomorphism from \( \Delta_{M_2} / (\Delta_{M_2} \cap \partial_1(R_1(X))) \) onto \( H(X, S, Z) \). But since \( PM_2 = N_2 P \), we have also \( PM_2^k = N_2^k P \) for every \( k \geq 1 \). Thus the projection \( v \mapsto Pv \) maps \( \Delta_{M_2} \) onto \( \Delta_{N_2} \) and we obtain
\[
H(X, S, Z) \simeq \frac{\Delta_{M_2}}{\Delta_{M_2} \cap \partial_1(R_1(X))} \simeq \Delta_{N_2}.
\]

Similarly, we have \( H^+(X, S, Z) \simeq \Delta_{N_2}^+ \). Finally, the map \( \sigma \) sends 1 to 1 and we conclude that \( K^0(X, S) \) is isomorphic to \( (\Delta_{N_2}, \Delta_{N_2}^+, P_{1_{M_2}}) \).

We give two examples of computation of the dimension group of a substitution shift. Other examples are treated in the exercises.

Example 5.6.10. Let \( \varphi : a \mapsto ab, b \mapsto a \) be the Fibonacci substitution and let \( (X, S) \) be the Fibonacci shift. As seen in Example 5.6.5 we have \( N_2 = M \) and
\[
M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]

The maximal eigenvalue of \( M \) is \( \lambda = (1 + \sqrt{5})/2 \) and the vector \( [\lambda \ 1] \) is a left eigenvector of \( M \). Thus \( H(X, T, Z) = \mathbb{Z}^2 \) and \( H^+(X, T, Z) = \{ (\alpha, \beta) \mid \alpha \lambda + \beta \geq 0 \} \). The order unit is
\[
P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Thus, using the map \( (\alpha, \beta) \mapsto \alpha \lambda + \beta \), we see that the dimension group of \( (X, S) \) is isomorphic the group of algebraic integers \( \mathbb{Z} + \mathbb{Z} \lambda \) with the order induced by the reals and the order unit \( 2 \lambda + 1 \). (see Example 4.3.6 and Example 4.4.2).
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To obtain a normalized subgroup with 1 as unit, we consider the automorphism of \( \mathbb{Z}^2 \) such that \((\alpha, \beta) \mapsto (\alpha - \beta, 2\beta - \alpha)\). This is actually the automorphism defined by the matrix \( M^{-2} \) since

\[
M^{-2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha - \beta \\ 2\beta - \alpha \end{bmatrix}.
\]

Then \((2, 1)\) maps to \((1, 0)\). Since \(\alpha\lambda + \beta = \lambda^2(\lambda(\alpha - \beta) + (2\beta - \alpha))\), we do not change the order. Thus we conclude that the dimension group of \((X, S)\) is isomorphic with \(\mathbb{Z} + \lambda\mathbb{Z}\) with the order induced by the reals and 1 as order unit.

**Example 5.6.11** Let \(\varphi: a \mapsto ab, b \mapsto ba\) be the Morse substitution on the alphabet \(A = \{a, b\}\) and let \((X, S)\) be the Morse shift. We have \(L_2(X) = A^2\).

Set \(A_2 = \{x, y, z, t\}\) and let \(f: A_2 \to A^2\) be the bijection \(x \mapsto aa, y \mapsto ab, z \mapsto ba, t \mapsto bb\). Then \(\varphi_2\) is the substitution \(x \mapsto yz, y \mapsto yt, z \mapsto zx, t \mapsto zy\). The Rauzy graph \(\Gamma_2(X)\) is represented in Figure 5.6.5.

![Figure 5.6.5: The Rauzy graph \(\Gamma_2(X)\).](image)

The matrices \(M, M_2, P\) and \(N_2\) are

\[
M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]

The eventual range \(R_{N_2}\) of \(N_2\) is generated by the vectors

\[
v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},
\]

the first one being an eigenvector for the maximal eigenvalue 2 and the second one for the eigenvalue \(-1\). Now, for \(\alpha v + \beta w \in R_{N_2}\), since

\[
N_2^k(\alpha v + \beta w) = 2^k \alpha v + (-1)^k \beta w = \begin{bmatrix} 2^k \alpha + (-1)^k \beta \\ 2^{k+1} \alpha + (-1)^{k+1} \beta \\ 2^k \alpha + (-1)^k \beta \end{bmatrix},
\]
we have $N^k_2(\alpha v + \beta w) \in \mathbb{Z}^3$ if and only if $\alpha = \frac{m}{3} = \frac{n}{3}$ and $\beta = \frac{m}{3}$ with $m, n \in \mathbb{Z}$ and $m + n \equiv 0 \mod 3$. The order unit is
\[
P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = v
\]

Thus,
\[
\Delta_{N_2} \simeq \{ (\alpha, \beta) \mid 3\alpha \in \mathbb{Z}[1/2], 3\beta \in \mathbb{Z}, 3\alpha + 3\beta \equiv 0 \mod 3 \},
\]
with
\[
\Delta_{N_2}^+ \simeq \{ (\alpha, \beta) \in \Delta_{N_2} \mid \alpha > 0 \} \cup \{ (0, 0) \}.
\]

The group $\Delta_{N_2}$ is actually isomorphic to $\mathbb{Z}[1/2] \times \mathbb{Z}$, using the map $(\alpha, \beta) \rightarrow (\alpha + \beta, 3\beta)$.

To close the loop, let us express the unique trace on $K^0(X, S)$ which, by Proposition [4.9.3] has the form $\alpha_\mu$ where $\mu$ is the unique invariant probability measure on $(X, S)$ (see Example [4.8.16]). We have
\[
\alpha_\mu(\alpha v + \beta w) = \alpha
\]
since this map is a positive unital morphism from $K^0(X, S)$ to $(\mathbb{R}, \mathbb{R}^+, 1)$. We find for example (in agreement with the value given in Example [4.8.16]) $\mu([aa]) = 1/6$ since the characteristic function of the cylinder $[aa]$ can be identified with the vector
\[
P \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ -1/2 \end{bmatrix} = 1/6(v + 2w) + \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}
\]
were the last expression is the decomposition in $\mathcal{R}_{N_2} \oplus \mathcal{K}_{N_2}$.

### 5.7 Exercises

#### Section [5.1]

5.1 Let $\Psi = \{ T^j B_i \mid 1 \leq i \leq m, 0 \leq j < h_i \}$ be a partition in towers nested in a partition $\Psi' = \{ T^h B'_k \mid 1 \leq k \leq m', 0 \leq h \leq h'_k \}$. Show that if
\[
T^j B_i \subset T^h B'_k
\]
then
\[
0 \leq j - h \leq h_i - h'_k.
\]
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5.2 Let $\sigma$ be the substitution $\sigma : a \mapsto abb, b \mapsto aab$. Let $\mathfrak{P}(n)$ be the partition with basis $\sigma^n(X)$ associated with $\sigma^n$. Let $\mathfrak{P}'(n)$ be the partition obtained by merging the two towers of $\mathfrak{P}(n)$, that is

$$\mathfrak{P}'(n) = \{T^j\sigma^n(X) \mid 0 \leq j < 3^n\}$$

Show that the sequence $\langle \mathfrak{P}'(n) \rangle$ satisfies (KR1), (KR2) but not (KR3).

5.3 Let $\sigma$ be the substitution $a \mapsto acb, b \mapsto bcb, c \mapsto abb$. Let $\mathfrak{P}(n)$ be the partition associated with $\sigma^n$. Show that the sequence $\langle \mathfrak{P}(n) \rangle$ satisfies (KR2) and (KR3) but not (KR1).

Section 5.5

5.4 Let $G$ be a strongly connected graph. Show that the sequence

$$0 \to \mathbb{Z}(\Gamma) \xrightarrow{\kappa} \mathbb{Z}(E) \xrightarrow{\beta} \mathbb{Z}(V) \xrightarrow{\gamma} \mathbb{Z} \to 0,$$

where $\kappa$ is the composition map, $\beta(e) = r(e) - s(e)$ and $\gamma(v) = 1$ identically, is exact.

Section 5.6

5.5 Consider the Chacon ternary substitution $\tau : a \mapsto aabc, b \mapsto bcb, c \mapsto abc$. Show that the dimension group of the associated shift space $(X, S)$ is isomorphic to $\mathbb{Z}[1/3] \times \mathbb{Z}$ with positive cone $\mathbb{Z}_+[1/3] \times \mathbb{Z}$ and unit $(3, -1)$.

5.6 Let $\varphi : a \mapsto ab, b \mapsto ac, c \mapsto a$ be the Tribonacci morphism and let $(X, S)$ be the corresponding substitution shift. Show that the dimension group of $(X, S)$ is the group $\mathbb{Z}[\lambda]$ where $\lambda$ is the positive real solution of $\lambda^3 = \lambda^2 + \lambda + 1$ (see also Example 5.4.5 where we found the same result using return words).

5.8 Solutions

Section 5.1

5.1 Let $B, B'$ be the bases of $\mathfrak{P}, \mathfrak{P}'$. Since $\mathfrak{P}$ is nested in $\mathfrak{P}'$, we have $B \subset B'$. If $h > j$, then $T^{h-j}B_k'$ contains an element of $B \subset B'$, a contradiction. Thus $0 \leq j - h$. Next, set $\ell = h_i - j$. Then $T^{h+j}B_k' \subset B'$ implies $h + \ell \geq h'_k$. Thus $j - h \leq j - (h'_k - \ell) = h_i - h'_k$.

5.2 Since $\sigma$ is primitive and proper and $X(\sigma)$ is not periodic, the sequence $\langle \mathfrak{P}(n) \rangle$ is a refining sequence of partitions in towers by Lemma 7.2.4 (to be proved in Chapter 7). The sequence $\mathfrak{P}'(n)$ satisfies (KR1) (since $B'(n) = B(n)$) and (KR2) but it cannot satisfy (KR3). Indeed, otherwise, $X(\sigma)$ would have a
BV-representation with one vertex at each level and would be an odometer (by Theorem 7.1.1).

5.3 Each $\mathcal{P}(n)$ is a partition in towers since $\sigma$ is primitive and $X(\sigma)$ is infinite. The sequence satisfies (KR2) as any sequence of partitions built in this way. It satisfies (KR3) because $\sigma^n[a]$ and $\sigma^n[c]$ tend both to $\sigma^\omega(b \cdot a)$ while $\sigma^n[b]$ tends to $\sigma^\omega(b \cdot b)$. But condition (KR1) is not satisfied since there are two admissible fixed points.

Section 5.5

5.4 Set $\text{Card}(V) = n$ and $\text{Card}(E) = m$. Clearly, $\kappa$ is injective and $\mathbb{Z}(\Gamma) = \text{Im}(\kappa) \subset \ker(\beta)$. Next $\text{Im}(\beta) = \ker(\gamma)$ and $\dim(\ker(\gamma)) = n - 1$. Thus

$$m = \dim(\mathbb{Z}(E)) = \dim(\ker(\beta)) + \dim(\text{Im}(\beta)) = \dim(\ker(\beta)) + n - 1.$$  

This implies that the dimension of $\ker(\beta)$ is $m - n + 1$. On the other hand, since $G$ is strongly connected, any covering tree $T$ of $G$ has $n - 1$ elements. Since $\Gamma$ has a basis of $\text{Card}(E) - \text{Card}(T) = m - n + 1$ elements (see Appendix D), this implies that the dimension of $\mathbb{Z}(\Gamma)$ is $m - n + 1$. We conclude that $\mathbb{Z}(\Gamma) = \ker(\beta)$.

Section 5.6

5.5 We have $\mathcal{L}_2(X) = \{aa, ab, bc, ca, cb\}$. Set $A_2 = \{x, y, z, t, u\}$ with the bijection taken in alphabetic order. The morphism $\tau_2$ is $x \mapsto xyzt, y \mapsto xyzu, z \mapsto zt, t \mapsto yzt, u \mapsto yzy$. The Rauzy graph $\Gamma_2(X)$ is represented in Figure 5.8.1.

![Figure 5.8.1: The Rauzy graph $\Gamma_2(X)$](image)

The matrices $M, M_2, P$ and $N_2$ are

$$M = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix},$$
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\[ P = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

The eigenvalues of \( N_2 \) are 3, 1, 0 and eigenvectors for 3, 1 are
\[ v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \]

One has \( N_2^n(\alpha v + \beta w) \in \mathbb{Z}^3 \) if and only if \( 23^n\alpha \in \mathbb{Z} \) and \( 2\beta \in \mathbb{Z} \). Thus \( \Delta_{N_2} = \{(\alpha, \beta) | 2\alpha \in \mathbb{Z}[1/3], 2\beta \in \mathbb{Z}\}, \Delta_{N_2}^+ = \{(\alpha, \beta) \in \Delta_{N_2} | \alpha > 0\} \cup \{(0, 0)\}. \)

The unit is \( P[1 \ 1 \ 1 \ 1 \ 1]^t = [4 \ 3 \ 1]^t = 3/2v - 1/2w. \) Using the isomorphism \( (\alpha, \beta) \mapsto (2\alpha, 2\beta) \), we obtain the desired result.

5.6 Let \( A_2 = \{x, y, z, t, u\} \) be an alphabet in order preserving bijection with \( \mathcal{L}_2(X) = \{aa, ab, ac, ba, ca\} \). The morphism \( \varphi_2 : A_2^* \to A_2^* \) is \( x \mapsto yt, y \mapsto yt, z \mapsto yt, t \mapsto zu, u \mapsto x. \) The matrices \( M \) and \( M_2 \) are
\[ M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

The Rauzy graph \( \Gamma_2(X) \) is represented in Figure 5.8.2

![Figure 5.8.2: The Rauzy graph \( \Gamma_2(X) \).](image)

Thus the matrix \( P \) is
\[ P = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \]
and \( N_2 = M. \) The matrix \( M \) is invertible and its dominant eigenvalue is the positive real number \( \lambda \) such that \( \lambda^3 = \lambda^2 + \lambda + 1 \). A corresponding row eigenvector is \( [\lambda^2, \lambda, 1] \). Thus the dimension group is \( \mathbb{Z}[\lambda]. \)
5.9 Notes

Kakutani-Rokhlin partitions owe their name to a result in ergodic theory called \textit{Roklin's Lemma} or \textit{Kakutani-Rokhlin Lemma} which states that every aperiodic measure-theoretic dynamical system can be represented by an arbitrary high tower of measurable sets \cite{Rohlin1948, Kakutani1943}.

5.9.1 Partitions in towers

Proposition \ref{prop:partition} is \cite[Lemma 3.1]{Putnam1989}, where the credit of the construction is given to Vershik. We follow the presentation of \cite[Proposition 6.4.2]{Durand2010}. Theorem \ref{thm:easy} is due to \cite{Hermanetal1992}. Theorem \ref{thm:dimension} is also from \cite{Hermanetal1992}. The proof of the fact that the dimension group is simple is from \cite[Theorem 2.14]{Putnam2010}.

5.9.2 Dimension groups and Rauzy graphs

The notion of fundamental group of a graph used in Section \ref{sec:graph} is classical in algebraic topology \cite{LyndonSchupp2001} for a more detailed introduction to its direct definition on a graph and its connection with spanning trees of the graph.

5.9.3 Dimension group of a substitution shift

The results of this section, in particular Proposition \ref{prop:dimension} are from \cite{Host1995} (see also \cite{Host2000}).
Chapter 6

Bratteli diagrams

We now introduce Bratteli diagrams. We will see that, adding an order on the diagram and provided this order is what we will call proper, an ordered Bratteli diagram defines in a natural way a topological dynamical system. We prove the Bratteli-Vershik representation theorem: any minimal topological dynamical system defined on a Cantor set can be obtained in this way (Theorem 6.3.3). Thus every minimal Cantor system \((X, T)\) can be represented by an ordered Bratteli diagram, called a BV-representation of \((X, T)\).

We will next introduce an equivalence on dynamical systems called Kakutani equivalence and prove that it can be characterized by a transformation on BV-representations.

We then prove one of the major results presented in this book, namely the Strong Orbit Equivalence Theorem (Theorem 6.5.1). This result shows that the dimension group is a complete invariant for the so-called strong orbit equivalence. As a complement, the Orbit Equivalence Theorem (Theorem 6.5.3) shows that the quotient of the dimension group by the infinitesimal subgroup is a complete invariant for orbit equivalence.

In Section 6.6 we develop a systematic study of equivalences on Cantor spaces. We introduce the notion of étale equivalence relation and prove that both the relations of orbit equivalence and of cofinality in Bratteli diagrams are étale equivalences.

In the last section (Section 6.7), we discuss the link with the notion of entropy (which had not been considered before in this book).

6.1 Bratteli diagrams

A Bratteli diagram is an infinite directed graph \((V, E)\) where the vertex set \(V\) and the edge set \(E\) can be partitioned into nonempty finite sets

\[
V = V(0) \cup V(1) \cup V(2) \cup \cdots \quad \text{and} \quad E = E(1) \cup E(2) \cup \cdots
\]

with the following properties:
1. \( V(0) = \{ v(0) \} \) is a one-point set,

2. \( r(E(n)) \subseteq V(n), \ s(E(n)) \subseteq V(n-1), n = 1, 2, \ldots, \)

where \( r : E \rightarrow V \) is called the range map and \( s : E \rightarrow V \) the source map. They satisfy \( s^{-1}(v) \neq \emptyset \) for all \( v \in V \) and \( r^{-1}(v) \neq \emptyset \) for all \( v \in V \setminus V(0) \).

We use the terminology of graphs to handle Bratteli diagrams. In particular, the vertex \( v(0) \) is called the root. A successor of a vertex \( v \in V \) is a vertex \( w \) such that \( v = s(e) \) and \( w = r(e) \) for some \( e \in E \). Two edges \( e, f \in E \) are consecutive if \( r(e) = s(f) \). A path is a sequence \((e_1, e_2, \ldots, e_n)\) of consecutive edges. The source of the path is \( s(e_1) \) and its range is \( r(e_n) \). A descendant of a vertex \( v \) is a vertex \( w \) such that there is a path from \( v \) to \( w \), that is a path with source \( v \) and range \( w \).

It is convenient to represent the Bratteli diagram by a picture with \( V(n) \) the vertices at (horizontal) level \( n \), and \( E(n) \) the edges (downward directed) connecting the vertices at level \( n-1 \) with those at level \( n \). Also, if \( \text{Card}(V(n-1)) = t(n-1) \) and \( \text{Card}(V(n)) = t(n) \), then \( E(n) \) determines a \( t(n) \times t(n-1) \) adjacency matrix \( M(n) \) defined by

\[
M(n)_{s,r} = \text{Card}\{ e \in E(n) \mid s(e) = s, r(e) = r \} \quad (6.1.1)
\]

(see Figure 6.1.1).

![Adjacency matrices](image)

Figure 6.1.1: Representation of a diagram between the levels \( n-1 \) and \( n+1 \).

We say that two Bratteli diagrams \((V, E)\) and \((V', E')\) are isomorphic whenever there exists a pair of bijections \( f : V \rightarrow V' \), preserving the degrees, and \( g : E \rightarrow E' \), intertwining the respective source and range maps:

\[
s' \circ g = f \circ s \quad \text{and} \quad r' \circ g = f \circ r .
\]

Let \( k, l \in \mathbb{N} \) with \( 1 \leq k < l \) and let \( E_{k,l} \) denote the set of paths from \( V(k) \) to \( V(l) \). Specifically,

\[
E_{k,l} = \{ (e_k, \ldots, e_l) \mid e_i \in E(i), k \leq i \leq l, r(e_i) = s(e_{i+1}), k \leq i \leq l-1 \} .
\]
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\[
\begin{bmatrix}
2 & 1 \\
2 & 2 \\
\end{bmatrix}
\]

\[E_{n,n+1}\]

Figure 6.1.2: Telescoping between the levels \(n - 1\) and \(n + 1\) in the diagram of Figure 6.1.1

Remark that the adjacency matrix of \(E_{k,l}\) is \(M(l) \cdots M(k)\). We define \(r(e_k, \ldots, e_l) := r(e_l)\) and \(s(e_k, \ldots, e_l) := s(e_k)\).

6.1.1 Telescoping and simple diagrams

Given a Bratteli diagram \((V, E)\) and a sequence

\[m_0 = 0 < m_1 < m_2 < \ldots\]

in \(\mathbb{N}\), we define the \textit{telescoping} of \((V, E)\) with respect to \(\{m_n \mid n \in \mathbb{N}\}\) as the new Bratteli diagram \((V', E')\), where \(V'(n) = V(m_n)\) and \(E'(n) = E_{m_{n-1}+1,m_n}\) and the range and source maps are as above (see Figure 6.1.2).

We say that \((V, E)\) is a \textit{simple Bratteli diagram} if there exists a telescoping \((V', E')\) of \((V, E)\) such that the adjacency matrices of \((V', E')\) have only non-zero entries at each level.

We will use the following characterisation of simple diagrams. Let \((V, E)\) be a Bratteli diagram. A set \(W \subset V\) is \textit{directed} if every edge having its source in \(W\) has also its range in \(W\). In symbols, for every \(v \in V\)

\[r(e) \in W \Rightarrow s(e) \in W.\]

It is \textit{hereditary} if it satisfies for every \(v \in V\) the following condition. If every edge with source \(v\) has its range in \(W\), then \(v\) itself is in \(W\). In symbols, for every \(v \in V\)

\[r(e) \in W \text{ for every edge } e \text{ such that } v = s(e) \Rightarrow v \in W.\]

**Proposition 6.1.1** A Bratteli diagram is simple if and only if there is no nonempty set both directed and hereditary other than \(V\).

**Proof.** Assume first that \((V, E)\) is simple. Let \(W \subset V\) be a nonempty directed and hereditary set. Since \(W\) is nonempty, there is at least one \(w\) in \(W\). Let \(n\) be such that \(w \in V(n)\). Since \((V, E)\) is simple, there is an \(m > n\) such that there is a path from \(w\) to every vertex in \(V(m)\). Since \(W\) is directed, this implies
\( V(m) \subset W \). Since \( W \) is hereditary, this implies that all vertices of \( V(n) \) for \( n \leq m \) are in \( W \). Thus \( v(0) \in W \), which implies \( V = W \).

Conversely, assume that \((V,E)\) is not simple. Let \( v \in V(n) \) be such that for every \( m > n \) there is some \( w \in V(m) \) which cannot be reached from \( v \). Consider the set \( W \) of vertices \( w \in V(m) \) for some \( m \geq n \) for which there is an integer \( p = p(w) > n \) such that all descendants of \( w \) in \( V(p) \) are descendants of \( v \). It is a directed set by definition. Suppose that some vertex \( w \in V \) is such that all its successors belong to \( W \). Let \( p \) be the supremum of the integers \( p(u) \) for \( u \) successor of \( w \). Then all descendants of \( w \) in \( V(p + 1) \) are descendants of \( v \) and thus \( w \) is in \( W \). This shows that \( W \) is hereditary. Finally there is at least one vertex in \( V(n) \) which is not in \( W \) since otherwise taking the supremum of the integers \( p(u) \) for \( u \in V(n) \), we find that all vertices in \( V(p) \) are descendants of \( v \). Thus \( W \) is a nonempty directed and hereditary set strictly contained in \( V \).

**Example 6.1.2** Consider the Bratteli diagram represented in Figure 6.1.3.

![Figure 6.1.3: A non simple Bratteli diagram](image)

This diagram is not simple because the vertices of the lower level can never reach the top level. Accordingly, the vertices at lower level (excluding the root) form a directed and hereditary set.

We denote by \( \sim \) the *telescoping equivalence* on Bratteli diagrams as the equivalence relation generated by isomorphism and telescoping. It is not hard to show that \( (V^1, E^1) \sim (V^2, E^2) \) if and only if there exists a Bratteli diagram \((V, E)\) such that telescoping \((V, E)\) to odd levels \( 0 < 1 < 3 < \ldots \) yields a telescoping of either \((V^1, E^1)\) or \((V^2, E^2)\), and telescoping \((V, E)\) to even levels \( 0 < 2 < 4 < \ldots \) yields a telescoping of the other (Exercise 6.2).

**Example 6.1.3** Consider the Bratteli diagram of Figure 6.1.4 in the middle. Telescoping at even levels gives the diagram on the left and telescoping at odd levels gives the diagram on the right. Thus the left and right diagrams are equivalent.

### 6.1.2 Dimension group of a Bratteli diagram

Let \((V, E)\) be a Bratteli diagram. Let \( V(n) = \{ v_1, \ldots, v_{t(n)} \} \) and \( G(n) = \mathbb{Z}^{t(n)} \).

Let also \( n_j \) be the number of paths from \( v(0) \) to \( v_j \in V(n) \). We consider \( G(n) \) as
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Figure 6.1.4: Three equivalent Bratteli diagrams.

a unital ordered group with the usual order and the unit \( u(n) = [n_1 \ldots n_t]^t \).

The dimension group of \((V, E)\), denoted \(D(V, E)\) is the direct limit of the sequence

\[
G(0) \xrightarrow{M^{(1)}} G(1) \xrightarrow{M^{(2)}} G(2) \ldots
\]

defined by the adjacency matrices \( M(n) \).

Example 6.1.4 The dimension group of the diagram represented in Figure 6.1.4 on the left is the group \( \mathbb{Z}[1/2] \) (see Example 3.3.1).

The following result shows that the group \( D(V, E) \) is a complete invariant for the telescoping equivalence.

Theorem 6.1.5 Two Bratteli diagrams \((V, E)\) and \((V', E')\) are telescoping equivalent if and only if the unital ordered groups \( D(V, E) \) and \( D(V', E') \) are isomorphic.

Proof. Taking a subsequence (starting at 0) does not change the direct limit and thus telescoping does not change the dimension group.

Conversely, Let \((V, E)\) and \((V', E')\) be two Bratteli diagrams. Set \( V(n) = \{v_1, \ldots, v_{\ell(n)}\} \) and \( V'(n) = \{v'_1, \ldots, v'_{\ell'(n)}\} \). Set \( G = D(V, E) \) and \( G' = D(V', E') \).

We shall construct a Bratteli diagram \((W, F)\) that contracts to a contraction of \((V, E)\) on odd levels and to a contraction of \((V', E')\) on even levels. It suffices to give the sets of vertices \( W(n) \) and the incidence matrices \( N(n) \) between consecutive levels.

We set \( W(1) = V(1) \) and \( N(1) = M(1) \). Looking at the canonical generators of \( \mathbb{Z}^{\ell(1)} \) as elements of \( G' \), we can consider that they are elements of some \( \mathbb{Z}^{\ell'(n_2)} \).

We set \( W(2) = V'(n_2) \), and denote \( N(2) \) the matrix of the map it defines from \( \mathbb{Z}^{\ell(1)} \) to \( \mathbb{Z}^{\ell'(n_2)} \). Again, the elements of \( \mathbb{Z}^{\ell(n_2)} \) can be considered as elements of \( G \), and thus belong to some \( \mathbb{Z}^{\ell(n_3)} \). We set \( W(3) = V(n_3) \) and we call \( N(3) \) the
map that it defines from \( Z_t^{(n_2)} \) to \( Z_t^{(n_3)} \). Proceeding like this, we obtain the sequence

\[
Z \overset{N(1)}{\longrightarrow} Z_t^{(1)} \overset{N(2)}{\longrightarrow} Z_t^{(n_2)} \overset{N(3)}{\longrightarrow} Z_t^{(n_3)} \ldots
\]

that is sufficient to define the Bratteli diagram we are looking for.

We illustrate this result with the following example.

**Example 6.1.6** Consider the two diagrams of Figure 6.1.4 on the left and on the right. We have already seen in Example 6.1.4 that the dimension group of the first one is \( \mathbb{Z}[1/2] \) obtained as the direct limit of the sequence \( \mathbb{Z} \overset{2}{\longrightarrow} \mathbb{Z} \overset{2}{\longrightarrow} \ldots \).

The dimension group of the second one is the direct limit of the sequence \( \mathbb{Z}^2 \overset{M}{\longrightarrow} \mathbb{Z}^2 \overset{M}{\longrightarrow} \ldots \) where \( M \) is the matrix

\[
M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

Since \( \mathcal{R}_M = \{ [x \ x]^t \mid x \in \mathbb{R} \} \), the isomorphism of the dimension groups is consequence of the commutative diagram below.

\[
ex \longrightarrow [x \ x]^t \\
\downarrow 2 \quad \quad \downarrow M \\
2x \longrightarrow [2x \ 2x]^t
\]

Theorem 6.1.5 means that the properties of a Bratteli diagram, or at least of its equivalence class for telescoping should be read on its dimension group. A first step in the direction is the following statement.

**Proposition 6.1.7** A Bratteli diagram is simple if and only if its dimension group is simple.

**Proof.** Let \((V, E)\) be a Bratteli diagram and \( G = D(V, E) \). Let us first suppose that \((V, E)\) is simple. We have to show that every nonzero element \( g \in G^+ \) is an order unit. Let indeed \( g \in G^+ \) be nonzero and let \( h \in G^+ \). We can choose \( n \geq 1 \) such that \( g = i_n(x) \) and \( h = i_n(y) \) with \( x, y \in G(n)^+ \). Let \( v \in V(n) \) be such that \( x_v > 0 \). Since \((V, E)\) is simple, we have \( M(m) \cdots M(n+1)x > 0 \) for all \( m \) large enough. Then \( M(m) \cdots M(n+1)y \leq NM(m) \cdots M(n+1)x \) for \( N \) large enough. This implies that \( h < Ng \). Thus \( G \) is simple.

Conversely, assume that \((V, E)\) is not simple. Let \( W \) be a nonempty directed and hereditary set strictly contained in \( V \). Let \( H \) be the set of \( h \in D(V, E) \) which correspond to an \( x = (x_n) \) with \( x_n \in \mathbb{Z}^{V(n)} \) having the property that for some \( n \geq 1 \) we have \( x_{n,v} = 0 \) for every \( v \notin W \) and \( x_{n+1} = M(m+1)x(m) \) for all \( m \geq n \). Since \( W \) is directed, every matrix \( M(m) \) has the form

\[
M(m) = \begin{bmatrix} W \\ 0 \end{bmatrix}
\]
Thus, if \( x \) satisfies this property for \( n \), it holds for every \( m \geq n \). Thus \( H \) is a subgroup of \( G \) which is clearly an ideal. Since \( W \neq \emptyset \), we can choose \( w \in W \cap V(n) \) and \( x \) such that \( x_{m,v} > 0 \). Then \( x_{m,v} > 0 \) for every descendant of \( w \), which implies that the class of \( x \) is not 0. Therefore, we have \( H \neq \{0\} \). Since \( W \) is strictly contained in \( V \) and since it is hereditary, we have \( H \neq G \). Thus \( G \) is not simple.

**Example 6.1.8** Consider again the nonsimple Bratteli diagram of Example 6.1.2. All matrices \( M(n) \) for \( n \geq 1 \) are equal to

\[
M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

The dimension group \( G \) is \( \mathbb{Z}^2 \) with the lexicographic order, that is \( (\mathbb{Z}^2, \mathbb{Z}^+ \times \mathbb{Z} \cup \{0\} \times \mathbb{Z}_+, 0) \) (see Example 3.2.9). The set \( \{0\} \times \mathbb{Z} \) is an order ideal.

### 6.1.3 Ordered Bratteli diagrams

An ordered Bratteli diagram \((V, E, \leq)\) is a Bratteli diagram \((V, E)\) together with a partial order \( \leq \) on \( E \) such that edges \( e, e' \) in \( E \) are comparable if, and only if, \( r(e) = r(e') \), in other words, we have a linear order on each set \( r^{-1}(\{v\}) \), where \( v \) belongs to \( V \setminus V(0) \) (see Figure 6.1.5).

![Figure 6.1.5: Order on the diagram of Figure 6.1.1](image)

Note that if \((V, E, \leq)\) is an ordered Bratteli diagram and \( k < l \) in \( \mathbb{Z}^+ \), then the set \( E_{k+1,l} \) of paths from \( V(k) \) to \( V(l) \) may be given an induced (lexicographic) order as follows:

\[
(e_{k+1}, e_{k+2}, \ldots, e_l) > (f_{k+1}, f_{k+2}, \ldots, f_l)
\]

if, and only if, for some \( i \) with \( k+1 \leq i \leq l \), \( e_j = f_j \) for \( i < j \leq l \) and \( e_i > f_i \).

It is a simple observation that if \((V, E, \leq)\) is an ordered Bratteli diagram and \((V', E')\) is a telescoping of \((V, E)\) as defined above, then with the induced order
≤', \( (V', E', \leq') \) is again an ordered Bratteli diagram. We say that \( (V', E', \leq') \) is a telescoping of \( (V, E, \leq) \) (see Figure 6.1.6).

Again there is an obvious notion of isomorphism between ordered Bratteli diagrams. Let \( \approx \) denote the equivalence relation on ordered Bratteli diagrams generated by isomorphism and by telescoping. One can show that \( G^1 \approx G^2 \), where \( G^1 = (V^1, E^1, \leq^1) \), \( G^2 = (V^2, E^2, \leq^2) \), if, and only if, there exists an ordered Bratteli diagram \( G = (V, E, \leq) \) such that telescoping \( G \) to odd levels \( 0 < 1 < 3 < \ldots \) yields a telescoping of either \( G^1 \) or \( G^2 \), and telescoping \( G \) to even levels \( 0 < 2 < 4 < \ldots \) yields a telescoping of the other (Exercise 6.2). This is analogous to the situation for the equivalence relation \( \sim \) on Bratteli diagrams as we discussed above.

**Example 6.1.9** The two first ordered diagrams of Figure 6.1.7 are equivalent. The third one is not (although all three are equivalent as unordered Bratteli diagrams).

![Figure 6.1.7: Two properly ordered Bratteli diagrams and a non properly ordered one.](image-url)
The following notion will be important when we will deal with Bratteli diagrams and subshifts. Fix \( n \geq 1 \) and let us consider \( V(n-1) \) and \( V(n) \) as alphabets. For every letter \( a \in V(n) \), consider the ordered list \((e_1, \ldots, e_k)\) of edges of \( E(n) \) which range at \( a \), and let \((a_1, \ldots, a_k)\) be the ordered list of the labels of the sources of these edges. This defines a morphism \( \tau(n) : a \mapsto a_1 \cdots a_k \) from \( V(n)^* \) to \( V(n-1)^* \) we call the morphism read on \( E(n) \). For example in Figure 6.1.5 the morphism we read on:

- \( E(n) \) is \( \tau(n) : 0 \mapsto AA, 1 \mapsto B, 2 \mapsto BA, 3 \mapsto A, 4 \mapsto B, \)

- \( E(n+1) \) is \( \tau(n+1) : a \mapsto 01, b \mapsto 312, \)

and on Figure 6.1.6 the morphism we read on \( E_{n,n+1} \) is \( \sigma : a \mapsto AAB, b \mapsto ABBA. \) We can check of course that we have \( \sigma = \tau_n \circ \tau_{n+1} \). Note that the matrix \( M(n) \) is the composition matrix of the morphism \( \tau(n) \).

6.2 Dynamics for ordered Bratteli diagrams

We shall see now how one can define a dynamics on the set of paths in a Bratteli diagram.

6.2.1 The Bratteli compactum

Let \((V,E,\leq)\) be an ordered Bratteli diagram. Let \( X_E \) denote the associated infinite path space, that is,

\[
X_E = \{(e_1, e_2, \ldots) \mid e_i \in E(i), r(e_i) = s(e_{i+1}), i = 1, 2, \ldots\}.
\]

We exclude trivial cases and assume henceforth that \( X_E \) is an infinite set.

Two paths in \( X_E \) are said to be cofinal if they have the same tails, i.e., the edges agree from a certain level on. We denote by \( R_E \) this equivalence, called the equivalence of cofinality on the set \( X_E \).

The set \( X_E \) is a closed subset of \( \prod_{i \geq 1} E(i) \). Since every \( E(i) \) is finite, the product is compact and thus \( X_E \) is compact. A basis for the topology is the family of cylinder sets

\[
[e_1, e_2, \ldots, e_k]_E = \{(f_1, f_2, \ldots) \in X_E \mid f_i = e_i, 1 \leq i \leq k\}.
\]

Each \([e_1, \ldots, e_k]_E \) is also closed, as is easily seen. When it will be clear from the context we will write \([e_1, \ldots, e_k]\) instead of \([e_1, \ldots, e_k]_E\). Endowed with this topology, we call \( X_E \) the Bratteli compactum associated with \((V,E,\leq)\). Let \( d_E \) be the distance on \( X_E \) defined by \( d_E((e_n)_n, (f_n)_n) = \frac{1}{k^*} \) where \( k = \inf\{i \mid e_i \neq f_i\} \). It clearly defines the topology of the cylinder sets.

If \((V,E)\) is a simple Bratteli diagram, then \( X_E \) has no isolated points, and so is a Cantor space (recall that we assume \( X_E \) to be infinite, see Exercise 6.4). Moreover, each class of the equivalence \( R_E \) (corresponding to cofinality) is dense in \( X_E \) (Exercise 6.5).
Let $x = (e_1, e_2, \ldots)$ be an element of $X_E$. We will call $e_n$ the $n$th label of $x$ and denote it by $x(n)$. We let $X_E^{\text{max}}$ denote those elements $x$ of $X_E$ such that $x(n)$ is a maximal edge for all $n$ and $X_E^{\text{min}}$ the analogous set for the minimal edges.

It is not difficult to show that $X_E^{\text{max}}$ and $X_E^{\text{min}}$ are non-empty (see Exercise 6.6). Moreover, for every $v \in V$, the set of minimal edges forms a spanning tree of the graph $(V, E)$. This means that for every $v \in V$, there is a unique path formed of minimal edges from $v$ to $v(0)$ (Exercise 6.6). The same holds for maximal edges.

The ordered Bratteli diagram $(V, E, \leq)$ is properly ordered if it is simple and if $X_E^{\text{max}}$ and $X_E^{\text{min}}$ both are a one point set: $X_E^{\text{max}} = \{x^{\text{max}}\}$ and $X_E^{\text{min}} = \{x^{\text{min}}\}$.

Example 6.2.1 The Bratteli diagrams of Figure 6.1.7 on the left and center are properly ordered while the diagram on the right is not. Indeed, there are two paths labeled with $0, 0, 0, \ldots$ and two paths labeled $1, 1, 1, \ldots$.

Note that every simple Bratteli diagram can be properly ordered (Exercise 6.7).

6.2.2 The Vershik map

We can now define, for a properly ordered Bratteli diagram $(V, E, \leq)$, a map $T_E : X_E \to X_E$, called the Vershik map (or the lexicographic map), associated with $(V, E, \leq)$.

We let $T_E(x^{\text{max}}) = x^{\text{min}}$. If $x = (e_1, e_2, \ldots) \neq x^{\text{max}}$, let $k$ be the least integer such that $e_k$ is not a maximal edge. Let $f_k$ be the successor of $e_k$ (relative to the order $\leq$ so that $r(e_k) = r(f_k)$). Define $T_E(x) = y = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$, where $(f_1, \ldots, f_{k-1})$ is the minimal edge in $E_{1,k-1}$ with range equal to $s(f_k)$.

Thus, the image by $T_E$ of a point $x \neq x^{\text{max}}$ is its successor in the lexicographic order.

The map $T_E$ is clearly continuous. It is moreover one-to-one (Exercise 6.8).

We call the resulting pair $(X_E, T_E)$ a Bratteli-Vershik dynamical system. Since $X_E$ is a Cantor space, it is a Cantor dynamical system.

Proposition 6.2.2 Let $(V, E, \leq)$ be a properly ordered Bratteli diagram. The system $(X_E, V_E)$ is a minimal Cantor dynamical system.

The proof is left as an exercise (Exercise 6.9).

In the sequel BV will refer to Bratteli-Vershik.

Example 6.2.3 Consider the Bratteli diagram $(V, E)$ of Figure 6.1.7 on the left. The system $(X_E, T_E)$ is isomorphic to the odometer $(\mathbb{Z}_2, T)$ where $T(x) = x + 1$.

Indeed, as well-known, the addition of 1 in base 2 consists, on the representation in base 2 of numbers with a fixed number of digits, in taking the next sequence in the lexicographic order. Since the representation in base 2 of 2-adic numbers is written with the least significant digit on the left, addition of 1 corresponds to the next element in the reverse lexicographic order for right infinite sequences (see Figure 6.2.1 below).
6.3. THE BRATTELI-VERSHIK MODEL THEOREM

Let $(X, T)$ be a minimal Cantor dynamical system. The properly ordered Bratelli diagram $(V, E, \leq)$ is a BV-representation of $(X, T)$ if $(X, T_E)$ is isomorphic to $(X, T)$. We will show, as a main result of this chapter, that every Cantor minimal system has a BV-representation.

6.3.1 From partitions in towers to Bratteli diagrams

We will first show how to associate to any nested sequence of partitions of a system $(X, T)$ an ordered Bratteli diagram.

Let

\[ \mathcal{P}(n) = \{ T^j B_i(n) \mid 0 \leq j < h_i(n), 1 \leq i \leq t(n) \} \]

be a nested sequence of KR-partitions of $(X, T)$. We may suppose that \( \mathcal{P}(0) = \{ X \} \). Hence \( t(0) = 1, h_1(0) = 1 \) and \( B_1(0) = X \).

Let \( V(n) = \{(n,1), \ldots, (n,t(n))\} \), for \( n \geq 0 \). Thus the set of vertices is, at each level \( n \), the set of towers of the partition \( \mathcal{P}(n) \).

The set of edges records the inclusions of the elements of \( \mathcal{P}(n) \) in the elements of \( \mathcal{P}(n-1) \). Specifically, let \( E(n) \) be the set of quadruples \((n, t', t, j)\) satisfying

\[ T^j B_t(n) \subseteq B_{t'}(n-1) \quad (6.3.1) \]

for \( 1 \leq t' \leq t(n-1), 1 \leq t \leq t(n), 0 \leq j \leq h_t(n) - 1 \) and \( n \geq 1 \). Note that, in particular,

1. the index \( j \) is such that \( T^j B_t(n) \) is contained in the basis \( B(n-1) \) of \( \mathcal{P}(n-1) \),
2. the index \( j \) is the return time of every element of \( B_t(n) \) to \( B(n-1) \).
3. not all indices \( j \) with \( 0 \leq j \leq h_t(n) - 1 \) appear in these quadruples.

The range and source maps are given by

\[ s(n, t', t, j) = (n-1, t') \quad \text{and} \quad r(n, t', t, j) = (n, t) \quad (6.3.2) \]
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Two edges $e_1 = (n_1, t'_1, t_1, j_1)$ and $e_2 = (n_2, t'_2, t_2, j_2)$ are comparable whenever $n_1 = n_2$ and $t_1 = t_2$. In this case we define $e_1 \geq e_2$ if $j_1 \geq j_2$. It is straightforward to verify that $(V, E, \leq)$ is an ordered Bratteli diagram.

It is useful to remark, from (6.3.2), that $((n, t'_n, t_n, j_n))_n$ is an infinite path of $(V, E, \leq)$ if, and only if, $t_{n-1} = t'_n$ for all $n \geq 1$. Hence the paths of the Bratteli diagram have the form $((n, t_{n-1}, t_n, j_n))_n$ with $1 \leq t_{n-1} \leq t(n-1)$, $1 \leq t_n \leq t(n)$, $0 \leq j_n \leq h_t(n)$ and

$$T^{j_n}B_{t_n}(n) \subset B_{t_{n-1}}(n-1) \quad \text{(6.3.3)}$$

Note that (6.3.3) implies (by Exercise 6.1) that

$$0 \leq j_n \leq h_t(n) - h_t_{n-1}(n-1).$$

Note also that $(n, t_{n-1}, t_n, j_n)$ is a minimal edge if and only if $j_n = 0$ and is maximal if and only if

$$j_n = h_t(n) - h_t_{n-1}(n-1). \quad \text{(6.3.4)}$$

Additionally, if $(n, t_{n-1}, t_n, j_n)$ is not a maximal edge, its successor is an edge $(n, t'_{n-1}, t_n, j'_n)$ with

$$j'_n = j_n + h_t_{n-1}(n-1) \quad \text{(6.3.5)}$$

since $j'_n$ is the least integer such that $T^{j'_n}B_{t_n}(n) \subset B(n-1)$.

![Bratteli Diagram](image)

Figure 6.3.1: The partition $\Psi(1)$ consists of two towers called $A$ and $B$. The dynamics $T$ acts vertically except for the last levels where it goes back to the base. The elements of the partition $\Psi(2)$ can be seen as the piling up of vertical pieces of the towers $A$ and $B$.

For example suppose that $\Psi(n)$ is a refining sequence of KR-partitions such that (see Figure 6.3.1)

1. $\Psi(1) = \{B_1(1), TB_1(1), B_2(1), TB_2(1), T^2B_2(1)\}$ and

2. $\Psi(2) = \{T^jB_i(2) \mid 0 \leq j < h_i(2), 1 \leq i \leq t(2)\}$ with
   
   (a) $t(2) = 3, h_1(2) = 9, h_2(2) = 4, h_3(2) = 7,$
   
   (b) $B_1(2) \subset B_1(1), T^2B_1(2) \subset B_1(1), T^4B_1(2) \subset B_2(1), T^7B_1(2) \subset B_1(1),$
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(c) \( B_2(2) \subseteq B_1(1) \), \( T^2B_2(2) \subseteq B_1(1) \),
(d) \( B_3(2) \subseteq B_2(1) \), \( T^3B_3(2) \subseteq B_1(1) \), \( T^5B_3(2) \subseteq B_1(1) \).

The corresponding Bratteli diagram is represented in Figure 6.3.2 (with the value of \( j \) indicated on the edge).

Figure 6.3.2: Diagrammatic representation of \( \mathcal{P}(0) \), \( \mathcal{P}(1) \) and \( \mathcal{P}(2) \).

The next two lemmas describe properties of the Bratteli diagram associated with the nested sequence of partitions \( \mathcal{P}(n) \).

Lemma 6.3.1 For every infinite path \( (e_n) \) with \( e_n = (n, t_{n-1}, t_n, j_n) \) in \( (V,E,\leq) \), the following assertions hold.

(i) We have \( 0 \leq \sum_{i=1}^{n} j_i \leq h_{t_n}(n) - 1 \) for every \( n \geq 1 \).

(ii) For \( 1 \leq m \leq n \), the waiting time for an element of \( B_{t_m}(m) \) to access \( B_{t_n}(n) \) is \( \sum_{i=m}^{n} j_i \).

(iii) The sequence

\[
C_n = T^{\sum_{i=1}^{n} j_i} B_{t_n}(n)
\]

is decreasing, that is

\[
C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots
\]

and the map \( \phi : (e_n) \to \cap_{n \geq 1} C_n \) sends distinct paths to disjoint sets.

(iv) If the sequence \( \mathcal{P}(n) \) generates the topology of \( X \), then \( \text{Card}(\cap_{n \geq 1} C_n) = 1 \).

Proof. (i) Let us prove the first assertion by induction on \( n \). It is true for \( n = 1 \) since \( j_1 \leq h_{t_1}(1) - 1 \). Next, for \( n \geq 2 \), since \( T^{j_n} B_{t_n}(n) \subseteq B_{t_{n-1}}(n-1) \), we have \( j_n \leq h_{t_n}(n) - h_{t_{n-1}}(n-1) \) (see Exercise 5.1). Since, by induction hypothesis, we have \( j_{n-1} + \ldots + j_1 \leq h_{t_{n-1}}(n-1) \) we conclude that \( j_n + \ldots + j_1 \leq h_{t_n}(n) \).

(ii) The statement is true for \( n = m \). Next, we argue by induction on \( n - m \). Since \( j_{n+1} \) is the return time from \( B_{t_{n+1}}(n+1) \) to \( B_{t_n}(n) \) and since, by induction
hypothesis $\sum_{i=m}^{n} j_i$ is the return time from $B_{t_n}(n)$ to $B_{t_m}(m)$, the conclusion follows.

(iii) By (6.3.3), we have for every $n \geq 1$

$$C_{n+1} = T^{\sum_{i=1}^{n+1} j_i} B_{t_{n+1}}(n) = T^{\sum_{i=1}^{n+1} j_i} T^{j_{n+1}} B_{t_{n+1}}(n + 1) \\ < T^{\sum_{i=1}^{n+1} j_i} B_{t_n}(n) = C_n.$$  

Since $(C_n)_{n \geq 1}$ is a decreasing sequence of closed sets and since $X$ is compact, its intersection is nonempty. Since $0 \leq \sum_{i=1}^{n} j_i \leq h_{t_n}(n) - 1$ for every $n \geq 1$, $C_n$ is an element of the partition $\mathcal{P}(n)$. Let $(e_n)$ and $(e'_n)$ be distinct paths and let $C_n, C'_n$ be the associated sequences. We have $e_k \neq e'_k$ for some $k \geq 1$. Since $C_m$ and $C'_m$ are distinct elements of the partition $\mathcal{P}(m)$, they are disjoint. Thus $\phi(e_n) \cap \phi(e'_n) = \emptyset$.

(iv) If the sequence of partitions generates the topology, the intersection of all $C_n$ is reduced to one point.

**Lemma 6.3.2** If the intersection of the bases of the partitions $\mathcal{P}(n)$ has only one element, the Bratteli diagram $(V, E, \leq)$ is properly ordered.

**Proof.** First, the diagram $(V, E)$ is simple. Indeed, it is enough to prove that there is an $n \geq 1$ such that there is a path from every vertex at level $n$ to every vertex at level 1. For this, it enough to take $n$ such that all the $h_{t_n}(n)$ are larger than the maximum of the waiting times to access an element of the partition $B(1)$.

Let us show now that $X_E^{\text{min}}$ consists of a single path. Let $(e_n)$ with $e_n = (n, t_{n-1}, t_n, j_n)$ be an infinite path of $X_E^{\text{min}}$. The edges comparable to $e_n$ are the edges of the form $(n, t, t_n, j)$ for some $t$ and exactly one of them is of the form $(n, t, t_n, 0)$. It is clearly a minimal edge. Hence $j_n = 0$ for all $n$. But, by Lemma 6.3.1

$$\phi(e_n) = \cap_n T^0 B_{t_n}(n) \subseteq \cap_n B(n)$$

which consists of a single point by hypothesis. Since $\phi$ sends distinct paths to disjoint sets, the path $(e_n)$ is the unique path of $X_E^{\text{min}}$.

Similarly, if $(e_n) \in X_E^{\text{max}}$, then $j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)$ by (6.3.4). Then

$$\phi(e_n) = \cap_n T^{h_{t_n}(n)} B_{t_n}(n) \subseteq \cap_n B(n - 1)$$

which is reduced to one point, whence the conclusion again.

**6.3.2 The BV-representation theorem**

We can now state and prove the BV-representation theorem.

**Theorem 6.3.3 (Herman, Putnam, Skau)** For every minimal Cantor system $(X, T)$, there exists a properly ordered Bratteli diagram $(V, E, \leq)$ such that $(X, T)$ is isomorphic to $(X_E, T_E)$. 

More precisely, for every refining sequence \((\mathcal{P}(n))\) of KR-partitions of \((X, T)\), the Bratteli diagram \((V, E, \leq)\) associated with \(\mathcal{P}(n)\) is such that \((X, T)\) is isomorphic to \((X_E, T_E)\).

**Proof.** By Theorem 5.1.7, there exists a be refining sequence of partitions \((\mathcal{P}(n))\) of \((X, T)\). By Lemma 6.3.2 the ordered Bratteli diagram \((V, E, \leq)\) associated to \((\mathcal{P}(n))\) is properly ordered. This allows to consider the Cantor system \((X_E, T_E)\). It is minimal by Proposition 6.2.2.

Consider the map \(\phi : X_E \rightarrow X\) defined by

\[
\phi((n, t_{n-1}, t_n, j_n)) = x \quad \text{where} \quad \{x\} = \cap_{n \geq 1} C_n
\]

with \(C_n = T^{\sum_{i=1}^{n} j_i} B_{t_n}(n)\). It is well defined (Lemma 6.3.1) and is a homeomorphism (see Exercise 6.11). Note that \((\mathcal{P}(n))\) being a decreasing sequence of partitions, we also have \(\{x\} = \cap_{n \geq N} C'_n\) for all \(N\).

There remains to show that it commutes with the dynamics. Let \(e = (e_n)_n\) be an infinite path of \(X_E\) with \(e_n = (n, t_{n-1}, t_n, j_n)\). Suppose first that \(e\) is not the maximal path. Then there exists \(n_0\) such that \(T_E(e) = e'_1 \cdots e'_{n_0-1} e'_n e_{n_0+1} e_{n_0+2} \cdots\) where \(e'_n = (n, t'_{n-1}, t'_n, j'_n)\), with \(j'_n = 0, 1 \leq n \leq n_0 - 1\) and \(e'_n = (n_0, t'_{n_0-1}, t_{n_0}, j'_n)\) is the successor of \(e_{n_0}\). Note that, for \(1 \leq n \leq n_0 - 1\), the edges \(e_n\) being maximal we have \(j_n = h_{t_n}(n) - h_{t_{n-1}}(n-1)\).

Since \((e'_{n_0})\) is the successor of \(e_{n_0}\), we have by (6.3.7), \(j'_n = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1)\). Hence

\[
\sum_{1 \leq n \leq n_0} j_n = j_{n_0} + \sum_{1 \leq n \leq n_0} h_{t_n}(n) - h_{t_{n-1}}(n-1) = j_{n_0} + h_{t_{n_0-1}}(n_0 - 1) - 1.
\]

while

\[
\sum_{1 \leq n \leq n_0} j'_n = j_{n_0} + h_{t_{n_0-1}}(n_0) - 1 = \sum_{1 \leq n \leq n_0} j_n + 1.
\]

Set \(C'_n = \cap_{1 \leq n \leq n_0} T^{\sum_{i=1}^{n} j_i} B_{t_n}(n)\). Then \(\phi \circ T_E(e) = \phi(e') = \cap_{n \geq 1} C'_n\). Since \(\sum_{n=1}^{n_0} j'_i = \sum_{i=1}^{n_0} j_i + 1\) and \(j'_n = j_n\) for \(n > n_0\), we have \(C'_n = TC_n\) for \(n \geq n_0\).

This shows that \(\phi \circ T_E(e) = T \phi(e)\) and thus the conclusion.

Suppose now that \(e\) is the maximal path. Let \(x_{\text{min}}\) be the minimal path of \((V, E, \leq)\). Then we have to prove that \(\phi(x_{\text{min}}) = T(\phi(e))\). But since \(\mathcal{P}(0) = \{X\}\), we have \(h_{t_0}(0) = 1\) and consequently

\[
T(\phi(e)) = T \left( \bigcap_{n \geq 1} T^{\sum_{i=1}^{n} h_{t_i}(i) - h_{t_{i-1}}(i-1)} B_{t_n}(n) \right) = \bigcap_{n \geq 1} T^{h_{t_n}(n)} B_{t_n}(n) \subseteq \bigcap_{n \geq 1} \bigcup_{1 \leq i \leq t(n)} B_i(n) = \{\phi(x_{\text{min}})\}.
\]
For every minimal invertible Cantor system \((X, T)\), the Bratteli diagram \(G = (V, E)\) build from a refining sequence of KR-partitions is such that \((X_E, T_E)\) is, by Theorem 6.3.3, conjugate to \((X, T)\). Now the system \((X_E, T_E)\) itself has a natural sequence of KR-partitions \((P(n))\), namely the partitions \([e_1, \ldots, e_n]\) for \((e_1, \ldots, e_n) \in E_{1,n}\). The basis of the partition \(P(n)\) is the union of the cylinders corresponding to minimal paths and can be identified with \(V(n)\). The height of the tower \(B_{t(n,t)}\) with \(1 \leq t \leq t(n)\) is

\[ h_t(n) = (M(n) \cdots M(1))^t. \]

This sequence of partitions is a refining sequence. It is easy to verify that the Bratteli diagram associated to this sequence is the original diagram \((V, E)\) (Exercise 6.12).

### 6.3.3 Dimension groups and BV-representation

We have showed in Proposition 5.3.1 that for any minimal invertible Cantor system \((X, T)\), the dimension group \(K^0(X, T)\) is a direct limit of groups \(G(n)\) associated with a sequence \((P(n))\) of KR-partitions. We will now see how this group is defined directly in terms of a BV representation of \((X, T)\).

**Theorem 6.3.4** Let \((V, E)\) be a properly ordered Bratteli diagram. The group \(K^0(X_E, T_E)\) is the dimension group \(D(V, E)\) of the diagram \((V, E)\).

**Proof.** Consider the partition \(P(n) = \{[e_1, \ldots, e_n]\}\) of \(X_E\) in cylinder sets for every path \((e_1, e_2, \ldots, e_n)\) of length \(n\) starting from \(V(0)\) in \((V, E)\). The basis of the partition \(P(n)\) is the union of the cylinders corresponding to minimal paths and can identified with \(V(n)\). By Proposition 5.3.1 the group \(K^0(X_E, T_E)\) is the direct limit of the groups \((G(n), G^+(n), 1_n)\) where \(G(n)\) can be identified with \(Z_t^{(n)}\) and with the morphisms \(I(n, n-1)\) defined (by Equation (5.2.1)) by the matrices

\[ M(n, n-1)_{t_n, t_{n-1}} = \text{Card}\{ j \mid 0 \leq j \leq h_{t_n}(n), T_E^j B_{t_n}(n) \subset B_{t_{n-1}}(n-1) \}. \]

Thus, by (5.3.3), we have also

\[ M(n, n-1)_{t_n, t_{n-1}} = \text{Card}\{ e \in E(n) \mid r(e) = t_n, s(e) = t_{n-1} \} = M(n)_{t_n, t_{n-1}}. \]

Thus the matrix of the maps \(I(n, n+1)\) is precisely the adjacency matrix \(M(n)\) of \((V, E)\). This completes the proof.

**Example 6.3.5** Let \((V, E)\) be the properly ordered Bratteli diagram represented in Figure 6.3.3. The adjacency matrix at each level is
Thus the dimension group $K^0(X_E, T_E)$ is $\mathbb{Z} + \lambda \mathbb{Z}$ with $\lambda = (1 + \sqrt{5})/2$ (see Example 3.3.6).

6.4 Kakutani equivalence

The minimal Cantor dynamical systems $(X, T)$ and $(Y, S)$ are Kakutani equivalent if they have (up to isomorphism) a common induced system, that is, there exist nonempty clopen sets $U \subseteq X$ and $V \subseteq Y$ such that the induced systems $(X_U, T_U)$ and $(Y_V, S_V)$ are isomorphic (see Exercise 6.14 for a proof that it is really an equivalence relation).

Two systems which are conjugate are Kakutani equivalent but the converse is false. For example, The system with two points is Kakutani equivalent with the system with one point but they are not conjugate.

Let us relate Kakutani equivalence to Bratteli diagrams. If $(V, E, \leq)$ is a properly ordered Bratteli diagram we may change it into a new properly ordered Bratteli diagram $(V', E', \leq')$ by making a finite change, that is, by adding and/or removing any finite number of edges (vertices), and then making arbitrary choices of linear orderings of the edges meeting at the same vertex (for a finite number of vertices). So $(V, E, \leq)$ and $(V', E', \leq')$ are cofinally identical, that is, they only differ on finite initial portions. (Observe that this defines an equivalence relation on the family of properly ordered Bratteli diagrams.) We have the following nice characterisation of the Kakutani equivalence.

**Theorem 6.4.1 (Giordano, Putnam, Skau)** Let $(X_E, T_E)$ be the dynamical system associated with the properly ordered Bratteli diagram $(V, E, \leq)$. Then the minimal Cantor dynamical system $(X, T)$ is Kakutani equivalent to $(X_E, T_E)$ if
and only if \((X,T)\) is isomorphic to \((X_{E'},T_{E'})\), where \((V',E',\leq')\) is obtained from \((V,E,\leq)\) by a finite change as described above.

An interesting particular case of this result is the following. Let \(U\) be a clopen set of \((X_E,T_E)\). It is a finite union of cylinder sets. We can suppose that they all have the same length, that is, for some \(n\), \(U = \bigcup_{p \in P} [p]\) where \(P\) is a set of paths from level \(n\) to level 0. To obtain a BV-representation of the induced system on \(U\) it suffices to take the properly ordered Bratteli diagram \((V',E',\leq')\) which consists of all the paths starting with an element of \(P\) endowed with the induced ordering. It is not too much work to prove that the induced system on \(U\) is isomorphic to \((X_{E'},T_{E'})\).

Conversely, we will have several occasions to use the following statement which describes the construction of a BV-representation for a system from one of an induced system.

Proposition 6.4.2 Let \((X,T)\) be a minimal Cantor system and let \((U,T_U)\) be the induced system on a clopen set \(U \subset X\). Let \((X_E,T_E)\) be a BV-representation of \((U,T_U)\) corresponding to a Bratteli diagram \((V,E)\) such that for every vertex \(v \in V(1)\) the return time to \(U\) is constant and equal to \(f(v)\) for every element of \([v]\). Then \(X\) has a BV-representation \((X_{E'},T_{E'})\) obtained from \((X_E,T_E)\) by replacing each edge from 0 to \(v \in V(1)\) by \(f(v)\) edges.

Proof. The system \((X_{E'},T_{E'})\) is isomorphic to the primitive of \((X_E,T_E)\) relative to the function \(x \mapsto f(v)\) when \(x \in [v]\). Thus \((X,T)\) and \((X_{E'},T_{E'})\) are isomorphic. □

We illustrate Proposition 6.4.2 with the following simple example.

Example 6.4.3 Let \(X\) be the set of integers of the form \(x + 3y\) where \(x = 0, 1, 2\) and \(y \in \mathbb{Z}_2\) and \(U\) be the set of those for which \(x = 0\). The systems \((X,T)\) with \(T\) being the addition of 1 and the system induced on \(U\) have the BV-representations shown in Figure 6.4.1. The diagram on the right is (up to the first level) the usual BV-representation of \(\mathbb{Z}_2\). The diagram on the left is the same except for the first level made of 3 edges in agreement with Proposition 6.4.2.

6.5 The Strong Orbit Equivalence Theorem

We say that two dynamical systems \((X,T)\) and \((Y,S)\) are orbit equivalent whenever there exists a homeomorphism \(\phi : X \to Y\) sending orbits to orbits

\[
\phi \left( \{T^n x \mid n \in \mathbb{Z} \} \right) = \{S^n \phi(x) \mid n \in \mathbb{Z} \},
\]

for all \(x \in X\). This induces the existence of maps \(\alpha : X \to \mathbb{Z}\) and \(\beta : X \to \mathbb{Z}\) satisfying for all \(x \in X\)

\[
\phi \circ T(x) = S^{\alpha(x)} \circ \phi(x) \quad \text{and} \quad \phi \circ T^{\beta(x)}(x) = S \circ \phi(x).
\]
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These maps are called the orbit cocycles associated to $\phi$.

When $\alpha$ and $\beta$ have at most one point of discontinuity, we say that $(X,T)$ and $(Y,S)$ are strongly orbit equivalent (SOE). It is natural to consider such a definition because one can show that if $\alpha$ is continuous then $(X,T)$ is conjugate to $(Y,S)$ or to $(Y,S^{-1})$ (Exercise 6.16).

The following result characterises strong orbit equivalence by means of Bratelli diagrams and dimension groups.

Recall from Chapter 5 that two Bratteli diagrams $(V,E)$ and $(V',E')$ are equivalent if they have a common intertwining that is, a Bratteli diagram $(W,F)$ such that telescoping to odd levels gives a telescoping of $(V,E)$ and telescoping to even levels gives a telescoping of $(V',E')$.

**Theorem 6.5.1** (Giordano,Putnam,Skau) Let $(X,T)$ and $(X',T')$ be two minimal Cantor dynamical systems. The following are equivalent:

1. There exist two BV-representations, $(V,E,\leq)$ of $(X,T)$ and $(V',E',\leq')$ of $(X',T')$, which have a common intertwining.

2. There exist two BV-representations, $(V,E,\leq)$ of $(X,T)$ and $(V',E',\leq')$ of $(X',T')$, and a homeomorphism $\psi: X_E \to X_{E'}$ such that $\psi(x)(n)$ depends only on $x(1)\ldots x(n)$ and $\psi(x_u) = x'_u$, $u \in \{\min, \max\}$, and having the property that if $x$ and $y$ are cofinal from level $n$, then $\psi(x)$ and $\psi(y)$ are cofinal from level $n + 1$.

3. $(X,T)$ and $(X',T')$ are strong orbit equivalent.

4. The dimension groups $K^0(X,T)$ and $K^0(X',T')$ are isomorphic as unital ordered groups.

**Proof.** Let us show that 1 implies 2. Let $(W,F)$ be a common intertwining of $(V,E)$ and $(V',E')$. Moreover, from Theorem 6.3.3 we can also suppose that all incidence matrices have entries greater than two. This means that every pair of vertices in consecutive levels has at least two connecting edges.

Let $x_{\min}$ and $x_{\max}$ be the minimal and maximal paths of $(V,E,\leq)$, and let $x'_{\min}$ and $x'_{\max}$ be those for $(V',E',\leq')$. There are unique paths $\tilde{x}_{\min}$ and $\tilde{x'}_{\min}$
in \((W, F)\) that contract respectively to \(x_{\min}\) and \(x'_{\min}\). Choose a path \(z_{\min}\) in \((W, F)\) passing through the same vertices as \(x_{\min}\) does at odd levels and through the same vertices as \(x'_{\min}\) at even levels. We similarly construct a path \(z_{\max}\) by taking care that it does not share any common edge with \(z_{\min}\). This is possible because the incidence matrices have entries larger than two.

Let us define two homeomorphisms \(\phi : X_F \to X_E\) and \(\phi' : X_F \to X_{E'}\). In constructing \(z_{\min}\), for each even \(n\), we matched a pair of edges in \(F_n \circ F_{n+1}\) with an edge in \(E_{n/2}\), namely
\[
(z_{\min}(n), z_{\min}(n+1)) \to x_{\min}(n/2),
\]
and we match a pair in \(F_{n+1} \circ F_{n+2}\) with an edge in \(E'_{(n+2)/2}\), namely
\[
(z_{\min}(n+1), z_{\min}(n+2)) \to x'_{\min}((n+2)/2).
\]

In the same way \((z_{\max}(n), z_{\max}(n+1))\) is matched with \(x_{\max}(n/2)\) and \((z_{\max}(n+1), z_{\max}(n+2))\) with \(x'_{\max}((n+2)/2)\). Now, for all even \(n\), we extend these matchings in an arbitrary way to bijections respecting the range and source maps from \(F(n) \circ F(n+1)\) to \(E(n/2)\) and from \(F(n+1) \circ F(n+2)\) to \(E'((n+2)/2)\). This defines two homeomorphisms
\[
\phi : X_F \to X_E\quad \text{and} \quad \phi' : X_F \to X_{E'}.
\]
The homeomorphism \(\psi = \phi' \circ \phi^{-1}\) has the desired properties.

Let us show that \(\text{\#}\) implies \(\text{\#}\). We will show that \((X_E, T_E)\) and \((X_{E'}, T_{E'})\) are SOE. In a minimal BV-representation, two points belong to the same orbit if and only if they are cofinal, except when it is the orbit of the minimal path. This implies that \(\psi\) maps orbits to orbits with the possible exception of the orbit of the minimal paths. But since \(\psi(x_u) = x'_u\), \(u \in \{\min, \max\}\), this is also true for the orbit of the minimal paths. Consequently, there are maps \(\alpha : X_E \to \mathbb{Z}\) and \(\beta : X_E \to \mathbb{Z}\) uniquely defined by the relations
\[
\psi \circ T_E(x) = T_E(x) \circ \psi(x)\quad \text{and} \quad \psi \circ T_E(x) = T_E(x) \circ \psi(x)
\]
for all \(x \in X_E\). It remains to prove that \(\alpha\) and \(\beta\) are continuous with the possible exception of \(x_{\max}\) and \(x'_{\max}\). We do it for \(\alpha\). It is similar for \(\beta\).

Let \(x = (x_n)_n \in X_E \setminus \{x_{\max}\}\) and \(k = \alpha(x)\). Let \(n_0\) be such that \((x_1, \ldots, x_{n_0})\) has a non-maximal edge and the minimum number of paths from any vertex in \(V_{n_0-1}\) to \(V_0\) is greater than \(k\).

Let \(y\) belonging to the cylinder \([x_1, \ldots, x_{n_0+1}]\). It suffices to show that \(\alpha(y) = k\). The paths \(T_E(x)\) and \(T_E(y)\) start with the same \(n_0 + 1\) first edges. Thus, from the property of \(\psi\), \(\psi \circ T_E(x)\) and \(\psi \circ T_E(y)\) start with the same \(n_0 + 1\) first edges \(f_1, f_2, \ldots, f_{n_0+1}\):
\[
\psi \circ T_E(x) = (f_1, f_2, \ldots, f_{n_0+1}, x'_{n_0+2}, \ldots)\quad \text{and} \quad \psi \circ T_E(y) = (f_1, f_2, \ldots, f_{n_0+1}, y_{n_0+2}, \ldots).
\]
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For the same reason, and because $x$ and $T_E(x)$, and $y$ and $T_E(y)$ are cofinal from $n_0 + 1$, $\psi(x)$ and $\psi(y)$ start with the same edges $g_1, g_2, \ldots, g_{n_0+1}$ and

$$\psi(x) = (g_1, g_2, \ldots, g_{n_0+1}, x'_{n_0+2}, \ldots)$$

and

$$\psi(y) = (g_1, g_2, \ldots, g_{n_0+1}, y'_{n_0+2}, \ldots).$$

But since there are at least $k$ paths from any vertex in $V_{n_0-1}$ to $V_0$, we deduce that $T_E^k(g_1, g_2, \ldots, g_{n_0+1}) = [f_1, f_2, \ldots, f_{n_0+1}]$ because $\psi \circ T_E(x) = T_E^k \circ \psi(x)$. Therefore $\psi \circ T_E(y) = T_E^k \circ \psi(y)$ and $\alpha(y) = k$.

Let us show that $\text{3}$ implies $\text{4}$. Let $\psi: (X, T) \rightarrow (X', T')$ be a SOE map. Remark that $(X', T')$ is isomorphic to $(X, \psi^{-1} \circ T' \circ \psi)$. Hence we can suppose $X' = X$ and set $S = \psi \circ T' \circ \psi$. Then we have

$$T(x) = S^{n(x)}(x)$$

and $S(x) = T^{\beta(x)}(x)$

where $\alpha$ and $\beta$ are continuous everywhere with $y$ as a possible exception.

Let $A$ be a clopen set not containing $y$. Since $\alpha$ is continuous on $A$, the set $\alpha(A)$ is compact and consequently finite: there exist $n_1, \ldots, n_k$ such that $A = \bigcup_{1 \leq i \leq k} A \cap \alpha^{-1}(\{n_i\})$. Recall that the indicator function (or characteristic function) of the set $A$ is denoted by $\chi_A$. Hence

$$TA = \bigcup_{1 \leq i \leq k} S^{n_i}(A \cap \alpha^{-1}(\{n_i\}))$$

and thus, taking the characteristic function of each side,

$$\chi_A \circ T^{-1} = \sum_{1 \leq i \leq k} \chi_{A \cap \alpha^{-1}(\{n_i\})} \circ S^{-n_i}.$$ 

But since $f - f \circ S^{-n} = (\sum_{1 \leq i \leq n} f \circ S^{-i}) \circ S - (\sum_{1 \leq i \leq n} f \circ S^{-i})$, we deduce that $\chi_A \circ T^{-1} - \chi_A$ belongs to $\partial S(C(X, Z))$.

Now suppose that $A$ contains $y$. Remark that $\chi_A \circ T^{-1} - \chi_A = \chi_{X \setminus A} - \chi_{X \setminus A} \circ T^{-1}$. But since $y$ is not contained in $X \setminus A$ we deduce from the previous case that $\chi_A \circ T^{-1} - \chi_A$ belongs to $\partial S(C(X, Z))$. Since $T$ is invertible we proved that for all clopen set $E$, $\chi_E \circ T - \chi_E$ belongs to $\partial S(C(X, Z))$ and consequently that $\partial_T(C(X, Z)) \subseteq \partial_S(C(X, Z))$. Proceeding similarly with the equality $S(x) = T^{\beta(x)}(x)$ we obtain $\partial_T(C(X, Z)) = \partial_S(C(X, Z))$.

This shows that $H(X, T, Z) = H(X, S, Z)$ and thus $K^0(X, T) = K^0(X, S)$.

Finally the implication $\text{3}$ implies $\text{1}$ results of Theorem $6.1.3$

\[\text{\blacksquare}\]

We give below an example of strongly orbit equivalent minimal Cantor systems which are not conjugate.

**Example 6.5.2** Consider the primitive substitution $\sigma: a \mapsto ab, b \mapsto a^2b^2$. As we shall see in Chapter $\text{7}$ the shift $X(\sigma)$ is conjugate to $(X_E, T_E)$ where $(V, E)$
is the Bratteli diagram represented in Figure 6.5.1 on the right (the morphism \(\sigma\) is read on \((V,E)\)). The incidence matrix of the diagram is

\[
M = \begin{bmatrix}
1 & 1 \\
2 & 2
\end{bmatrix}
\]

which has eigenvalues 0, 3. We claim that the dimension group is isomorphic to \(\mathbb{Z}[1/3]\) with the usual ordering but with order unit 2, that is, the group \(\frac{1}{2}\mathbb{Z}[1/3]\). Indeed, in the basis

\[
u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

we have

\[
\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{3}u + \frac{1}{3}v
\]

Since multiplication by 3 is an automorphism, this proves the claim. Thus \(X(\sigma)\) has the same dimension group as the odometer in base \(p_n = 2.3^{n-1}\). Indeed, the dimension group of this odometer is, by Proposition 7.1.2, the subgroup of \(\mathbb{Q}\) formed of the \(p/q\) with \(q\) dividing some \(2.3^n\), which is \(\frac{1}{2}\mathbb{Z}[1/3]\).

The shift \(X(\sigma)\) and the odometer in base \(2.3^{n-1}\) are thus strong orbit equivalent by Theorem 6.5.1. The BV-representation of this odometer is represented in Figure 6.5.1 on the left. An intertwinning of the Bratteli diagrams is represented in the middle.

Concerning orbit equivalence, we have the following additional result that we quote without proof. Recall from Chapter 3 that we denote by \(\text{Inf}(G)\) the infinitesimal subgroup of a unital ordered group \(G\).

**Theorem 6.5.3** Let \((X,T)\) and \((X',T')\) be two Cantor minimal systems and let \(G = K^0(X,T), G' = K^0(X',T')\) be their dimension groups. The following conditions are equivalent.
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(i) The systems \((X, T)\) and \((X', T')\) are orbit equivalent.

(ii) The groups \(G / \text{Inf}(G)\) and \(G' / \text{Inf}(G')\) are isomorphic.

We give below an example of orbit equivalent shifts which are not strongly orbit equivalent.

Example 6.5.4 Let \(\sigma\) be the primitive and proper substitution \(a \rightarrow aab, b \rightarrow abb\). The corresponding substitution shift is a Toeplitz shift (in the same way as in Example 2.6.1). A BV-representation of \(X(\sigma)\) is given in Figure 6.5.2 on the right. The incidence matrix of this Bratteli diagram is

\[
M = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

It has eigenvalues \(-1, 3\). The dimension group is \(G = \mathbb{Z}[1/3] \times \mathbb{Z}\) with \(G^+ = \mathbb{Z}_+[1/3] \times \mathbb{Z}\) and \(1_G = (1, 1)\). Thus (see Example 3.2.10), the quotient \(G / \text{Inf}(G)\) is isomorphic to \(\mathbb{Z}[1/3]\). By Theorem 5.5.3 the shift \(X(\sigma)\) is orbit equivalent to the odometer \(\mathbb{Z}_3\) whose BV-representation is shown on the left.

6.6 Equivalences on Cantor spaces

We systematically consider in this section pairs \((X, R)\) of a topological space \(X\) and a relation \(R\) on \(X\). After all, the orbit equivalence suggests the idea of studying this notion for its own sake. We will show that this can be successfully realized.

We have in mind two basic examples.

The first one is is the cofinality equivalence \(R_E\) on the space \(X_E\) of paths in Bratteli diagram \((V, E)\). Recall that

\[
R_E = \{(x, y) \in X_E \times X_E \mid \text{for some } N \geq 1, x_n = y_n \text{ for all } n \geq N\}.
\]
Note that $X_E$ is actually a one-sided shift space and that $R_E$ is contained in the orbit equivalence on this shift.

The second one is the orbit equivalence

$$R_T = \{(x, y) \in X \times X \mid T^n x = y \text{ for some } n \in \mathbb{Z}\}$$

when $(X, T)$ is a topological dynamical system.

Let us give a first example of how one may study pairs $(X, R)$ of an equivalence $R$ on a space $X$ on their own.

A relation $R$ on a space $X$ is minimal if for every $x \in X$, the equivalence class of $x$ is dense in $X$.

A set $Y \subset X$ is invariant, or $R$-invariant if it is saturated by $R$, that is, $Y$ is a union of classes of $R$.

We have then the following statement.

**Proposition 6.6.1** If $R$ is minimal, the only closed $R$-invariant sets in $X$ are $X$ and $\emptyset$.

**Proof.** Assume that $Y$ is a closed nonempty $R$-invariant subset of $X$. Then $Y$ is dense in $X$ and thus $Y = X$. ■

We will see below that the converse is also true under an additional condition.

The orbit equivalence $R_T$ of a dynamical system $(X, T)$ is obviously minimal if and only if the system is minimal.

More, interestingly, the cofinality equivalence $R_E$ on a Bratteli diagram is minimal if and only if the diagram is simple.

Indeed, let $(V, E)$ be a simple Bratteli diagram and let $x, y \in X_E$. For every $n \geq 1$ there is an $m \geq n$ such that there is a path $(z_{n+1}, \ldots, z_{m-1})$ from $r(y_n)$ to $s(x_m)$. Let $z^{(n)} = (y_0, \ldots, y_n, z_{n+1}, \ldots, z_{m-1}, x_m, x_{m+1}, \ldots)$. Then $x$ and $z^{(n)}$ are cofinal and $\lim z^{(n)} = y$. This shows that the class of $x$ is dense in $X_E$.

Conversely, assume that $(V, E)$ is not simple. Then we can find a proper subset $W$ of $V$ which contains all its successors and such that if a vertex has all its successors in $W$, then it is in $W$ (such sets will appear again in Chapter 10 under the name of directed and hereditary sets). Let $Y$ be the set of infinite paths $y$ in $X_E$ which pass by a vertex $w$ in $W$. By definition, the set $Y$ contains all paths which coincide with $y$ until $w$. Thus $Y$ is open. Its complement is a closed set $Z$. The set $Z$ is also invariant because if $z \in Z$ is cofinal to $t$, then $t$ cannot pass by a vertex in $W$ and thus $t$ is in $Z$. Finally, the definition of $W$ implies that the set $Z$ is nonempty and strictly included in $X_E$. This shows that $R_E$ is not minimal.

### 6.6.1 Local actions and étale equivalences

Let $X, Y$ be topological spaces. If $U \subset X$ and $V \subset Y$ are clopen sets, a homeomorphism $\gamma : U \to V$ is called a partial homeomorphism.

We denote $s(\gamma) = U$ the source of $\gamma$ and by $r(\gamma) = V$ the range of $\gamma$. 
If $\gamma_1 : U_1 \to V_1$ and $\gamma_2 : U_2 \to V_2$ are such partial homeomorphisms, we denote $\gamma_1 \cap \gamma_2$ the map equal to $\gamma_1$ on the set where the two functions agree. Thus for the intersection to be a partial homeomorphism, we require this set to be open.

A function $f : X \to Y$ is a local homeomorphism if for every $x \in X$, there is a clopen set $U \subset X$ containing $x$ such that $f(U) \subset Y$ is open and that $f|_U$ is a partial homeomorphism.

For a relation $\rho$ on $X$, we may consider its inverse which is the set of pairs $(y,x)$ for $(x,y) \in \rho$. The composition $\sigma \circ \rho$ of two relations $\sigma, \rho$ on $X$ is defined as usual by

$$\sigma \circ \rho = \{(x,y) \in X \times X \mid (x,z) \in \sigma \text{ and } (z,y) \in \rho \text{ for some } z \in X\}.$$ 

Finally the intersection is well defined since a relation on $X$ is just a subset of $X \times X$.

We will find convenient to consider a map $\gamma : X \to X$ as a relation on $X$, via the identification of $\gamma$ and its graph $\{(x,\gamma(x)) \mid x \in X\}$. When $\gamma : U \to V$ is a partial homeomorphism, its inverse (considered as a function or as a relation) $\gamma^{-1} : V \to U$ is again a partial homeomorphism. When $\gamma' : U' \to V'$ is another partial homeomorphism, the composition $\gamma \circ \gamma'$ always exists, even without the requirement that $U' = V$ as usual for maps (note that the notation of composition for relations reverses the order of the factors). If $U' \cap V$ is empty, then $\gamma \circ \gamma'$ is empty.

We denote by $s, r$ the two canonical projections from $X \times X$ onto $X$ defined by $s(x,y) = x$ and $r(x,y) = y$. This is consistent with the notation $s(\gamma) = U$ and $r(\gamma) = V$ for a partial homeomorphism $\gamma : U \to V$. For an open set $U$, we denote $id_U$ the identity map on $U$, that is, $id_U = \{(x,x) \mid x \in U\}$.

A collection $\Gamma$ of partial homeomorphisms of $X$ is a local action if

(i) The collection of sets $U \subset X$ such that $id_U \in \Gamma$ forms a base of the topology.

(ii) The family $\Gamma$ is closed under taking inverses, composition and intersection.

Proposition 6.6.2 If $\Gamma$ is a local action, then

1. $\cup \Gamma$ is an equivalence relation.

2. $\Gamma$ is a basis for a topology on $\cup \Gamma$. 

Note that for $\gamma_1, \gamma_2 \in \Gamma$ the intersection $\gamma_1 \cap \gamma_2$ is the map which is equal to $\gamma_1$ (and $\gamma_2$) on the set of points $x$ where $\gamma_1(x) = \gamma_2(x)$. Thus the condition $\gamma_1 \cap \gamma_2 \in \Gamma$ implies that if $\gamma_1$ and $\gamma_2$ agree on some point $x$, they agree on a neighborhood of $x$.

Regarding a local action $\Gamma$ as a set of binary relations on $X$, we can consider the union $\cup \Gamma'$ of all its elements. Thus $(x,y) \in \cup \Gamma$ if and only if $y = \gamma(x)$ for some $\gamma \in \Gamma$. 

Proposition 6.6.2 If $\Gamma$ is a local action, then

1. $\cup \Gamma$ is an equivalence relation.

2. $\Gamma$ is a basis for a topology on $\cup \Gamma$. 

3. With this topology, the source and range maps \( s, r : \cup \Gamma \to X \) are local homeomorphisms.

**Proof.** 1. For every \( x \in X \), there is by condition (i) a set \( U \subseteq X \) containing \( x \) such that \( \text{id}_U \in \Gamma \). Thus \( \cup \Gamma \) is reflexive. Since \( \Gamma \) is closed by inverse, \( \cup \Gamma \) is symmetric. Finally, since \( \Gamma \) is closed by composition, the relation \( \cup \Gamma \) is transitive (note that we did not use the closure by intersection).

2. This results from the fact that the elements of \( \Gamma \) cover \( \cup \Gamma \) and that \( \Gamma \) is closed under intersection.

3. For \( \gamma \in \Gamma \), denote \( s_\gamma = s \mid _\gamma \). Then \( s_\gamma : (x, \gamma(x)) \mapsto x \) is a bijection from \( \gamma \) to \( s(\gamma) \). We claim that \( s_\gamma \) is a homeomorphism from \( \gamma \) (as a subset of \( R \) with the topology from \( \Gamma \)) to \( s(\gamma) \) (as a subset of \( X \) with usual topology).

To show that \( s_\gamma \) is continuous, consider a clopen set \( U \subseteq X \) such that \( \text{id}_U \in \Gamma \). The set \( s_\gamma^{-1}(U \cap s(\gamma)) = \text{id}_U \circ \gamma \) is in \( \Gamma \) since \( \Gamma \) is closed by composition. Thus it is an open set for the topology of \( \cup \Gamma \). Since such sets \( U \) generate the topology of \( X \) by definition of a local action, we conclude that \( s_\gamma \) is continuous.

To show that \( s_\gamma^{-1} \) is continuous, consider \( \gamma' \in \Gamma \). We have \( s_\gamma(\gamma \cap \gamma') = s(\gamma \cap \gamma') \). Since \( \gamma \cap \gamma' \in \Gamma \), the set \( s(\gamma \cap \gamma') \) is clopen. Thus, \( s_\gamma^{-1} \) is also continuous. This proves the claim.

The proof concerning \( r \) is symmetric.

We are now going to show that each of the two examples of relations on a Cantor space given above are \( \acute{e}tale \) relations.
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6.6.2 The étale relation \( R_E \)

We first consider the cofinality relation \( R_E \) on Bratteli diagram \((V, E)\). We first prove the following statement which associates to it a local action. Let \((V, E)\) be a Bratteli diagram. For \( n \geq 1 \) and \( p, q \in E_{0,n} \) with \( r(p) = r(q) \), let
\[
\gamma(p, q) = \{(x, y) \in X_E \times X_E \mid x \in [p], y \in [q], x_k = y_k(k > n)\}.
\]
Then each \( \gamma(p, q) \) is a partial homeomorphism from \([p]\) to \([q]\).

**Proposition 6.6.4** The set \( \Gamma_E \) of all partial homeomorphisms \( \gamma(p, q) \) is a local action.

*Proof.* Since \( \gamma(p, p) = \text{id}_{[p]} \), the family \( \Gamma_E \) contains all \( \text{id}_{[p]} \) and thus the family of \( U \subset X_E \) such that \( \text{id}_U \) is in \( \Gamma_E \) is a basis of the topology of \( X \). We have \( \gamma(p, q)^{-1} = \gamma(q, p) \) and thus \( \Gamma_E \) is closed under taking inverses. Next \( \gamma(p, q) \circ \gamma(q, r) = \gamma(p, r) \) and thus \( \Gamma_E \) is closed under composition. Finally, \( \gamma(p, q) \cap \gamma(p', q') \) is empty or equal to \( \gamma(p, q) \) if \( p = p' \) and \( q = q' \). Thus \( \Gamma_E \) is closed under intersection. ■

**Theorem 6.6.5** Let \((V, E)\) be a Bratteli diagram. The cofinality equivalence \( R_E \) is an étale equivalence relation with respect to the topology defined by \( \Gamma_E \).

*Proof.* Let us show that \( R_E = \cup \Gamma_E \). If \( x, y \) are cofinal, there is an \( N \geq 1 \) such that \( x_n = y_n \) for all \( n \geq N \). Set \( p = x_0 x_1 \cdots x_{n-1} \) and \( q = y_0 y_1 \cdots y_{n-1} \). Then \((x, y), n \gamma(p, q) \) and thus \((x, y) \in \Gamma_E \). The converse is obvious. ■

6.6.3 The étale relation \( R_T \)

We will prove that the orbit equivalence \( R_T \) on a minimal Cantor system is étale. We first prove that one may associate to \( R_T \) a local action.

**Proposition 6.6.6** Let \((X, T)\) be a minimal Cantor system. Let \( \Gamma_T \) be the set of all partial homeomorphisms of the form \( T^n |_U \) for \( n \in \mathbb{Z} \) and \( U \subset X \) clopen. Then \( \Gamma_T \) is a local action.

*Proof.* Since \( X \) is a Cantor space, the clopen sets form a basis of the topology. And for every clopen set \( U \), we have \( \text{id}_U = T^n |_U \) and thus \( \text{id}_U \) is in \( \Gamma_T \). Next, if \( \gamma = T^n |_U \), then \( \gamma^{-1} \) is the restriction of \( T^{-n} \) to \( T^n U \). Thus \( \Gamma_T \) is closed by taking inverses. It is also closed under composition since for \( \gamma = T^n |_U \) and \( \gamma' = T^n |_{U'} \) with \( U' = T^n U \), we have \( \gamma' \circ \gamma = T^{n+n'} |_U \). Finally, since \((X, T)\) is minimal \( T^n x = T^n x \) implies \( n = n' \). Thus \( \gamma \cap \gamma' \) is either empty or equal to \( \gamma \). ■

**Theorem 6.6.7** Let \((X, T)\) be a minimal Cantor dynamical system. The equivalence \( R_T \) on \((X, T)\) defined by
\[
R_T = \{(x, T^n x) \mid x \in X, n \in \mathbb{Z}\}
\]
is étale with respect to the topology defined by the local action \( \Gamma_T \).
Proof. This follows from the fact that \( R_T = \bigcup \Gamma_T \).

6.7 Entropy and Bratteli diagrams

The topological entropy of a shift space \((X, S)\), denoted by \( h(X, S) \), or simply \( h(S) \), is the growth rate of the number \( p_n(X) \) of finite words of length \( n \) occurring in elements of \( X \), that is

\[
h(X, S) = \limsup_n \frac{\log p_n(X)}{n}.
\]

Thus \( 0 \leq h(X, S) \leq \log \text{Card}(A) \) for a shift on the alphabet \( A \). One can show that the \( \limsup \) is actually a limit (see Exercise 6.17).

Entropy is invariant under conjugacy (Exercise 6.18). It is a measure of the ‘size’ of a shift space. For example, an edge shift on a graph with adjacency matrix \( M \) has entropy \( \log \lambda \) where \( \lambda \) is the dominant eigenvalue of \( M \) (Exercise 6.19). One could expect that minimal shifts have entropy zero. However, one can show that there are minimal shifts of arbitrary entropy.

We will show that, far from being invariant by strong orbit equivalence, every minimal Cantor dynamical system is SOE to a minimal Cantor dynamical system of entropy zero.

To give a convenient way to compute the entropy \( h(T_E) \) of \((X_E, T_E)\) we need some notation.

For \( n \geq 1 \), let \( P(n) \) (paths in Bratteli diagrams) be the set of paths from \( V(0) \) to \( V(n) \). We define \( \pi_n \) on \( X_B \) by \( \pi_n((e_k)_{k \geq 1}) = (e_1, \ldots, e_n) \). We will consider the set \( A_n = \pi_n(X_B) \) as an alphabet. We call \( S_n \) the shift on \( A_n^\mathbb{Z} \). The set

\[
X_n = \left\{ \left( \pi_n \left( V_B^k(x) \right) \right)_{k \in \mathbb{Z}} \mid x \in X_B \right\}
\]

is included in \( A_n^\mathbb{Z} \), \( S_n \)-invariant and compact. Hence \((X_n, S_n)\) is a subshift.

Lemma 6.7.1 One has

\[
h(X_E, T_E) = \lim_{n \to +\infty} h(S_n).
\]

Proof.

Hence, we need first to compute \( h(S_n) \). To this end we will need the following subshifts. When \( W \) is a set of finite words, we denote by \( \Omega(W) \) the subset of all bi-infinite words formed by concatenation of finite words belonging to \( W \). Let \( S_W \) denote the shift map on \( \Omega(W) \). It is clear \((\Omega(W), S_W)\) is a subshift.

Lemma 6.7.2 Let \( W \) be a set of \( m \) finite words of length at least \( l \). Then

\[
h(S_W) \leq \frac{\log m}{l}.
\]
Proof. Let \( k \) be the greatest length of the finite words in \( W \). Let \( w \) be a finite word of length \( n \) occurring in some word of \( \Omega(W) \). Then there exist \( r \) finite words \( m_1, \ldots, m_r \) of \( W \), a prefix \( s \) and suffix \( p \) of some finite words in \( W \) such that \( w = s m_1 \cdots m_r p \). Since \( r \leq \frac{n}{k} \), we deduce that there are at most \( k^2 m^{2+\frac{n}{k}} \) finite words of length \( n \) in \( L_n(\Omega(W)) \), which ends the proof.

An ordering on a Bratteli diagram is a **consecutive ordering** if whenever edges \( e, f \) and \( g \) have the same range, \( e \) and \( g \) have the same source and \( e \leq f \leq g \), then \( e \) and \( f \) have the same source (see Figure 6.7.1).

![Figure 6.7.1: An example of a consecutive ordering viewed from a vertex \( v \in V(n+1) \) to \( V(n) \).](image)

**Proposition 6.7.3** Let \((V, E, \leq)\) be a properly ordered Bratteli diagram where \( \leq \) is a consecutive ordering. Suppose that

\[
\lim_{n \to +\infty} \frac{\log(\eta(n+1) \text{Card}(V(n)))}{\eta(n+1)} = 0,
\]

where \( \eta(n), n \geq 1 \), is the minimum number of edges from a vertex at level \( n-1 \) to a vertex at level \( n \). Then, \( h(T_E) = 0 \).

Proof. Let \( A_n \) and \( X_n \) be defined as above when we described \( h(V_B) \). Let \( u \) be a vertex at level \( n \), and let \( p_1, \ldots, p_s \) be the paths from level 0 to \( u \), listed in increasing order. We set \( W(u) = p_1 \cdots p_s \). We can consider that it belongs to \( A^*_n \). Now, assume that \( v \) is a vertex at level \( n+1 \), that \( a_1, \ldots, a_t \) are the edges from \( u \) to \( v \), listed in increasing order, and that \( y \) is an infinite path in the Bratteli diagram such that \( y_1 \cdots y_{n+1} = p_1 a_1 \). Then,

\[
(\pi_n (V_B^k(y)))_{0 \leq k \leq st-1} = W(u)^t.
\]

Note that \( t \) is greater than \( \eta(n+1) \). Therefore, \( X_n \) is included in \( \Omega(W) \) where

\[
W = \{ W(u)^t \mid u \in V(n), \ \eta(n+1) \leq t < 2\eta(n+1) \},
\]

and consequently, \( h(S_n) \leq h(S_W) \). As \( W \) consists of at most \( \text{Card}(V(n))\eta(n+1) \) finite words of length at least \( \eta(n+1) \), from Lemma 6.7.2 we obtain

\[
h(S_W) \leq \frac{\log(\eta(n+1) \text{Card}(V(n)))}{\eta(n+1)}.
\]

This completes the proof.
CHAPTER 6. BRATTELI DIAGRAMS

**Theorem 6.7.4** Any minimal Cantor dynamical system is strongly orbit equivalent to a minimal Cantor dynamical system of entropy zero.

*Proof.* From Theorem 6.3.3 it suffices to consider a minimal Bratteli-Vershik dynamical system \((X, T)\). Let \(\eta(n), n \geq 1\), be the minimum number of edges from a vertex at level \(n-1\) to a vertex at level \(n\).

From Theorem 6.3.3 we know, by contracting if needed, that we can assume the incidence matrices of \(B = (V, E, \leq)\) to have strictly positive entries. Hence, contracting again if needed, we can suppose that

\[
\lim_{n \to +\infty} \frac{\log(\eta(n+1) \text{Card}(V(n)))}{\eta(n+1)} = 0.
\]

Consider \((V', E', \leq')\) where \(V' = V, E' = E\) and \(\leq'\) is a consecutive ordering. Then, from Proposition 6.7.3 \((X_{E'}, T_{E'})\) has zero entropy and, from Theorem 6.5.1 is strongly orbit equivalent to \((X, T)\).

In this proof, we find all the arguments to prove that all minimal BV-dynamical systems with a consecutive ordering have entropy zero.

**Theorem 6.7.5** Let \(\alpha \in [1, +\infty[\) and \((X, T)\) a minimal Cantor dynamical system. There exists a minimal shift space of entropy \(\log \alpha\) which is strongly orbit equivalent to \((X, T)\).

### 6.8 Exercises

**Section 6.1**

6.1 Show that the following conditions are equivalent for a Bratteli diagram \((V, E)\).

(i) For each \(m \geq 0\) and every vertex \(v\) in \(V(m)\), there exists \(n > m\) such that, for every \(w \in V(n)\), there is a path from \(v\) to \(w\).

(ii) For each \(m \geq 0\), there exists \(n > m\) such that, and every vertex \(v\) in \(V(m)\) and for every \(w \in V(n)\), there is a path from \(v\) to \(w\).

(iii) The Bratteli diagram \((V, E)\) is simple.

6.2 Show that the equivalence on Bratteli diagrams (ordered or not) generated by telescoping is given by \((V, E) \equiv (V', E')\) if there exists a Bratteli diagram \((W, F)\) such that telescoping to odd levels gives a telescoping of \((V, E)\) and telescoping to even levels gives a telescoping of \((V', E')\).

6.3 Two nonnegative integer square matrices \(M, N\) are said to be \(C^*\)-equivalent if there are sequences \(R_n, S_n\) of nonnegative matrices and \(k_n, \ell_n\) of integers such that

\[
M^{k_n} = R_n S_n, \quad N^{\ell_n} = S_n R_{n+1}. \quad (6.8.1)
\]
for all $n \geq 1$. Show that $C^*$-equivalence is an equivalence relation containing shift equivalence. Hint: show that $M, N$ are $C^*$-equivalent if and only if the stationary Bratteli diagrams with matrices $M$ and $N$ are equivalent modulo intertwining.

Let $M, N$ be the matrices

$$
M = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1
\end{bmatrix} = I + P, \quad N = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1
\end{bmatrix} = QM \quad (6.8.2)
$$

where $P$ and $Q$ denote the matrices of the permutations $(12345)$ and $(123541)$ respectively. Show that $M, N$ are $C^*$-equivalent (it can be shown that they are not shift equivalent, see the notes for a reference).

**Section 6.2**

6.4 Show that the Bratteli compactum $X_E$ of a simple Bratteli diagram $(V, E, \leq)$ is a Cantor space.

6.5 Show that a Bratteli diagram $(V, E)$ is simple if and only if each class of the equivalence $R_E$ of cofinality is dense in $R_E$.

6.6 Show that $X_E^{\text{max}}$ and $X_E^{\text{min}}$ are non-empty sets for every ordered Bratteli diagram $(V, E, \leq)$ and more precisely that the set of maximal (resp. minimal) edges forms a spanning tree of $(V, E)$.

6.7 Show that every simple Bratteli diagram can be properly ordered.

6.8 Let $(V, E, \leq)$ be properly ordered Bratteli diagram. Show that the Vershik map $T_E : X_E \to X_E$ is one-to-one.

6.9 Show that for every properly ordered Bratteli diagram $(V, E, \leq)$, the system $(X_E, T_E)$ is minimal.

6.10 Show that a telescoping of Bratteli diagrams from $(V, E, \leq)$ to $(V', E', \leq')$ induces a conjugacy from $(X_E, T_E)$ to $(X_{E'}, T_{E'})$.

**Section 6.3**

6.11 Prove that the map $\phi$ in the proof of Theorem 6.3.3 is a homeomorphism.

6.12 Let $(X, T)$ be a minimal invertible Cantor system. Let $\mathcal{P}(n)$ be a refining sequence of partitions in towers with bases $B(n) = \bigcup_{1 \leq t \leq t(n)} B_t(n)$ and let $(V, E)$
be the corresponding Bratteli diagram. Let $\phi : X_E \to X$ be the conjugacy from $(X_E, T_E)$ onto $(X, T)$ defined by (6.3.7). Show that

$$B_t(n) = \phi([e_1, \ldots, e_n])$$

(6.8.3)

where $(e_1, \ldots, e_n)$ is the minimal path from $V(0)$ to $(n, t_n) \in V(v)$. Conclude that $\phi$ sends the partition $[e_1, \ldots, e_n]$ for $(e_1, \ldots, e_n) \in \mathcal{E}_{1,n}$ to the partition $\mathcal{P}(n)$.

Section 6.4

6.13 Let $(X_1, T_1)$ and $(X_2, T_2)$ be minimal topological dynamical systems. Say that $(X_1, T_1)$ is a derivative of $(X_2, T_2)$ if $(X_1, T_1)$ is isomorphic to an induced transformation of $(X_2, T_2)$. We also say in this case that $(X_2, T_2)$ is a primitive of $(X_1, T_1)$. Show that two minimal transformations $S, T$ have a common derivative if and only if they have a common primitive.

6.14 Show that Kakutani equivalence is an equivalence relation.

6.15 A Bratteli diagram is stationary if all its adjacency matrices are equal. Prove that the family of systems isomorphic to a stationary BV-dynamical system is stable under Kakutani equivalence.

Section 6.5

6.16 Let $(X, T)$ and $(Y, S)$ be two Cantor minimal systems such that if there is homeomorphism $\phi : X \to Y$ which sends orbits to orbits, that is such that there are two maps $\alpha, \beta$ such that $\phi \circ T(x) = S^{\alpha(x)} \circ \phi(x)$ and $\phi \circ T^{\beta(x)}(x) = S \circ \phi(x)$. Our aim is to show that if $\alpha$ is continuous, then $(X, T)$ is conjugate to $(Y, S)$ or to $(Y, S^{-1})$.

1. Show that, replacing $S$ by $\phi^{-1} \circ S \circ \phi$, we may assume that $X = Y$. Set $T_k(x) = S^{f_k(x)}(x)$. Show that $k \mapsto f_k(x)$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$ which satisfies the cohomological equation

$$f_k(T^j(x)) = f_{k+j}(x) - f_j(x).$$

(6.8.4)

2. Show that for every integer $M > 0$ there is an integer $\tilde{M}$ such that $[-M, M] \subset \{f_k(x) \mid \tilde{M} \leq k \leq \tilde{M}\}$ for all $x \in X$.

3. For $m > 0$, set

$$A_m = \{x \mid \forall n \geq m, f_n(x) > 0 \text{ and } f_{-n}(x) < 0\},$$

$$B_m = \{x \mid \forall n \geq m, f_n(x) < 0 \text{ and } f_{-n}(x) > 0\}.$$

Show that there is $K > 0$ such that $X = A_K$ or $X = B_K$. Assume that $X = A_K$ (the other case is symmetric).
4. Set
\[ P_m(x) = \text{Card}\{f_i(x) \mid f_i(x) > 0, |i| \leq m\}, \]
\[ N_m(x) = \text{Card}\{f_i(x) \mid f_i(x) < 0, |i| \leq m\}, \]
and \( a(x) = (N_M(x) - P_M(x))/2 \) (which is independent of \( M \) for \( M \geq K \)).
Show that \( a(Sx) = a(x) - j + 1 \) where \( j \) is such that \( f_j(x) = 1 \).

5. Show that \( g(x) = T^{a(x)}x \) is a conjugacy from \((X,T)\) to \((X,S)\).
Conclude that \((X,T)\) is conjugate to \((Y,S)\) or to \((Y,S^{-1})\).

Section 6.7

6.17 Let \((u_n)\) be a sequence of real numbers such that \( u_{n+m} \leq u_n + u_m \) for all \( n,m \geq 0 \) (such a sequence is called \textit{subadditive}). Show that the limit \( \lim u_n/n \) exists and is equal to \( \inf u_n/n \) (\textit{Fekete’s Lemma}). Conclude that the \( \limsup \) in the definition of topological entropy can be replaced by a limit.

6.18 Show that entropy is invariant under conjugacy.

6.19 Let \((X,S)\) be the edge shift on a graph \( G \). Let \( M \) be the adjacency matrix of \( G \) and let \( \lambda \) be its dominant eigenvalue. Show that \( h(X,S) = \log \lambda \).

6.9 Solutions

Section 6.1

6.1 (i)⇒(ii) is clear since every \( V(m) \) is finite. (ii)⇒(iii) Denote \( f : \mathbb{N} \to \mathbb{N} \) the function defined by \( f(n) = n \) if \( n > m \) is the least integer such that (ii) holds. Then the telescoping of \((V,E)\) with respect to this sequence has the desired property. (iii)⇒(i) is clear.

6.2 We have to show that the relation is transitive. For this, consider Bratteli diagrams \( B_1, B_2, B_3 \) such that \( T_1 \) is an intertwining of \( B_1, B_2 \) and \( T_2 \) is an intertwining of \( B_2, B_3 \). We will build an intertwining \( T_3 \) of \( B_1, B_3 \). This will prove that the relation is transitive.

Denote by \( B_1(n,m) \) the matrix of \( B_1 \) between levels \( n \) and \( m \) and similarly for \( B_2, B_3, T_1, T_2 \). For every \( n \geq 1 \), there is an integer \( \ell_1(n) \) such that
\[
T_1(n,0) = \begin{cases} 
B_1(\ell_1(n),0) & \text{if } n \text{ is odd} \\
B_2(\ell_1(n),0) & \text{if } n \text{ is even}
\end{cases}
\]
and there is a similar integer \( \ell_2(n) \) for \( T_2 \) related to \( B_2 \) and \( B_3 \). We start with \( T_3(1,0) = T_1(1,0) \). Next, by telescoping \( T_2 \), we can obtain \( \ell_2(1) \geq \ell_1(2) \). We define
\[
T_3(2,1) = T_2(2,1)B_2(\ell_2(1),\ell_1(2))T_1(2,1).
\]
This corresponds to the path from level 2 in $T_2$ to level 1 in $T_1$ indicated in Figure 6.9.1. We verify that, with this choice, $T_3(2,0) = B_3(\ell_2(2),0)$. This follows by inspection of Figure 6.9.1 or by a patient verification as below.

\[
T_3(2,0) = T_3(2,1)T_3(1,0) \\
= (T_2(2,1)B_2(\ell_2(1),\ell_1(2))T_1(2,1))T_3(1,0) \\
= (T_2(2,1)B_2(\ell_2(1),\ell_1(2))T_1(2,1))T_1(1,0) \\
= T_2(2,1)B_2(\ell_2(1),\ell_1(2))T_1(2,0) \\
= T_3(2,1)B_2(\ell_2(1),0) \\
= T_2(2,0) = B_3(\ell_2(2),0).
\]

We perform one more step to convince everybody that the construction can continue in the same way forever. We may assume, again by telescoping $T_2$ if necessary, that $\ell_1(4) \geq \ell_2(3)$. Then we define

\[
T_3(3,2) = T_1(5,4)B_2(\ell_1(4),\ell_2(3))T_2(3,2)
\]

and the reader may verify as above that $T_3(3,1) = B_1(\ell_1(5),\ell_1(1))$.

The fact that $C^*$-equivalence is the same as the existence of an intertwining of the stationary Bratteli diagrams with matrices $M,N$ appears clearly on the diagram of Figure 6.9.2. As a consequence, the $C^*$-equivalence is an equivalence relation since the existence of an intertwining is an equivalence (Exercise 6.2).
The matrices $M, N$ given by Equation (6.8.2) have the same maximal eigenvalue 2 and the same left and right eigenvectors $[1 \ 1 \ 1 \ 1 \ 1]$. We may conjugate to write

$$M = \begin{bmatrix} 2 & 0 \\ 0 & M_1 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 0 \\ 0 & N_1 \end{bmatrix}$$

Let $\lambda$ be the maximal modulus of the eigenvalues of $M_1, M_2$ and let $\mu$ be the maximal modulus of eigenvalues of their inverses. For every $k \geq 1$, we have

$$N^{-k}M^c = \begin{bmatrix} 2^{(c-1)k} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N_1^{-k}M_1^c \end{bmatrix}$$

After we conjugate back, the entries of the first term will be at least $c_12^{(c-1)k}$ and those of the second term will be at most $c_2\lambda^k\mu^k$. Thus, if we choose $c$ such that $2^{(c-1)k} > \lambda^k\mu^k$, all entries of $N^{-k}M^c$ will be positive. This shows that for all large enough $k$, the matrix $N^{-k}M^n$ is positive for all $n$ large enough. It is also integral because $M, N$ have both determinant equal to 2.

Consider now the construction of sequences $(R_n, S_n)$ and $(k_n, \ell_n)$ such that Equation (6.8.1) holds for all $n \geq 1$. Start with $S_1 = I$. Equations (6.8.1) are equivalent to the equations

$$\begin{align*}
R_1 &= M^{k_1} \\
R_2 &= M^{-k_1}N^{\ell_1} \\
S_2 &= N^{-\ell_1}M^{k_1+k_2} \\
R_3 &= M^{-k_1-k_2}N^{\ell_1+\ell_2}
\end{align*}$$

and so on. By the preceding remark, we can successively choose $k_1, \ell_1, k_2, \ell_2, \ldots$ in such a way that all $R_n, S_n$ are nonnegative and integral. Thus $M, N$ are $C^*$-equivalent.

Section 6.2

6.4 The $X_E$ is compact. If $(V', E')$ is obtained by telescoping $(V, E)$, the induced map from $X_E$ to $X_{E'}$ is a homeomorphism. Thus, to show that, if
(V, E) is simple then $X_E$ has no isolated points, we can assume that there is an edge between any vertices in $V(n-1)$ and $V(n)$. Assume that $x = (e_1, e_2, \ldots)$ is an isolated point. Then $\{e_1, \ldots, e_n\} = \{x\}$ for some $n \geq 1$. But then $E(m)$ has to be reduced to $\{e_m\}$ for all $m > n$, a contradiction with our hypothesis that $X_E$ is infinite.

6.5 Assume that $(V, E)$ is simple. Let $e = (e_n)_{n \geq 1}$ and $f = (f_n)_{n \geq 1}$ be elements of $X_E$. We show that $f$ belongs to the closure of the class of $e$. Since $(V, E)$ is simple, for every $n \geq 1$, there is by Exercise 6.1 an integer $m > n + 1$ such that there is a path $(g_{n+1}, \ldots, g_{m-1})$ from $r(f_n)$ to $s(e_m)$. Then $h^{(n)} = (f_1, \ldots, f_n, g_{n+1}, \ldots, g_{m-1}, e_m, e_{m+1}, \ldots)$ is a path in $X_E$ which is in the cofinality class of $e$. Since the sequence $h^{(n)}$ tends to $f$ when $n \to \infty$, the claim is proved.

6.6 For each $n \geq 1$, set

$$F_n = \{[e_1, \ldots, e_n] \in E_{1,n} \mid \text{every } e_i \text{ is maximal}\}$$

An easy induction on $n$ shows that for every vertex $v \in V(n + 1)$ there is an element $[e_1, \ldots, e_n]$ in $F_n$ such that $r(e_n) = v$. This proves that the set of maximal edges forms a spanning tree. Next $F_n$ is closed. Thus, the set $X_E^{\text{max}}$ is the intersection of the nonempty closed sets $F_n$. Since $X_E$ is compact, it is nonempty. The argument for $X_E^{\text{min}}$ is similar.

6.7 Let $(V, E)$ be a simple Bratteli diagram. Fix an order on $V(n)$ for each $n \geq 1$. We order the edges $e \in E(n)$ in such a way that, for two edges $e, f$ with the same range, one has $s(e) < s(f) \Rightarrow e < f$. Then any long enough path made of minimal edges leads to the minimal vertex.

6.8 The inverse of $T_E$ can be described as follows. First $T_E^{(-1)}(x_{\text{min}}) = x_{\text{max}}$. Next, for $e = (e_n)_{n \geq 1}$ distinct of $x_{\text{min}}$, let $k \geq 1$ be the minimal index such that $e_k$ is not a minimal edge. Then $T_E^{(-1)}(e) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$ where $f_k$ is the antecedent of $e_k$ and $(f_1, \ldots, f_{k-1})$ is the maximal path from $v(0)$ to $s(f_k)$.

6.9 We claim that if $e_m = f_m$ for $m > n$, and if $e_n < f_n$, then there is a $k \geq 0$ such that $T_E^k(e) = f$. This proves that the classes of cofinality are contained in the orbits of $T_E$ and thus implies the statement since the classes of $R_E$ are dense by Exercise 6.5.

One proves the claim by induction on $n \geq 1$. It is clearly true for $n = 1$. Next, let $(g_1, \ldots, g_{n-1})$ be a path of maximal edges from $v(0)$ to $s(e_n)$. Then there is an integer $k \geq 1$ and a path $(h_1, \ldots, h_{n-1})$ of minimal edges such that $T_E^k(g_1, \ldots, g_{n-1}, e_n, e_{n+1}, \ldots) = (h_1, \ldots, h_{n-1}, f_n, f_{n+1}, \ldots)$. By induction hypothesis, there is an $\ell \geq 0$ such that $T_E^\ell(h_1, \ldots, h_{n-1}, f_n, f_{n+1}, \ldots) = f$. This proves the claim.
Let \( m_0 = 0 < m_1 < m_2 < \ldots \) be the sequence defining the telescoping from \((V, E, \leq)\) to \((V', E', \leq')\). Let \( \varphi : X_E \to X_{E'} \) be the map defined by \( y = \varphi(x) \) if \( x = (e_1, e_2, \ldots) \) and \( y = (f_1, f_2, \ldots) \) with \( f_n = (e_{m_n+1}, \ldots, e_{m_{n+1}}) \).

The map \( \varphi \) is clearly a homeomorphism from \( X_E \) onto \( X_{E'} \). To show that it is a conjugacy, we have to show that \( \varphi(T_E x) = T_{E'} \varphi(x) \). Consider first the case of \( x = x_{\text{max}} \). Then \( \varphi(T_E x) = \varphi(x_{\text{min}}) = y_{\text{min}} \) while \( T_{E'} \varphi(x) = T_{E'} y_{\text{max}} = y_{\text{min}} \).

Next, let \( x = (e_1, e_2, \ldots) \neq x_{\text{max}} \) and \( y = (e_1', e_2', \ldots) = \varphi(x) \). Let \( k \) be the least index \( n \) such that \( e_n \) is not maximal and let \( f_k \) be the successor of \( e_k \). Let \( n \) be such that \( m_n < k \leq m_{n+1} \).

Then

\[
\varphi(T_E x) = \varphi(e_1, \ldots, f_k, \ldots) = (e_1', \ldots, f_k', \ldots)
\]

with \( f_k' = (e_{m_{n+1}}, \ldots, f_k, \ldots, e_{m_{n+1}}) \) while

\[
T_{E'} \varphi(x) = T_{E'} (e_1', e_2', \ldots) = (e_1', \ldots, f_k', \ldots)
\]

since \( f_k' \) is the successor of \( e_k' \) in \( E_k' \). Thus \( \varphi \) is a conjugacy.

**Section 6.3**

Let \( \psi : X \to X_E \) be defined as follows. For every \( x \in X \) and \( n \geq 1 \), since \( \mathcal{P}(n) \) is a partition of \( X \), there is a unique pair \( (k_n, t_n) \) with \( 1 \leq k_n \leq h_{i_n}(n) - 1 \) and \( 1 \leq t_n \leq t(n) \) such that \( x \in T^{k_n} B_{t_n}(n) \). We set \( k_0 = 0 \) and \( t_0 = 1 \). Then \( k_n \geq k_{n-1} \) for all \( n \geq 1 \). The map \( \psi(x) = (n, t_{n-1}, t_n, k_n - k_{n-1})_{n \geq 1} \) is well defined and continuous. Since \( \psi \) is obviously the inverse of \( \varphi \), the result follows.

By definition of \((V, E)\), we have \( e_i = (i, t_{i-1}, t_i, j_i) \) for \( 1 \leq i \leq n \). Since each \( e_i \) is minimal, we have \( j_i = 0 \). Thus, by definition of \( \phi \),

\[
\phi([e_1, \ldots, e_n]) = T^{\Sigma_{i=1}^n j_i} B_{t_n}(n) = B_{t_n}(n).
\]

**Section 6.4**

Let first \((Z, U)\) be a common primitive of \((X, S)\) and \((Y, T)\). Thus there are nonempty clopen sets \( A, B \subset Z \) such that \((X, S) \simeq (A, U_A)\) and \((Y, T) \simeq (B, U_B)\). For every \( n \geq 0 \), \((A, U_A)\) is isomorphic to \((U^n A, U^n A)\). Since \( U \) is minimal, we may assume that \( C = U^n A \cap B \neq \emptyset \). Thus \((C, U_C)\) is a common derivative of \((X, S)\) and \((Y, T)\).

Conversely, let \((Z, U)\) be a common derivative of \((X, S)\) and \((Y, T)\). Then, using the tower notation of Exercise 2.7, we have \( S \simeq (Z^g, U^g) \) and \( T \simeq (Z^h, U^h) \). Let \((W, R)\) be the tower corresponding to \( g + h \). Then \((W, R)\) is isomorphic to \((X, S)^k\) with \( k \) defined by

\[
k(z, i) = \begin{cases} 1 & \text{if } i < g(z) \\ h(z) + 1 & \text{if } i = g(z) \end{cases}
\]
through the map
\[
\varphi(z, \ell) = \begin{cases} 
(z, \ell, 1) & \text{if } \ell < g(z) \\
(z, g(z), \ell + 1) & \text{otherwise}
\end{cases}
\]
Thus \((W, R)\) is a primitive of \((X, S)\). In the same way, \((W, R)\) is a primitive of \((Y, T)\).

6.14 We have to show the transitivity of the relation \((X, S) \sim (Y, T)\) if \((X, S)\) and \((Y, T)\) are Kakutani equivalent. Suppose that \((X, S) \sim (Y, T)\) and \((Y, T) \sim (Z, U)\). Let \((X_1, S_1)\) be a common derivative of \((X, S)\) and \((Y, T)\) and let \((X_2, S_2)\) be a common derivative of \((Y, T)\) and \((Z, U)\) (see Figure 6.9.3). Since \((X_1, S_1)\) and \((Z_1, U_1)\) have a common primitive, namely \((Y, T)\) they have
\[
(X, S) \quad (Y, T) \quad (Z, U)
\]
\[
\downarrow \quad \downarrow \quad \downarrow
\]
\[
(X_1, S_1) \quad (Z_1, U_1)
\]
\[
\downarrow \quad \downarrow
\]
\[
(X_2, S_2)
\]
Figure 6.9.3: Transitivity of Kakutani equivalence
by Exercise 6.13 a common derivative \((X_2, S_2)\). Thus \((X, S) \sim (Z, U)\).

6.15 Let \((V, E)\) be a stationary Bratelli diagram. By Theorem 6.4.1 a system \((X, T)\) Kakutani equivalent to \((X_E, T_E)\) is isomorphic to a BV-system \((X_{E'}, V_{E'})\) where \((V', E')\) is obtained from \((V, E)\) by a finite number of changes. Then a telescoping of \((V', E')\) gives again a stationary Bratelli diagram. Thus \((X, T)\) is conjugate to a stationary BV-system.

Section 6.5

6.16
1. Set \(\tilde{S} = \phi^{-1} \circ S \circ \phi\). Then \(T(x) = \tilde{S}^{\alpha(x)}(x)\). Thus, replacing \(S\) by \(\tilde{S}\), we may assume \(X = Y\). Since \((X, T)\) is a Cantor system, there are no periodic points. Thus \(k \mapsto f_k(x)\) is a bijection. It satisfies (6.8.4) since
\[
T^{k+j}x = S^{f_k(T^jx)}(T^jx) = S^{f_k(T^jx)+f_j(x)x}.
\]
2. Follows from compactness since the \(f_k\) are continuous.
3. Set \(n_0 = \sup_{x \in X} |\alpha(x)|\) and choose \(K\) such that \([-n_0, n_0] \subset \{f_i(x) \mid -K \leq i \leq K\}\) for all \(x \in X\). Then \(X = A_K \cup B_K\). Since \(A_K, B_K\) are closed and invariant, the conclusion follows.
4. This follows by a counting argument since $f_{j+k}(x) = f_k(Sx) + 1$ for every $k$ by (6.8.4).

5. We have

\[
g \circ S(x) = T^{a(Sx)}(Sx) = T^{a(x) - j + 1}(T^j(x)) = T^{a(x)+1}(x) = T(T^{a(x)}(x)) = T \circ g(x).
\]

Section 6.7

6.17 Set $\alpha = \inf u_n/n$. We show that for every $\varepsilon > 0$, we have $(u_n/n) - \alpha \leq \varepsilon$ for all large enough $n$. Let $k$ be such that $u_k/k < \alpha + \varepsilon/2$. Then, for $0 \leq j < k$ and $m \geq 1$, we have

\[
\frac{u_{mk+j}}{mk+j} \leq \frac{u_{mk}}{mk} + \frac{u_j}{mk+j} \leq \frac{u_{mk}}{mk} + \frac{u_j}{mk} \\
\leq \frac{mu_k}{mk} + \frac{jv_1}{mk} \leq \frac{u_k}{k} + \frac{v_1}{m} \leq \alpha + \frac{\varepsilon}{2} + \frac{v_1}{m}.
\]

Hence, if $nmk + j$ is large enough so that $u_1/m < \varepsilon/2$, then $u_n/n < \alpha + \varepsilon$.

Set $v_n = p_n(X)$. Then $v_{n+m} \leq v_nv_m$ since a word of length $n + m$ is determined by its prefix of length $n$ and its suffix of length $m$. Thus the sequence $u_n = \log v_n$ is subadditive.

6.18 Let us show that if $\varphi : X \to Y$ is a surjective morphism from $(X,T)$ onto $(Y,S)$, then $h(Y) \leq h(X)$. By Theorem 2.2.10, $\varphi$ is the sliding block code associated to some block map $f : \mathcal{L}_{n+m+1} \to B$, extended to a map from $\mathcal{L}_{n+m+k}$ to $\mathcal{L}_k(Y)$ for every $k \geq 1$. Since $\varphi$ is surjective, $f$ is surjective. Thus $p_k(Y) \leq p_{n+m+k}(X)$ for every $k \geq 1$, which implies

\[
\frac{p_k(Y)}{k} \leq \frac{p_k(X)}{k} + \frac{p_n(X) + p_m(X)}{k}.
\]

The second term of the right hand side tends to 0 when $k \to \infty$ and thus $h(Y) \leq h(X)$.

6.19 It is easy to show that

\[
c\lambda^n \leq \sum_{i,j} M^n_{i,j} \leq d\lambda^n.
\]

for some constants $c, d$. Since $p_n(X) = \sum_{i,j} M^n_{i,j}$, the result follows.
6.10 Notes

Let us begin by a historical overview of the contents of this chapter.

In 1972 Ola Bratteli [Bratteli, 1972] introduced special infinite graphs subsequently called Bratteli diagrams which conveniently encoded the successive embeddings of an ascending sequence \((A_n)_{n \geq 0}\) of finite-dimensional semi-simple algebras over \(\mathbb{C}\) (“multi-matrix algebras”). The sequence \((A_n)_{n \geq 0}\) determines a so-called approximately finite-dimensional (AF) \(C^*\)-algebra (see Chapter 10). Bratteli proved that the equivalence relation on Bratteli diagrams generated by the operation of telescoping is a complete isomorphism invariant for AF-algebras.

From a different direction came the extremely fruitful idea of A. M. Vershik [Vershik, 1985] to associate dynamics (called adic transformations) with Bratteli diagrams (Markov compacta) by introducing a lexicographic ordering on the infinite paths of the diagram. By a careful refining of Vershik’s construction, R. H. Herman, I. F. Putnam and C. F. Skau [Herman et al., 1992] succeeded in showing that every minimal Cantor dynamical system is isomorphic to a Bratteli-Vershik dynamical system.

6.10.1 Bratteli diagrams

The BV representation theorem (Theorem 6.3.3) saying that \((X, T)\) can be topologically realised as a BV-dynamical system is the main result of Herman et al. (1992). We recall that A. M. Vershik obtained in Vershik (1985) such a result in a measure-theoretic context.

Theorem 6.1.5 is due to Elliott [Elliott, 1976] (see also Krieger [1980b]).

6.10.2 Kakutani equivalence

Kakutani equivalence was introduced in the context of measure-theoretic systems [Kakutani, 1943]. Theorem 6.4.1 is from Giordano et al. (1995). Exercises 6.13 and 6.15 are from Ornstein et al. (1982) (the context is for measure-theoretic systems but the arguments transpose easily).

6.10.3 Strong orbit equivalence

The Strong Orbit Equivalence theorem (Theorem 6.5.1) is due to Giordano et al. (1995). We follow the proof proposed in Glasner and Weiss (1995), giving more details.

Exercise 6.16 is due to Boyle (1983) (see Giordano et al., 1995, Theorem 2.4)). See also Boyle and Tomiyama (1998). Two systems \((X, T)\) and \((Y, S)\) such that \((X, T)\) is conjugate to \((Y, S)\) or to \((Y, S^{-1})\) are called flip equivalent.
6.10.4 Equivalences on Cantor spaces

The consideration of etale equivalence relations to formulate results on orbit equivalence in topological dynamical systems is due to Putnam (2010). Etale equivalence relations are a particular case of the notion of etale groupoid introduced by Renault (2014). We follow here the beautiful book of Putnam (2018).

6.10.5 Entropy

We suggest Walters (1982) or Lind and Marcus (1995) for an introduction to topological entropy.

The fact that there exist minimal systems of arbitrary entropy is due to Grillenberger (1972/73) who proved that for every alphabet $A$ with $d \geq 2$ letters and every $h \in [0, \log d)$ there exists a uniformly recurrent word on $A$ with entropy $h$. The fact that every minimal Cantor dynamical system is SOE to a minimal Cantor dynamical system of entropy zero (Theorem 6.7.4) is from Boyle and Handelman (1994).

In Boyle and Handelman (1994) the authors use a different lemma instead of Lemma 6.7.2. They show that $h(S_W) = \log m^l$, whenever $W$ is a set of $m$ distinct finite words of length $l$.

Theorem 6.7.4 shows that there can be dynamical systems with different entropies in a strong orbit equivalence class. Hence it is natural to ask whether all entropies can be realised inside a given class. M. Boyle and D. Handelman showed in Boyle and Handelman (1994) that it is true in the class of the odometer ($\mathbb{Z}_2, x \mapsto x + 1$). Later, F. Sugisaki proved in Sugisaki (2003) that it is true in any strong orbit equivalence class. Theorem 6.7.5 showing that the realizations can be chosen to be subshifts, is from Sugisaki (2007).

6.10.6 Exercises

The solution of Exercise 6.2 has been kindly provided to us by Ian Putnam. Exercise 6.3 is from Bratteli et al. (2000) where a proof that the matrices of Equation (6.8.2) are not shift equivalent is given (we have not included it here as it uses technical tools from algebraic number theory which are out of our scope). The decidability of $C^*$-equivalence of matrices was proved in Bratteli et al. (2001).
Chapter 7

Substitution shifts

In this chapter, we give examples of BV-representations for some classical dynamical systems. We first treat the case of odometers and show that they can be represented by Bratteli diagrams with one vertex at each level.

We describe next the construction of BV-representations for substitutive shifts. The main result is Theorem 7.2.1 which states that the family of Bratteli-Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to isomorphism) the disjoint union of the family of infinite substitution minimal systems and the family of stationary odometer systems.

In the next sections, we treat the cases of linearly recurrent shifts (Section 7.3) and of \( S \)-adic shifts (Section 7.4). In Section 7.5 we give a description of the dimension group of unimodular \( S \)-adic shifts (Theorem 7.5.4). This will be used, in Chapter 8, to the case of dendric shifts. Finally, in Section 7.6 we prove a result characterizing substitutive sequences by a finiteness property of the set of their derivatives (Theorem 7.6.1).

7.1 Odometers

Let \((p_n)_{n \geq 1}\) be a strictly increasing sequence of natural integers such that \(p_n\) divides \(p_{n+1}\) for all \(n \geq 1\). We endow the set \(X = \prod_{n \geq 1} \mathbb{Z}/p_n \mathbb{Z}\) with the product topology of the discrete topologies. The set

\[ Z_{(p_n)} = \{(x_n)_{n \geq 1} \in X \mid x_n \equiv x_{n+1} \mod p_n\} \]

is a group for componentwise addition, called the group of \((p_n)\)-adic integers. It is a compact topological group (see Exercise 7.1). A basis of the topology is given by the sets

\[ [a_1, a_2, \ldots, a_m] := \{(x_n) \in Z_{(p_n)} \mid x_i = a_i, 1 \leq i \leq m\} \]

The neutral element is 0 = \((0,0,\ldots)\).
As a variant of the definition, one starts with an arbitrary sequence \((q_n)_{n \geq 1}\) of natural integers and considers the group

\[ Y = \{(y_n)_{n \geq 0} \mid 0 \leq y_n \leq q_{n+1} - 1\} \]

of \((q_n)\)-adic expansions using for addition the sum with carry from left to right as in an expansion of a real number with respect to an integer basis. This is equivalent to the previous definition using \(q_1 = p_1\) and \(q_{n+1} = p_{n+1}/p_n\) for \(n \geq 1\) (Exercise 7.4).

When \(p_n = p^n\) for all \(n \geq 1\), this defines the classical group of \(p\)-adic integers \(\mathbb{Z}_p\). The corresponding odometer has already been met several times (see Example 5.1.8).

When \(p_n = n!\) for all \(n \geq 1\), the group \(\mathbb{Z}_{(n!)}\) is called the group of profinite integers. The corresponding expansion of integers denoted \(x = (c_1, c_2, \ldots)\) if \(x = c_1 + c_2! + c_3! + \ldots\) with \(0 \leq c_i < i\) is called the factorial number system.

Let \(T : \mathbb{Z}_{(p_n)} \to \mathbb{Z}_{(p_n)}\) be the map \(x \mapsto x + 1\) where \(1 = (1, 1, \ldots)\). The pair \((\mathbb{Z}_{(p_n)}, T)\) is called odometer in base \(p_n\). It is a minimal dynamical system (indeed, the orbit of 0 is dense and 0 is in the closure of the orbit of every point).

The odometer in base \(n!\) is called the universal odometer.

Let us compute the BV-representation of an odometer. For all \(n\), we set \(B(n) = \{[0^{n-1}]\}\), \(t(n) = 1\), \(h(n) = p_n\) and

\[ \mathfrak{P}(n) = \{T^jB(n) \mid 0 \leq j \leq h(n) - 1\}. \quad (7.1.1) \]

Then \((\mathfrak{P}(n))_n\) is a refining sequence of KR-partitions (see Exercise 6.8).

Remark that \(R'B(n) = [j_0 j_1 \cdots j_{n-1}]\) where \(j_i = j_i \mod p_i\). The edges of the BV-representation of \((\mathbb{Z}_{(p_n)}, R)\) given in Section 6.3 are of the form \((n, 1, 1, l)\), with \(0 \leq l \leq q_n - 1 = \frac{p_n}{p_{n-1}} - 1\).

Thus an odometer has a BV-representation with one vertex at each level. The converse is also clearly true. We can thus state the following simple nice result.

**Theorem 7.1.1** A Cantor dynamical system is an odometer if and only if it has a BV-representation with one vertex at each level.

For example if \(p_1 = 2\), \(p_2 = 10\) and \(p_3 = 30\), the first three levels are given in Figure 7.1.1.

![Figure 7.1.1: The three first levels of the BV representation of \((\mathbb{Z}_{(p_n)}, R)\).](image-url)
7.1. ODOMETERS

Proposition 7.1.2 The dimension group of the odometer in base \((p_n)\) is the subgroup of \(\mathbb{Q}\) formed of the \(p/q\) with \(q\) dividing some \(p_n\).

Proof. The dimension group of \((\mathbb{Z}(p_n), T)\) is the direct limit of the sequence
\[
\mathbb{Z}^p \rightarrow \mathbb{Z}^p \rightarrow \mathbb{Z} \rightarrow \cdots
\]
whence the statement. Note that the order unit is 1.

For example, the dimension group of the odometer in base \((2^n)\) is the group of dyadic rationals.

We will use in the next section the following statement, which identifies systems equivalent to an odometer in a non obvious way.

Proposition 7.1.3 Let \((V, E, \leq)\) be an ordered Bratteli diagram such that

(i) The vertex set for \(n \geq 1\) is \(V(n) = \{1, 2, \ldots, n\}\).

(ii) The incidence matrices \(M(n)\) are all equal for \(n \geq 2\) to a square matrix of rank 1.

(iii) The order is such that \(e \leq e'\) if \(r(e) = r(e')\) and \(s(e) \leq s(e')\).

Then \((X_E, T_E)\) is topologically conjugate to an odometer in base \((qp^n)\).

Proof. Set \(M = M(n)\) for \(n \geq 2\). Since \(M\) has rank 1, we have \(M = xy\) with \(x\) a nonnegative integer column vector and \(y\) a nonnegative integer row vector. Consider the ordered Bratteli diagram \((V', E', \leq')\) with incidence matrices \(M'(1) = M(1)\) and \(M'(2n) = x, M'(2n + 1) = y\) for \(n \geq 1\). By condition (iii), we can choose the order on \((V', E', \leq')\) such that the telescoping with respect to 0, 1, 3, 5, \ldots gives \((V, E, \leq)\). Now the telescoping of \(G'\) with respect to 0, 2, 4, \ldots is the odometer in basis \((pq^n)\) with \(p = yM(1)\) and \(q = yx\).

For example, the odometer represented in Figure 7.1.2 has incidence matrices
\[
M(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad M(n) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}
\]
for \(n \geq 2.\)

Note finally the following simple but important observation.

Proposition 7.1.4 The family of odometers is disjoint from the family of minimal shift spaces.

Proof. Any odometer \(\mathbb{Z}(p_n)\) is infinite and has a partition in towers with a basis formed of one element, namely \(\mathbb{Z}(p_n) = \bigcup_{j=0}^{h-1} T^j B\) where \(h = p_1\) and \(B = [0]\). This is not possible for a minimal shift space \((X, S)\). Indeed, assume that \(X = \bigcup_{j=0}^{h-1} T^j B\) where \(B\) is a clopen set. Then there is an integer \(n\) and a partition of \(L_n(X)\) in \(h\) sets \(W_0, \ldots, W_{h-1}\) such that \(T^j B = [W_j]\) for \(0 \leq j < h\).
Let $w \in W_0$. Then the periodic point $\cdots ww \cdots$ belongs to $X$. Since $(X,S)$ is minimal, it must be periodic and thus finite, a contradiction.

A different proof uses the notion of expansive system. A dynamical system $(X,T)$, endowed with the distance $d$, is expansive if there exists $\epsilon$ such that for all pairs of points $(x,y)$, $x \neq y$, there exists $n$ with $d(T^n x, T^n y) \geq \epsilon$. We say that $\epsilon$ is a constant of expansivity of $(X,T)$. Expansivity is a property invariant by conjugacy (but the constant may not be the same).

A shift space is expansive while an odometer is not. Actually, a topological dynamical system is a shift space if and only if it is expansive (Exercise 7.12).

A Bratteli diagram has the equal path number property if for all $n \geq 1$ and $u,v \in V(n)$ we have $\text{Card}(r^{-1}(u)) = \text{Card}(r^{-1}(v))$. Note that this implies that the number of paths from $u,v$ to $v(0)$ are the same.

Note that the representation of odometers given in Section 7.1 shares this property.

**Theorem 7.1.5** A minimal shift is Toeplitz if and only if it has a BV-representation $(X_E, V_E)$ where $(V, E, \leq)$ has the equal path number property. Moreover, there exist BV-systems having the equal path number property that are neither expansive nor equicontinuous.

One of the directions is easy. Indeed, let $(X_E, V_E)$ be a BV-representation of the minimal shift $(X,T)$ which has the equal path number property. Let $\phi : X_E \to X$ be a conjugacy. We may assume that the initial partition $\mathcal{P}(0)$ is such that $B(0)$ separates the $x \in X$ with different letter $x_0$. Then is clear that $x = x_{\text{min}}$ is a Toeplitz sequence. For example, one has $x_{pk} = x_0$ for all $p \in \mathbb{Z}$ with $k = \text{Card}(r^{-1}u)$ for all $u \in V(1)$.

**Example 7.1.6** Let $\sigma : 0 \to 01, 1 \to 00$ be the substitution (generating the period-doubling sequence, see Example 2.6.1). A BV-representation of the cor-
responding shift is shown in Figure 7.1.3. The orbit of $x_{\text{min}}$ is shown in Figure 7.1.4.

Figure 7.1.3: The BV-representation of the period doubling shift.

Figure 7.1.4: The orbit of $x_{\text{min}}$.

7.2 Substitutions

We will now consider the BV-representation of substitutive shifts. Let us note that we work in all this section with bi-infinite words and two-sided shifts. We first need a new definition.

A Bratteli diagram $(V,E)$ is stationary if there exists $k$ such that $k = \text{Card}(V(n))$ for all $n$, and if (by an appropriate labelling of the vertices) the incidence matrices between level $n$ and $n+1$ are the same $k \times k$ matrix $M$ for all $n = 1, 2, \ldots$. In other words, beyond level 1 the diagram repeats itself. Clearly we may label the vertices in $V(n)$ as $v(n,a_1), \ldots, v(n,a_k)$, where $A = \{a_1, \ldots, a_k\}$ is a set of $k$ distinct symbols. The matrix $M$ is called the matrix of the stationary diagram.

The ordered Bratteli diagram $(V,E, \leq)$ is stationary if $(V,E)$ is stationary, and the ordering on the edges with range $v(n,a_i)$ is the same as the ordering
on the edges with range \( v(m, a_i) \) for \( m, n = 2, 3, \ldots \) and \( i = 1, \ldots, k \). In other words, beyond level 1 the diagram with the ordering repeats itself.

An odometer is *stationary* if it has a stationary BV-representation. It is easy to see that the odometer in base \( (p_n) \) is stationary if and only if \( p_n = pq^n \) for some \( p, q \geq 2 \).

Let \((V, E, \leq)\) be a stationary properly ordered Bratteli diagram. The morphism read on \( E(n) \) is constant from \( n \leq 2 \). We call it the *substitution read on\((V, E, \leq)\)*.

Given a substitution \( \sigma \), it is not possible in general to use the stationary Bratteli diagram \((V, E, \leq)\), with \( \sigma \) read on \((V, E, \leq)\), to represent the shift space corresponding to \( \sigma \). For example, in the case of the *Thue-Morse substitution* \( a \mapsto ab, b \mapsto ba \) the Bratteli diagram is given in Figure 7.2.1.

![Figure 7.2.1: The Thue–Morse substitution read on a Bratteli diagram.](image)

It is clear that it has two maximal and two minimal paths. Hence this representation does not give a properly ordered Bratteli diagram.

### 7.2.1 Main result

We will show that one has however the following result. It implies in particular that every infinite minimal substitution has a BV-representation.

**Theorem 7.2.1** The family \( B \) of Bratteli-Vershik systems associated with stationary, properly ordered Bratteli diagrams is (up to isomorphism) the disjoint union of the family of infinite substitution minimal systems and the family of stationary odometer systems.

Furthermore, we will see that the correspondence in question is given by an explicit and algorithmically effective construction.

A substitution \( \sigma \) on the alphabet \( A \) is *proper* if there are two letters \( r, l \in A \) such that, for every \( a \in A \), \( r \) is the last letter of \( \sigma(a) \) and \( l \) is the first letter of \( \sigma(a) \). It is called *eventually proper* if there is an integer \( p \geq 1 \) such that \( \sigma^p \) is
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An eventually proper substitution $\sigma$ has exactly one fixed point which is $\sigma^\omega(\ell \cdot r)$.

**Proposition 7.2.2** The substitution read on a stationary, properly ordered, Bratteli diagram is primitive and eventually proper.

**Proof.** Let $\sigma$ be the substitution read on $(V, E, \leq)$. Since $(V, E, \leq)$ is properly ordered, it is simple. For every $a, b \in A$, since $(V, E, \leq)$ is simple, there is a path from $(1, b)$ to some $(n, a)$. Then $b$ occurs in $\sigma^n(a)$ showing that $\sigma$ is primitive.

Let $i(a)$ be the first letter of $\sigma(a)$. For every $a \in A$ and $n \geq 1$, the source of the minimum edge with range $(n, a)$ is $(n - 1, i(a))$. Thus, if the minimal rank of the maps $i^n : A \to A$ were larger than 1, there would exist more than one minimal infinite path, a contradiction with the hypothesis that $(V, E, \leq)$ is properly ordered. This shows that there exists $n$ such that $i^n(a)$ is the same for all $a \in A$. A symmetric argument holds for the last letter. Thus $\sigma$ is eventually proper.

We recall that a shift space is said periodic if there exist $x \in X$ and an integer $k$ such that $X = \{x, Sx, \ldots, S^{k-1}x\}$. Otherwise it is said to be aperiodic. Thus a periodic shift is the same as a minimal finite shift space. Observe that the property of being periodic is decidable (see Exercise 2.27).

### 7.2.2 Diagrams with simple hat

We say that a Bratteli diagram $(V, E)$ has a simple hat whenever it has only simple edges between the top vertex and any vertex of the first level. Note that the Bratteli diagram associated with a nested sequence of partitions as in Section 6.3 has a simple hat.

The following result gives a proof of one direction of Theorem 7.2.1 in the particular case of diagrams with a simple hat.

**Proposition 7.2.3** Let $(V, E, \leq)$ be a stationary, properly ordered Bratteli diagram with a simple hat, let $\sigma : A^* \to A^*$ be the substitution read on $(V, E, \leq)$, and let $(X, S)$ be the substitution shift associated to $\sigma$.

1. If $(X, S)$ is not periodic, then it is isomorphic to $(X_E, T_E)$.

2. If $(X, S)$ is periodic, then $(X_E, T_E)$ is isomorphic to an odometer in base $(qp^n)_n$, for some $p, q \geq 2$.

We will use the following lemma.

**Lemma 7.2.4** Let $\sigma : A^* \to A^*$ be a primitive and eventually proper substitution. If $X(\sigma)$ is infinite, the family

$$\mathcal{P}(n) = \{T^j\sigma^n([a]) \mid a \in A, \ 0 \leq j < |\sigma^n(a)|\}$$

is a refining sequence of partitions in towers.
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Proof. By Proposition 7.2.2 the partition $\mathcal{P}(n)$ is for every $n \geq 1$ a KR-partition of $(X,S)$ with basis $\sigma^n(X)$. The sequence $\mathcal{P}(n)$ is clearly nested. Since $\sigma$ is eventually proper, the intersection of the bases is reduced to one point, namely the unique fixed point of $\sigma$. Finally, $\mathcal{P}(n)$ tends to the partition in points. Indeed, let $p \geq 1$ be such that all $\sigma_p(a)$ begin with $\ell$ and end with $r$. Then (see Figure 7.2.2) $\mathcal{P}(\sigma^n([a]) = [\sigma^{-p}(r) \cdot \sigma^n(a) \sigma^{-p}(\ell)]$.

Thus all words in $T^p \sigma^n([a])$ coincide on $[-m,m]$ for $m = \min_{b \in B} |\sigma^{-p}(b)|$. This shows that $\mathcal{P}(n)$ is a refining sequence of KR-partitions.

![Figure 7.2.2](image-url)

Figure 7.2.2: The sequence $\mathcal{P}(n)$ generates the topology.

Proof of Proposition 7.2.3 Assume first that $(X,S)$ is aperiodic. By Proposition 7.2.2 the substitution $\sigma$ is primitive and eventually proper. By Lemma 7.2.4 the family $\mathcal{P}(n)$ is a refining sequence of partitions in towers.

Since $(V,E,\leq)$ has simple hat, the Bratteli diagram associated to the sequence of partitions $\mathcal{P}(n)$ is clearly equal to $(V,E,\leq)$. Thus, by Theorem 6.3.3, $(X,S)$ is isomorphic with $(X,\Sigma,\tau)$.

Assume now that $(X,S)$ is periodic. Replacing $\sigma$ by some power does not modify $(X,S)$ (and replaces $(V,E,\leq)$ by a periodic telescoping) so that we may assume that all words $\sigma(a)$ for $a \in A$ begin with $\ell$ and end with $r$. The unique fixed point $x = \sigma^\omega(r \cdot \ell)$ of $\sigma$ is then periodic. Set $x = \cdots \cdot w w w \cdots$ with $w$ as short as possible. Then $w$ is a primitive word, that is, it is not a power of a shorter word. Let $n$ be large enough so that $|\sigma^n(a)| \geq |w|$ for every $a \in A$. Then each $\sigma^{2n}(a)$ is a word of period $|w|$ which

- begins with $w$ since it begins with $\sigma^n(\ell)$,
- ends with $w$ since it ends with $\sigma^n(r)$.

Since $w$ is primitive, this forces each $\sigma^{2n}(a)$ to be a power of $w$. Thus, replacing again $\sigma$ be one of its powers, we may assume that every $\sigma(a) = w^{k_a}$ is a power of $w$. In this case, the incidence matrix of the diagram $(V,E,\leq)$ is such that for every $a,b \in A$

$$M_{a,b} = k_a |w|_b$$

Thus the hypotheses of Proposition 7.1.3 are satisfied and $(X,\Sigma,\tau)$ is an odometer in basis $q p^n$ with $q = |w|$ and $p = \sum_{a,b \in A} k_a |w|_b$. 

$\blacksquare$
We illustrate the periodic case in the following example.

**Example 7.2.5** Let \((V, E, \leq)\) be the stationnary Bratteli diagram represented in Figure 7.2.3 on the left. The substitution read on \((V, E, \leq)\) is \(\sigma : a \to ab, b \to abc, c \to abc \). The substitution shift defined by \(\sigma\) is periodic since \(\sigma(abc) = (abc)^2\). The corresponding odometer is represented in Figure 7.2.3 on the right.

A **partition matrix** is a \(t \times \ell\)-matrix \(P\) with coefficients 0, 1 such that every column of \(P\) has exactly one coefficient equal to 1. Such a matrix defines a partition \(\theta\) of the set \(\{1, 2, \ldots, \ell\}\) of indices of the columns putting together \(x, y\) if \(P_{i,x} = P_{i,y} = 1\). It also defines a map \(\pi\) from \(\{1, 2, \ldots, \ell\}\) onto \(\{1, 2, \ldots, k\}\) by \(\pi(x) = i\) if \(i\) is the index such that \(P_{i,x} = 1\).

**Example 7.2.6** the matrix

\[
P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},
\]

is a partition matrix. The corresponding partition \(\theta\) is \(\{1\}, \{2, 3\}\) and the map \(\pi\) is \(1 \to 1, 2 \to 2, 3 \to 2\).

The following result will allow us to reduce to the case of a Bratteli diagram with a simple hat.

**Proposition 7.2.7** For every stationary and properly ordered Bratteli diagram \((V, E, \leq)\), there is a stationary properly ordered Bratteli diagram \((V', E', \leq')\) with a simple hat such that \((X_E, T_E)\) and \((X_{E'}, T_{E'})\) are isomorphic.

More precisely, for every ordered Bratteli diagram \(B\), there is an ordered Bratteli diagram \(B'\) such that the following holds. There exist two nonegative matrices \(P, Q\), with \(P\) a partition matrix, such that the matrices \(M, M'\) of \(B\) and \(B'\) satisfy

\[
M = PQ, \quad M' = QP.
\]
and the vector \( v = M(1) \) is such that \( v = Pw \) where \( w \) has all its components equal to 1. Moreover, if \( B \) is properly ordered, then \( B' \) is properly ordered.

**Proof.** Let \( M \) be the \( t \times t \)-matrix equal to the incidence matrices \( M(n) \) of the diagram \( B \), for all \( n \geq 2 \) and let \( v = M(1) \). Set \( \ell = \sum_{i=1}^{t} v_i \).

Let \( P \) be the \( t \times \ell \) partition matrix defined by

\[
P_{i,j} = \begin{cases} 
1 & \text{if } \sum_{k<i} v_k < j \leq \sum_{k\leq i} v_k \\
0 & \text{otherwise.}
\end{cases}
\]

Thus the rows of \( P \) are the characteristic vectors of elements of the partition of \( \{1, 2, \ldots, \ell\} \) into the \( t \) sets \( V_1 = \{1, 2, \ldots, v_1\}, \ldots, V_t = \{\ell - v_t + 1, \ldots, \ell\} \).

We may assume, replacing if necessary \( M \) by some power, that, for every \( i \), the set of edges with range \( i \) has more than \( v_i \) elements. We choose an \( \ell \times t \) matrix \( Q \) such that \( M = PQ \). This is equivalent to splitting the set of edges entering the vertex \( i \) in \( v_i \) nonempty subsets (the sum of the rows with index in \( V_i \) is then the row of index \( i \) of \( M \)). The edges in each subset keep the order induced by the order on \( B \). Let \( B' \) be the Bratteli diagram with incidence matrices \( M'(1) = w = [1 \ 1 \ldots 1]^t \) and \( M'(n) = QP \) for \( n \geq 2 \). We order the diagram \( B' \) by the order induced by that of \( B \). Since \( v = Pw \), \( B \) and \( B' \) can both be obtained by telescoping from the Bratteli diagram \( C \) with incidence matrices \( (w, P, Q, P, Q, \ldots) \). If \( B \) is properly ordered, \( C \) is properly ordered and consequently \( B' \) also.

We illustrate the construction in the following example.

**Example 7.2.8** Let \( (V, E, \leq) \) be the Bratteli diagram represented in Figure 7.2.4 on the left. The matrices \( M, P, Q, M' \) are

- \( M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \)
- \( P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \)
- \( Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \)
- \( M' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \)

![Figure 7.2.4: The transformation of \((V, E, \leq)\) by splitting vertices.](image)

The diagram with matrices \((w, P, Q, P, \ldots)\) is represented in the middle of Figure 7.2.4 and the diagram with simple hat \((V', E', \leq')\) with matrices \(w, M', M', \ldots\) on the right.
7.2. SUBSTITUTIONS

7.2.3 A useful result

The following result will be the key to build a BV-representation of an aperiodic minimal substitution shift. It allows one to replace a pair \((\tau, \phi)\) of morphisms with \(\tau\) primitive by a pair \((\zeta, \theta)\) with \(\zeta\) primitive and \(\theta\) letter-to-letter (see Figure 7.2.5).

\[\begin{array}{ccc}
B^\mathbb{Z} & \xrightarrow{\gamma} & C^\mathbb{Z} \\
\phi & \uparrow & \downarrow \theta \\
A^\mathbb{Z}
\end{array}\]

Figure 7.2.5: The eventually proper substitution \(\zeta\)

**Proposition 7.2.9** Let \(y \in B^\mathbb{Z}\) be an admissible fixed point of a primitive substitution \(\tau\) on the alphabet \(B\) and let \(\phi : B^* \rightarrow A^*\) be a non-erasing morphism. Set \(x = \phi(y)\) and let \((X, S)\) be the subshift spanned by \(x\).

There exist a primitive substitution \(\zeta\) on an alphabet \(C\), an admissible fixed point \(z\) of \(\zeta\), a morphism \(\gamma : B^* \rightarrow C^*\) and a map \(\theta : C \rightarrow A\) such that:

1. \(\theta(z) = x\) and \(\phi = \theta \circ \gamma\).
2. If \(\phi\) is injective and \(\phi(B)\) is a circular code, then \(\theta\) is a conjugacy from \((X(\zeta), S)\) onto \((X, S)\),
3. If \(\tau\) is eventually proper, then \(\zeta\) is eventually proper.

**Proof.** The proof below is very simple, but the notation is, in an unavoidable way, a bit heavy. By substituting \(\tau\) by a power of itself if needed, we can assume that \(|\tau(b)| \geq |\phi(b)|\) for all \(b \in B\). We define:

- an alphabet \(C\) by \(C = \{b_p \mid b \in B, 1 \leq p \leq |\phi(b)|\}\),
- a map \(\theta : C \rightarrow A\) by \(\theta(b_p) = (\phi(b))_p\),
- a map \(\gamma : B \rightarrow C^+\) by \(\gamma(b) = b_1b_2\cdots b_{|\phi(b)|}\).

Clearly \(\theta \circ \gamma = \phi\). We define a substitution \(\zeta\) on \(C\) as follows. For \(b\) in \(B\) and \(1 \leq p \leq |\phi(b)|\), we set

\[\zeta(b_p) = \begin{cases} 
\gamma \left( (\tau(b))_p \right) & \text{if } 1 \leq p < |\phi(b)| \\
\gamma \left( (\tau(b))_{|\phi(b)|} \right) & \text{if } p = |\phi(b)|.
\end{cases}\]
Hence, for every \( b \in B \), \( \zeta(\gamma(b)) = \zeta(b_1) \cdots \zeta(\phi(b)) = \gamma(\tau(b)) \), i.e.,

\[
\zeta \circ \gamma = \gamma \circ \tau
\]  

(7.2.1)

and it follows that

\[
\zeta^n \circ \gamma = \gamma \circ \tau^n \text{ for all } n \geq 0.
\]

We claim that \( \zeta \) is primitive. Let \( n \) be an integer such that \( b \) occurs in \( \tau^n(a) \) for all \( a, b \in B \). Let \( b_p \) and \( c_p \) belong to \( C \). By construction, \( \zeta(b_p) \) contains \( \gamma(\tau(b)) \) as a factor, thus \( \zeta^{n+1}(b_p) \) contains \( \zeta^n(\gamma(\tau(b))) = \gamma(\tau^n(\tau(b))) \) as a factor. By the choice of \( n, c \) occurs in \( \tau^n(\tau(b)) \), thus \( \gamma(c) \) is a factor of \( \gamma(\tau^n(\tau(b))) \), and also of \( \zeta^{n+1}(b_p) \). Since \( c_p \) is a letter of \( \gamma(c) \), \( c_p \) occurs in \( \zeta^{n+1}(b_p) \) and our claim is proved.

Let \( z = \gamma(y) \). By (7.2.1) we get \( \zeta(z) = \gamma(\tau(y)) = \gamma(y) = z \), and \( z \) is a fixed point of \( \zeta \). By construction, \( z \) is uniformly recurrent, thus it is an admissible fixed point of \( \zeta \). Moreover, \( \theta(z) = \theta(\gamma(y)) = \phi(y) = x \), and \( \mathbf{1} \) is proved.

**Proof of (3)** Since \( \theta \) commutes with the shift and maps \( z \) to \( x \), and by minimality of the subshifts, it maps \( X(\zeta) \) onto \( X \). There remains to prove that \( \theta : X(\zeta) \to X \) is one-to-one. Let \( \alpha \in X \). By definition of \( X \), there exist \( t \in X \), and an integer \( p \), with \( 0 \leq p < |\phi(t_0)| \), such that \( \alpha = \delta_S \phi(t) \). Let \( \beta \) be an element of \( X(\zeta) \) with \( \theta(\beta) = \alpha \). By definition of \( \gamma \), there exist some \( \delta \in X(\tau) \) and some integer \( q \), with \( 0 \leq q < |\gamma(\delta_0)| \), such that \( \beta = S^q \gamma(\delta) \). It follows that \( S^q \gamma(\delta) = \theta(\beta) = \alpha = \delta_S \phi(t) \). Since \( 0 \leq q < |\gamma(\delta_0)| = |\phi(\delta_0)| \) by construction of \( \gamma \), since \( \phi(B) \) is a circular code and since \( \phi \) is injective, it follows that \( \delta = t \) and \( q = p \), thus \( \beta = S^p \gamma(t) \); \( \beta \) is uniquely determined by \( \alpha \), and \( \theta \) is one-to-one.

**Proof of (4)** Let \( l \in B \) be the first letter of \( \tau(b) \) for every \( b \in R \). Let \( b_p \in C \), and \( c = \tau(b) \). By definition of \( \zeta \), the first letter of \( \zeta(b_p) \) is \( c_1 \), and the first letter of \( \zeta^2(b_p) \) is the first letter of \( \zeta(c_1) \), that is, \( l_1 \). By the same method, if \( r \) is the last letter of \( \tau(b) \) for every \( b \in B \), then the last letter of \( \zeta^2(b_p) \) is \( \tau(\phi(r)) \) for every \( b_p \in C \).

Note that the proof of Proposition (7.2.9) is very close to that of Proposition (7.2.7) (see Exercise (7.16)).

Consider a substitution \( \sigma : A \to A^* \) generating an aperiodic minimal shift \((X, S)\). Replacing \( \sigma \) by one of its powers, we can choose an admissible fixed point \( x \) of \( \sigma \). Set \( r = x_{-1} \) and \( l = x_0 \). Note that since \( x \) is a fixed point of \( \sigma \), \( \sigma(r) \) ends with \( r \) and \( \sigma(l) \) begins with \( l \).

Let \( R_X(rl) \) be the set of right return words to \( rl \). Every word in \( R_X(rl) \) ends with \( l \). Set

\[
R_X(r \cdot l) = lR_X(rl)l^{-1}
\]

which is the set of words of the form \( l \sigma(x) \) for \( x \in R_X(rl) \). Thus \( w \) is in \( R_X(r \cdot l) \) if and only if \( w \) is in \( E(X) \) and contains exactly two occurrences of \( rl \), one as a prefix and one as a suffix. In particular all words in \( R_X(r \cdot l) \) begin with \( l \) and end with \( r \). Moreover, the set \( R_X(r \cdot l) \) satisfies the following properties.

(i) any word in \( E(X) \) which begins with \( l \) and ends with \( r \) is a concatenation of words of \( R_X(r \cdot l) \).
Proposition 7.2.11
The substitution $R_x$ of $y$ is a unique element ally proper and aperiodic.

Set $B$ so that $τ$ is primitive.

It follows that $σ$ and thus that $σ_u$ in the finite word $σ$ of $σ^n(τ)$ begins with $a$ and ends with $b$. This implies that $σ^n(b) = φ(w)$ for some unique $w ∈ B^*$. We set $τ(b) = w$.

This defines a substitution $τ$ on the alphabet $B$, characterised by

$$φ ∘ τ = σ ∘ φ .$$ (7.2.2)

It follows that $φ ∘ τ^n = σ^n ∘ φ$ for each $n ≥ 0$.

Example 7.2.10
Consider the Fibonacci substitution $φ : a → ab, b → a$ generating the Fibonacci shift $X(φ)$. The sequence $x = σ^2ω(a cot a)$ is an admissible fixed point of $σ^2 : a → aba, b → ab$. We have

$$R'_X(a,a) = \{aba, ababa\}.$$

Set $B = \{a, b\}$ with $φ(a) = aba$ and $φ(b) = ababa$. Then

$$φ ∘ τ(a) = φ ∘ φ(a) = φ(aba) = abaababa = φ(ab)$$

so that $τ(a) = ab$. Similarly, we find $τ(b) = abb$.

Proposition 7.2.11
The substitution $τ$ defined by (7.2.2) is primitive, eventually proper and aperiodic.

Proof.
Let us show that $τ$ is eventually proper. For this, let $n$ be such that $|σ^n(l)| > |φ(y_0)|$. Then for every $b ∈ B$, the first letter of $τ^n(b)$ is $y_0$. Indeed, $φ ∘ τ^n(b) = σ^n ∘ φ(b)$ begins with $σ^n(l)$ (see Figure 7.2.1). But $σ^n(l)$ is a prefix of $x = σ^n(x) = σ^n ∘ φ(y)$. Thus, by the choice of $n$, $σ^n ∘ φ(y_0)l$ is a prefix of $σ^n(l)$. This implies that $τ^n(b)$ begins with $y_0$. A similar argument shows that if $n$ is such that $|σ^n(r)| > |φ(y_{n-1})|$, then every $τ^n(b)$ end with $y_{n-1}$. Thus $τ$ is eventually proper.

Let $k > 0$ be an occurrence of $r,l$ large enough so that every return word $w ∈ R$ appears in the decomposition of $x_{[0,k)}$, that is, every $b ∈ B$ occurs in the finite word $u ∈ B^+$ defined by $φ(u) = x_{[0,k)}$. Let $n$ be so large that $|σ^n(l)| > k$. Let $b, c ∈ B$. As above, $x_{[0,k)}l$ is a prefix of $σ^n(l)$, which is a prefix of $σ^n(φ(b)) = φ(τ^n(b))$. Thus $u$ is a prefix of $τ^n(b)$, and $c$ occurs in $τ^n(b)$, hence $τ$ is primitive.

Moreover,

$$φ(τ(y)) = σ(φ(y)) = σ(x) = x = φ(y) ,$$
thus $\tau(y) = y$ since $R_X(r \cdot l)$ is circular, and $y$ is the unique fixed point of $\tau$.
Since $\phi(y) = x$ is not periodic, $y$ is not periodic. Thus $\tau$ is aperiodic.

We are now ready to prove Theorem 7.2.1.

**Proof of Theorem 7.2.1**

Let first $(V, E, \leq)$ be a stationary properly ordered Bratteli diagram. By Proposition 7.2.7, we may assume that $(V, E, \leq)$ has a simple hat. Then, using Proposition 7.2.3, we conclude that $(X, T)$ is either an aperiodic minimal substitution shift or an odometer. This proves the theorem in one direction.

Let us now prove the converse implication. If $(X, T)$ is an odometer, we have seen (Section 7.1) that $(X, T)$ has a BV-representation with a stationary, properly ordered, Bratteli diagram. Moreover, $(X, T)$ cannot be at the same time a minimal substitution shift by Proposition 7.1.4.

Let finally $\sigma : A^* \to A^*$ be a substitution generating a minimal shift space $(X, S)$. Let $\tau : B^* \to B^*$ be the primitive, eventually proper and aperiodic substitution defined by Proposition 7.2.11.

Let $\zeta : C \to C^*$ be the substitution given by Proposition 7.2.9 with an admissible fixed point $z \in C^Z$ and a map $\theta : C \to A$. Since $\phi$ is injective and $R_X(r \cdot l) = \phi(B)$ is a circular code, by assertion 2, $\theta$ is an isomorphism from $(X, S)$ onto $(X, S)$. Moreover, by assertion 3, since $\tau$ is eventually proper, $\zeta$ is eventually proper. Let $(V, E, \leq)$ be the stationary properly ordered Bratteli diagram such that $\zeta$ is the morphism read on $(V, E, \leq)$. By Proposition 7.2.3, the systems $(X, S)$ and $(X, T)$ are isomorphic. This concludes the proof.

Note that the preceding results imply the following property of substitution shifts.

**Corollary 7.2.12** Every infinite minimal substitution shift is isomorphic to a primitive eventually proper substitution shift.
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Proof. A minimal substitution shift is, by Theorem 7.2.1, isomorphic to \((X_E, T_E)\) with \((V, E, \leq)\) a stationary properly ordered Bratteli diagram. By Proposition 7.2.3, the system \((X_E, T_E)\) is isomorphic to the substitution shift \((X_\sigma, S)\) where \(\sigma\) is the morphism read on \((V, E, \leq)\). But by Proposition 7.2.2, the morphism \(\sigma\) is primitive and eventually proper, which proves the statement.

Let us illustrate this result on the case of the binary Chacon substitution \(\sigma : 0 \rightarrow 0010, 1 \rightarrow 1\). The substitution is not primitive but the corresponding shift space is minimal (see Exercise 2.23).

Let \(x = \sigma^\omega(0.0)\). This is the Chacon binary sequence. Using the return words to 0.0 we see that \(x = \varphi(y)\) where \(\phi : \{a, b, c\}^* \rightarrow \{0, 1\}^*\) is defined by \(\phi(a) = 0\), \(\phi(b) = 010\) and \(\phi(c) = 01010\), and \(y = \tau^\omega(b, a)\) where \(\tau\) is defined by \(\tau(a) = ab\), \(\tau(b) = acb\) and \(\tau(c) = accb\). According to the proof of Proposition 7.2.9, we need to take \(\tau^2\) instead of \(\tau\) and we take

1. \(C = \{a_1, b_1, b_2, b_3, c_1, c_2, c_3, c_4, c_5\}\),
2. \(\theta : C \rightarrow \{0, 1\}\) given by the following table

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(a_1)</th>
<th>(b_1)</th>
<th>(b_2)</th>
<th>(b_3)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
<th>(c_4)</th>
<th>(c_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta(\alpha))</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
3. \(\gamma : \{a, b, c\} \rightarrow C^+\) defined by

\[
\gamma(a) = a_1, \quad \gamma(b) = b_1 b_2 b_3, \quad \gamma(c) = c_1 \cdots c_5,
\]
4. the substitution \(\zeta : C^* \rightarrow C^*\) defined by the following tables

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(a_1)</th>
<th>(b_1)</th>
<th>(b_2)</th>
<th>(b_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\zeta(\alpha))</td>
<td>(\gamma(abacb))</td>
<td>(\gamma(a))</td>
<td>(\gamma(b))</td>
<td>(\gamma(accbacb))</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>(c_1)</td>
<td>(c_2)</td>
<td>(c_3)</td>
<td>(c_4)</td>
</tr>
<tr>
<td>(\zeta(\alpha))</td>
<td>(\gamma(a))</td>
<td>(\gamma(b))</td>
<td>(\gamma(a))</td>
<td>(\gamma(c))</td>
</tr>
</tbody>
</table>

A BV-representation of the Chacon shift (that is, the subshift generated by \(x\)) is isomorphic to \((X_E, T_E)\) where \((V, E, \leq)\) a stationary properly ordered Bratteli diagram with a simple hat such that \(\zeta\) is the substitution read on it. This diagram is obtained by telescoping at odd levels the diagram of Figure 7.2.7.

As already mentioned (and developed in Exercise 7.16), an alternative route to obtain a Bratteli diagram with a simple hat is to use Proposition 7.2.7. In the present example, we would obtain

\[
M = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 3 & 3 \\ 4 & 4 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix},
\]
Figure 7.2.7: The BV-representation of the Chacon binary shift

and thus

\[ M' = \begin{bmatrix}
2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 4 & 4 & 4 & 4
\end{bmatrix} \]

7.2.4 Dimension groups and BV representation of substitution shifts

We now derive from the previous results a description of the dimension group of substitution shifts.

Let \( \sigma : A^* \to A^* \) be a substitution generating an aperiodic minimal shift \((X, S)\). As in the proof of Theorem 7.2.1, changing if necessary \( \sigma \) for some power of \( \sigma \), let \( x \in A^\mathbb{Z} \) be an admissible fixed point of \( \sigma \) and set \( r = x_{-1} \) and \( \ell = x_0 \). Let \( R_X(r \cdot \ell) = \ell R_X(r \ell) \ell^{-1} \) and let \( \phi : B \to R_X(r \cdot \ell) \) be a coding morphism for \( R_X(r \cdot \ell) \). Let \( \tau : B^+ \to B^* \) be the morphism such that \( \phi \circ \tau = \sigma \circ \phi \).

Theorem 7.2.13 Let \( \sigma \) be a substitution generating an aperiodic minimal shift \((X, S)\) and let \( \tau : B^+ \to B^* \) be as above. Let \( M \) be the composition matrix of \( \tau \). Then \( K^0(X, S) = (\Delta_M, \Delta_M^+, v) \) where \( v \) is the image in \( \Delta_M \) of the vector with components \( |\phi(b)| \) for \( b \in B \).

Proof. By Proposition 7.2.11 the substitution \( \tau \) is primitive, eventually proper and aperiodic. Thus, by Proposition 7.2.10 the system \((X_\tau, S)\) is isomorphic to \((X_E, T_E)\) where \( B = (V, E, \leq) \) is the Bratteli diagram with \( \tau \) read on \( B \). By Theorem 6.3.4 the dimension group of \((X_\tau, S)\) is isomorphic with \((\Delta_M, \Delta_M^+, 1_M)\). Since \((X_\tau, S)\) is the system induced by \((X, S)\) on the clopen set \([r \cdot \ell]\), the result follows from Proposition 4.7.2.
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The result can of course also be deduced from Proposition 6.4.2.

We will use the following simple result (note that it is a particular case of Theorem 6.4.1 on Kakutani equivalence).

Let $\tau : B^* \rightarrow B^*$ be a primitive eventually proper nonperiodic substitution and let $X(\tau)$ be the associated substitution shift. Let $(V,E,\leq)$ be a properly ordered simple Bratteli diagram with simple hat such that $\tau$ is the substitution read on $(V,E,\leq)$. By Proposition 7.2.3, the shift $X(\tau)$ is conjugate to $(X_E,T_E)$.

We identify the set $V \setminus \{0\}$ to $B \times \mathbb{N}$. Let $\phi : B^* \rightarrow A^*$ be a morphism recognizable on $X(\tau)$ and let $Y = X(\tau,\phi)$ be the corresponding substitutive shift.

Proposition 7.2.14 The shift $Y$ is conjugate to $(X_{E'},T_{E'})$ where $(V',E')$ is the Bratteli diagram obtained from $(X,E)$ by replacing each edge from $0$ to $(b,1)$ by $|\phi(b)|$ edges $(0,b,i)$ with $0 \leq i < |\phi(b)|$.

Proof. Let $U$ be the clopen subset of $X_{E'}$ formed of the paths with a first edge of the form $(0,b,0)$ for some $b \in B$. This system induced by $(X_{E'},T_{E'})$ on $U$ is clearly $X_E$. Thus $X_{E'}$ is the primitive of $X_E$ relative to the function $f(x) = |\phi(x_0)|$. On the other hand, since $\phi$ is recognizable on $X$, by Proposition 2.4.24, $Y$ is the primitive of $X$ relative to the function $f(x) = |\phi(x)|$. Thus $X_{E'}$ and $Y$ are conjugate.

We will see several examples of application of this result below.

7.2.5 Some examples

We will illustrate the preceding results on some classical examples.

Example 7.2.15 Let $X$ be the Fibonacci shift generated by $\sigma : a \rightarrow ab, b \rightarrow a$. Consider the fixed point $\sigma^2 \omega (a \cdot a)$. Let $\phi : \{u,v\} \rightarrow \{a,b\}^*$ be a coding morphism for $R_X (a \cdot a) = \{aba,ababa\}$. The morphism $\tau : B^* \rightarrow B^*$ such that $\phi \circ \tau = \sigma^2 \circ \phi$ is $\tau : u \rightarrow uv, v \rightarrow uvv$ (see Example 7.2.10). The Bratteli diagram with simple hat corresponding to $\tau$ is shown in Figure 7.2.8 on the left. The Bratteli diagram of $X$ (according to Proposition 7.2.14) is shown on the right. According to Theorem 6.3.4, the dimension group is the group $(\Delta_M,\Delta^+_M,v)$ with

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 3 \end{bmatrix},$$

(exchanging the letters $u,v$ to write $M$ and $v$). This is consistent with the fact that $K^0(X,S) = \mathbb{Z}[\lambda]$ with $\lambda = (1 + \sqrt{5})/2$ (see Examples 4.9.5 and 5.6.10) because

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \quad \text{and} \quad v = M^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Example 7.2.16 Let now $(X,S)$ be the Thue-Morse shift generated by $\sigma : 0 \rightarrow 01, 1 \rightarrow 10$. Let $x = \sigma^2 \omega (0 \cdot 0)$. Set $B = \{1,2,3,4\} = \{0,1\}^+$ and let $\phi : B^* \rightarrow \{0,1\}^*$.
be the coding morphism for $R_X(0 \cdot 0) = \{01101, 01110, 01011010, 010110\}$. Let
\[ \tau : B^* \to B^* \] be the substitution such that $\phi \circ \tau = \sigma^2 \circ \phi$. We have $\tau : 1 \to 1234, 2 \to 124, 3 \to 13234, 4 \to 1324$. Using again Proposition 7.2.14, we obtain
the BV-representation of Figure 7.2.9.

The dimension group $K^0(X, S)$ is thus $(\Delta_M, \Delta^+_M, v)$ with

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}, \quad \text{and} \quad v = \begin{bmatrix}
4 \\
3 \\
5 \\
4
\end{bmatrix}
\]
7.3. LINEARLY RECURRENT SHIFTS

This is consistent with the result found in Example 5.6.11 where we identified $K^0(X,S)$ to $\mathbb{Z}[1/2] \times \mathbb{Z}$ with the natural order of $\mathbb{R}^2$ and unit $(1,0)$. Indeed, the eventual range of $M$ is generated by the vectors

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}. $$

In this basis, the matrix $M$ takes the form

$$N = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

with eigenvectors $z = x+y$ for the eigenvalue 4 and $t = x-2y$ for the eigenvalue 1. This allows us to identify $K^0(X,S)$ via the map $\alpha x + \beta y \mapsto (\alpha, \beta)$ with the pairs $(\alpha, \beta)$ such that $2^n \alpha, \beta \in \mathbb{Z}$ for some $n \geq 1$. The order unit is $v = 3x+y \mapsto (3,1)$. Since $v = Nx$, we can normalize the unit to $(1,0)$.

In Example 5.6.11 we obtained the dimension group as the group of a matrix of dimension 3 instead of 4 as here. Actually the method used in Example 5.6.11 can be used also to build a BV-representation of the Thue-Morse shift with 3 vertices at each level $n \geq 1$ (Exercise 7.17).

7.3 Linearly recurrent shifts

In this section, we study the case of linearly recurrent shifts, already introduced in Chapter 2. This family contains primitive substitution shifts (Proposition 2.4.19). The main result is a characterisation of the BV-representation of aperiodic linearly recurrent shifts (Theorem 7.3.6).

7.3.1 Properties of linearly recurrent shifts

In what follows we show that a subshift is linearly recurrent if and only if it has a BV-representation where the incidence matrices have positive entries and belong to a finite set.

The class of linearly recurrent shift spaces is clearly closed under conjugacy (Exercise 7.12).

A simple example of a minimal shift which is not LR is given below.

**Example 7.3.1** Let $u_n$ be the sequence of words defined by $u_{-1} = b, u_0 = a$ and $u_{n+1} = u_n^R u_{n-1}$ for $n \geq 1$. The subshift generated by the infinite word $u$ having all $u_n$ as prefixes is not LR. This results from the property of LR shift spaces with constant to be $(K+1)$-th power free (Proposition 7.3.4 below).

We will use Theorem 7.2.1 to prove the following result, which gives an easily verifiable condition for a substitution shift to be LR (the condition is, for example, visibly satisfied by the Chacon binary substitution). It proves that the property of being LR is decidable for a substitution shift.
Theorem 7.3.2 (Damanik, Lenz) Let \( \sigma : A \to A^* \) be a substitution and let \((X, S)\) be the corresponding substitution subshift. Let \( e \in A \) be a letter such that \( \lim_{n \to \infty} |\sigma^n(e)| = \infty \). Then the following are equivalent.

(i) The letter \( e \) occurs with bounded gaps in \( L(X) \) and for every \( a \in A \), one has \( |\sigma^n(e)|_a \geq 1 \).

(ii) \((X, S)\) is minimal.

(iii) \((X, S)\) is linearly recurrent.

Proof. (i)⇒ (ii) We have to show that for every \( n \geq 1 \), the word \( \sigma^n(e) \) occurs in every long enough word of \( L(X) \). Since \( e \) occurs with bounded gaps, there is \( k \) such that every word of \( L_k(X) \) contains \( e \). Choose \( w \) of length \( |w| \geq (3+k) \max_{a \in A} |\sigma^n(a)| \). Then \( w \) contains a factor \( \sigma^n(u) \) for some word \( u \in L_k(X) \) and thus contains \( \sigma^n(e) \).

(i)⇒ (ii) By Corollary 7.2.12 \((X, S)\) is isomorphic to a primitive substitutive shift. Thus the assertion results from Proposition ??.

(iii)⇒ (i) is obvious.

A fourth equivalent statement could be added, namely that \((X, S)\) is uniquely ergodic (see the notes for a reference).

From Theorem 7.3.2 we know that all minimal substitution shifts are linearly recurrent. We will prove now some important properties of LR shifts.

We prove before a lemma recording a property of return words which had not been used before.

Lemma 7.3.3 Let \( X \) be a shift space and \( u \in L(X) \). Set \( U = R_X(u) \). Let \( x \in U^* \) be such that \( ux \in L(X) \) and \( y \in U \) be such that \( x = pyus \) for some \( p, s \in L(X) \). Then \( pu, s \) are in \( U^* \).

Proof. Since \( ux \in L(X) \), we have \( usu \in L(X) \) (see Figure 7.3.1 on the left). This implies \( su \in U^* \). Since \( U \) is a prefix code, \( pu, us \in U^* \) implies \( p \in U^* \).

![Figure 7.3.1: A property of return words.](image)

A dual statement holds for left return words: let \( U = R'_X(u) \). If \( x \in U^* \) is such that \( xu \in L(X) \) and \( y \in U \) is such that \( x = pyus \) for some \( p, s \in L(X) \) (see Figure 7.3.1 on the right). Then \( p, us \) are in \( U^* \).

Proposition 7.3.4 Let \( X \) be a non-periodic shift space and suppose that it is linearly recurrent with constant \( K \). Then:
7.3. LINEARLY RECURRENT SHIFTS

1. Every word of $L_n(X)$ appears in every word of $L_{(K+1)n-1}(X)$.

2. The number of distinct factors of length $n$ of $X$ is at most $Kn$.

3. $L(X)$ is $(K+1)$-power free, that is, for every nonempty word $u$, $u^n \in L(X)$ implies $n \leq K$.

4. For all $u \in L(X)$ and for all $w \in R_X(u)$ we have $\frac{1}{K}|u| < |w|$.

5. If $u \in L(X)$, then $\text{Card}(R_X(u)) \leq K(K+1)^2$.

Proof. The first three assertions are already in Proposition 2.2.7.

4. Suppose that for $u \in L(X)$ and $w \in R_X(u)$, we have $|u| \geq K|w|$. Since $u$ is a suffix of $uw$, the length of $w$ is a period of $u$ and thus $u^{K+1}$ is a suffix of $u$, a contradiction with assertion 3.

5. Let $u,v \in L(X)$ with $|v| = (K+1)^2|u|$. By assertion 1, every element of $L(X)$ of length $(K+1)|u|$ appears in $v$ and thus all words of $uR_X(u)$ are factors of $v$. By assertion 4, we have

$$\text{Card}(R_X(u))|u|/K \leq \sum_{w \in R_X(u)} |w| \quad (7.3.1)$$

Now, since every word in $uR_X(u)$ occurs in $v$. Considering all successive occurrences of $u$ in $v$, we have a factor of $v$ of the form $ux$ with $x = x_1x_2 \cdots x_k$ where all $x_i$ are in $R_X(u)$ and every occurrence of $u$ in $v$ is either as a prefix of $ux$ and as a suffix of some $x_i$. If $uy$ is a factor of $v$ with $y \in R_X(u)$, then it is a factor of $x$ and thus $x = puyx$. By Lemma 7.3.3, since $R_X(u)$ is a code, $y$ is equal to some of the $x_i$. Thus every word in $R_X(u)$ appears as some $x_i$. This implies that

$$\sum_{w \in R_X(u)} |w| \leq |v| = (K+1)^2|u| \quad (7.3.2)$$

whence the conclusion $\text{Card}(R_X(u)) \leq K(K+1)^2$.

Note the following corollary of Proposition 7.3.4.

Corollary 7.3.5 The factor complexity of a minimal substitution shift is at most linear.

Indeed, by Theorem 7.3.2, a minimal substitution shift is linearly recurrent. This statement is not true for a substitution shift which is not primitive (Exercise 7.19).

We now come back to return words. Let $X$ be a minimal shift space. For $u, v$ such that $uv \in L(X)$, we consider as in the proof of Theorem 7.2.1 the set $R_X(u)$ of right return words to $u$ and the set

$$R_X(u \cdot v) = vR_X(uv)v^{-1}$$

called the set of return words to $u \cdot v$. As we will manipulate return words, it is important to observe that a finite word $w \in A^+$ is a return word to $u \cdot v$ in $X$ if, and only if,
1. $uwv$ is in $L(X)$, and

2. $v$ is a prefix of $wv$ and $u$ is a suffix of $uw$, and

3. the finite word $uwv$ contains exactly two occurrences of the finite word $uv$.

If $v$ is the empty word, we have $R_X(u \cdot \nu) = R_X(u)$ and if $u$ is the empty word, then $R(u \cdot \nu) = R'_X(v)$, the set of left return words to $v$. When it will be clear from the context we will use $R(u \cdot \nu)$ in place of $R_X(u \cdot \nu)$.

Let $B_{u \cdot \nu}$ be a finite alphabet in bijection with $R(u \cdot \nu)$ by $\theta_{u \cdot \nu} : B_{u \cdot \nu} \to R(u \cdot \nu)$. The map $\theta_{u \cdot \nu}$ extends to a morphism from $B_{u \cdot \nu}^\ast$ to $A^\ast$ and the set $\theta_{u \cdot \nu}(B_{u \cdot \nu}^\ast)$ consists of all concatenations of return words to $u \cdot \nu$.

The set $R(u \cdot \nu)$ is a circular code and, more precisely, no word of $R(u \cdot \nu)$ overlaps no trivially a product of words of $R(u \cdot \nu)$. In particular, the map $\theta_{u \cdot \nu} : B_{u \cdot \nu}^\ast \to A^\ast$ is one-to-one.

### 7.3.2 BV-representation of linearly recurrent shifts

The following result characterises linearly recurrent shift spaces in terms of BV-representations. We first need to recall that a dynamical system $(X,T)$, endowed with the distance $d$, is expansive if there exists $\epsilon$ such that for all pairs of points $(x,y)$, $x \neq y$, there exists $n$ with $d(T^n x, T^n y) \geq \epsilon$. We say that $\epsilon$ is a constant of expansivity of $(X,T)$. It is easy to see that the shift spaces are expansive but that odometers are not.

**Theorem 7.3.6** An aperiodic shift space is linearly recurrent if, and only if, it has an expansive BV-representation satisfying:

1. its incidence matrices have positive entries and belong to a finite set of matrices,

2. for all $n \geq 1$ the substitution read on $E(n)$ is eventually proper.

**Proof.** Let $(X,S)$ be an aperiodic LR shift space. It suffices to construct a sequence of KR-partitions having the desired properties.

From Proposition 7.3.4 there exists an integer $K$ such that for all $u$ occurring in some $x \in X$ and all $w \in R(u)$, we have

$$\frac{|u|}{K} \leq |w| \leq K|u| .$$

We set $\alpha = (K + 1)^2$. Let $x = (x_n)_n$ be an element of $X$. For each non-negative integer $n$, we set $u_n = x_{-\alpha^n} \cdots x_{-2} x_{-1}$, $v_n = x_0 x_1 \cdots x_{\alpha^n-1}$, $R_n = R(u_n \cdot v_n)$, $B_n = B_{u_n \cdot v_n}$ and $\theta_n = \theta_{u_n \cdot v_n}$.

By the choice of $K$, every word of $R_X(u_n \cdot v_n)$ appears in $u_{n+1} \cdot v_{n+1}$. Indeed, $\alpha \geq (K + 1)^2$ implies $2\alpha^n \geq (K + 1)^2 2\alpha^n$ and thus $|u_{n+1} v_{n+1}| \geq (K + 1)^2 |u_n v_n|$.
Now define for all \( n \)
\[
\mathcal{P}(n) = \{ S^j[u_n, w v_n] \mid w \in R_n, \ 0 \leq j < |w| \}.
\]

The verification that \((\mathcal{P}(n))_n\) is a sequence of KR-partitions having the desired properties is left to the reader.

Let now \((V, E, \leq)\) be a properly ordered Bratteli diagram satisfying \(1\) and \(2\). Let \( \epsilon \) be a constant of expansivity of \( X_E \). Let \( n_0 \) be a level of \((V, E, \leq)\) such that all cylinders \([e_1, \ldots, e_{n_0-1}]\) are included in a ball of radius \( \epsilon/2 \). For any vertex \( v \in V(n_0-1) \) let \( h_v \) denote the number of edges from \( v \) to \( V(0) \). Now consider the alphabet
\[
A = \{(v, j) \mid v \in V(n_0-1), \ 0 \leq j < h_v\},
\]
the map \( C : X_E \to A \) defined by
\[
C(\epsilon(n))_n = (r(\epsilon_{n_0-1}), j),
\]
if \((\epsilon(n))_{1 \leq n \leq n_0-1}\) is the \( j \)th finite path of \((V, E, \leq)\) from \( r(\epsilon_{n_0-1}) \) to \( V(0) \) with respect to the order on \((V, E, \leq)\), and finally define \( \varphi : X_E \to A^2 \) by
\[
\varphi(x) = (C \circ T^n_E(x))_{n \in \mathbb{Z}}.
\]

We clearly have that \((X_E, T_E)\) is isomorphic to \((\Omega, S)\) where \( \Omega = \varphi(X_E) \) and \( S \) is the shift on \( A \). There remains to show that \((\Omega, S)\) is LR.

Let \( K = \sup_{n \geq n_0} \max_{v \in V(n)} \sum_{v' \in V(n-1)} M(n)_{v,v'} \). Condition \(1\) implies that \( K \) is finite. Let \( L = \max_{v,v' \in V(n_0-1)} (h_v/h_{v'}) \).

Let \( v \in V(n_0-1) \) and let \( w \) be a return word to \( u = (v, 0)(v, 1) \cdots, (v, h_v-1) \). Due to Condition \(1\) and Condition \(2\) we have
\[
|w| \leq 2K \max_{v' \in V(n_0-1)} h_{v'} \leq 2KL|u| \tag{7.3.3}
\]

Now for all \( n \geq n_0 \), let \( \tau_n : V(n) \to V(n-1)^* \) be the substitution read on \( E(n) \). We set \( W = \{(v, 0)(v, 1) \cdots, (v, h_v-1) \mid v \in V(n_0-1)\} \) and we define the morphism \( \sigma : V(n_0-1) \to A^* \) by \( \sigma(v) = (v, 0)(v, 1) \cdots, (v, h_v-1) \).

It is clear that all the elements of \( \Omega \) are concatenations of finite words belonging to \( W \). They are also concatenations of finite words belonging to \( \sigma \circ \tau_{n_0}(V(n)) \), and more generally, of finite words belonging to \( \sigma \circ \tau_{n_0} \circ \cdots \circ \tau_n(V(n)) \) for all \( n \geq n_0 \). As for \(7.3.3\), we can prove that all return words to some elements of \( \sigma \circ \tau_{n_0} \circ \cdots \circ \tau_n(V(n)) \) satisfy the same inequality.

Now let \( u \) be any non-empty finite word appearing in some word of \( \Omega \) and \( w \) be a return word to \( u \). There exists \( n \) such that
\[
\max_{v \in V(n)} |\sigma \circ \tau_{n_0} \circ \cdots \circ \tau_n(v)| \leq |u| < \max_{v \in V(n+1)} |\sigma \circ \tau_{n_0} \cdots \circ \tau_{n+1}(v)|.
\]

Then \( u \) is a factor of some \( \sigma \circ \tau_{n_0} \circ \cdots \circ \tau_{n+1}(vv') \), \( v \) and \( v' \) belonging to \( V(n+1) \). From Condition \(1\) and Condition \(2\) we deduce that \( vv' \) is a factor of
some $\tau_{n+2} \circ \tau_{n+3}(v'')$, $v'' \in V(n+3)$. Then, $u$ is a factor of $\sigma \circ \tau_n \circ \cdots \circ \tau_{n+3}(v'')$. Consequently

$$|w| \leq 2KL|\sigma \circ \tau_n \circ \cdots \circ \tau_{n+3}(v'')| \leq 2K|u|$$

and $(\Omega, S)$ is LR.

Let us call linearly recurrent any Cantor dynamical systems having a BV-representation satisfying 1 and 2 in Theorem 7.3.6. Actually, more can be said about these dynamical systems but we need the following theorem which can be seen as an extension of Proposition 7.2.3.

We say that a minimal Cantor dynamical system $(X, T)$ has topological rank $k$ if $k$ is the smallest integer such that $(X, T)$ has a BV-representation $(X_E, T_E)$ where the sequence of number of vertices $(\text{Card}(V(n)))_n$ bounded by $k$. When such a $k$ does not exist, we say that it has infinite topological rank. Of course, linearly recurrent BV-dynamical systems have finite topological rank.

A topological dynamical system $(X, T)$, endowed with the distance $d$, is said to be equicontinuous whenever

$$\forall \epsilon, \exists \delta > 0, \sup_{n \in \mathbb{Z}} d(T^n x, T^n y) < \epsilon \text{ if } d(x, y) < \delta .$$

**Theorem 7.3.7** Let $(X, T)$ be a minimal Cantor dynamical system with topological rank $k \in \mathbb{N}$. Then, $(X, T)$ is expansive if and only if $k \geq 2$. Otherwise it is equicontinuous.

Let us take the notation of the proof of Theorem 7.3.6. We call $\sigma_n$ the morphism $\sigma$ and $A_n$ the alphabet $A$. Let $X_n$ be the subset of $A_n^\mathbb{Z}$ consisting of all the words $x$ such that for all $i, j, x_i x_{i+1} \cdots x_j$ is a factor of $\sigma_n \circ \tau_n \cdots \circ \tau_{n+3}(v)$ for some $n \in \mathbb{N}$ and $v \in V(n)$. It can be checked that $(X_n, S)$ is a minimal subshift.

**Corollary 7.3.8** Let $(X_E, T_E)$ be a BV-dynamical system with finite topological rank. Then, $(X_E, T_E)$ is expansive if and only if there exists $n_0$ such that $(X_{n_0}, S)$ is not periodic.

Moreover, if the cylinders $[e_1, \ldots, e_{n_0-1}]$ of $X_E$ are all included in balls of radius $\epsilon/2$, $\epsilon$ being a constant of expansivity, then $(X_E, T_E)$ is isomorphic to $(X_{n_0}, S)$.

Note that once some $(X_{n_0}, S)$ is not periodic, then $(X_n, S)$ is aperiodic for all $n \geq n_0$.

### 7.4 $S$-adic representations

We introduce now the notion of an $S$-adic representation of a shift, which generalizes the representation of shifts as substitution shifts. We will replace the
iteration of a morphism by the application of an arbitrary sequence of morphisms. The concept is of course close to that of a BV-representation but more flexible. For example, every stationary Bratteli diagram defines a substitution shift (with the substitution read on the diagram), but not conversely since, as we have seen, one has to use a conjugacy to obtain a stationary Bratteli diagram for a substitution shift (when the substitution is not proper). The same situation occurs for $S$-adic representations.

In this section, all morphisms are assumed to be non-erasing and alphabets are always assumed to have cardinality at least 2.

### 7.4.1 Directive sequence of morphisms

For a morphism $\sigma : A^* \to B^*$, we denote as usual $|\sigma| = \max_{a \in A} |\sigma(a)|$ and $\langle \sigma \rangle = \min_{a \in A} |\sigma(a)|$.

Let $S$ be a family of morphisms. Let $(A_n)_{n \geq 0}$ be a sequence of finite alphabets and let $\tau = (\tau_n)_{n \geq 0}$ be a sequence of morphisms with $\tau_n : A_{n+1}^* \to A_n^*$ and $\tau_n \in S$. For $0 \leq n \leq N$, we define $\tau_{(n,N)} = \tau_n \circ \tau_{n+1} \circ \cdots \tau_{N-1}$ and $\tau_{[n,N]} = \tau_n \circ \tau_{n+1} \circ \cdots \tau_N$. For $n \geq 0$, the language $\mathcal{L}^{(n)}(\tau)$ of level $n$ associated with $\tau$ is defined by

$$\mathcal{L}^{(n)}(\tau) = \{ w \in A_n^* | w \text{ occurs in } \tau_{[n,N]}(a) \text{ for some } a \in A_N \text{ and } N > n \}.$$ 

The language $\mathcal{L}^{(n)}(\tau)$ defines a shift space $X^{(n)}(\tau)$ called the shift generated by $\mathcal{L}^{(n)}(\tau)$. More precisely, $X^{(n)}(\tau)$ is the set of points $x \in A_N^\mathbb{Z}$ such that $\mathcal{L}(x) \subseteq \mathcal{L}^{(n)}(\tau)$.

Note that it may happen that $\mathcal{L}(X^{(n)}(\tau))$ is strictly contained in $\mathcal{L}^{(n)}(\tau)$ (if, for example, all $\tau_n$ are equal to a morphism which is not a substitution) or even empty (if $|\tau_{[0,n]}|$ is bounded). We say that $\tau$ is a directive sequence if for every $n$, the language $\mathcal{L}^{(n)}(\tau)$ is extendable. This implies that $|\tau_{[0,n]}|$ tends to infinity with $n$ and that $\mathcal{L}(X(\mathcal{V}(\tau))) = \mathcal{L}^{(n)}(\tau)$ for every $n \geq 0$.

When $\tau$ is directive sequence, we set $X(\tau) = X^{(0)}(\tau)$ and $= \mathcal{L}(\tau) = \mathcal{L}^{(0)}(\tau)$. We call $(X(\tau), S)$ the $S$-adic shift with directive sequence $\tau$. We also say that $\tau$ is an $S$-adic representation of $X = X(\tau)$.

**Example 7.4.1** Every substitutive shift $X = X(\sigma, \phi)$ has an $S$-adic representation with $S = \{ \sigma, \phi \}$. Indeed, let $Y = X(\sigma)$ and $X = \pi(Y)$ where $\sigma : B^* \to B^*$ is a substitution and $\phi : B^* \to A^*$ is nonerasing. Then $X$ is the $S$-adic shift with directive sequence $(\phi, \sigma, \sigma, \ldots)$.

As for BV-representations, we have the notion of telescoping of a directive sequence. Given a sequence of morphisms $\tau = (\tau_n)$ and a sequence $(n_m)$ of integers with $n_0 = 0 < n_1 < n_2 < \ldots$, a telescoping of $\tau$ with respect to $(n_m)$ is the sequence of morphisms $\tau' = (\tau_{[n_m, n_{m+1}]})_{m \geq 0}$. If $\tau$ is a directive sequence, then $\tau'$ is a directive sequence. Moreover $\tau$ and $\tau'$ define the same shift $X(\tau) = X(\tau')$. 
As for substitution shifts, in which fixed points play an important role, we have for directive sequences, the notion of limit point. Let \( \tau = (\tau_n)_{n \geq 0} \) be an \( S \)-adic system with \( \tau_n : A_{n+1}^N \rightarrow A_n^N \). A sequence \( x \in A_0^N \) is called a limit point of \( \tau \) if there is a sequence \((w^{(n)})\) of sequences \( w^{(n)} \in A_n^N \) such that \( w^{(n)} = \tau_n(w^{(n+1)}) \) with \( x = w^{(0)} \).

**Example 7.4.2** Let \( \sigma \) be a substitution with fixed point \( x = \sigma^\omega(a) \). Then \( x \) is a limit point of the directive sequence \( \langle \sigma, \sigma, \ldots \rangle \). Indeed, the constant sequence \( w^{(n)} = \sigma^\omega(a) \) is such that \( w^{(n+1)} = \sigma(w^{(n)}) \) and \( x = w^{(0)} \).

Observe that, contrary to the case of substitutive sequences, every sequence has an \( S \)-adic representation (Exercise 7.21).

### 7.4.2 Primitive directive sequences

We say that the sequence \( \tau \) is primitive if, for any \( n \geq 1 \), there exists \( N > n \) such that for all \( a \in A_N \), \( \tau_{\langle n,N \rangle}(a) \) contains occurrences of all letters of \( A_n \).

Set \( M_{\tau_{\langle n,N \rangle}} = M_{\tau_n}M_{\tau_{n+1}}\cdots M_{\tau_{N-1}} \) where \( M_{\tau_n} \) is the composition matrix of \( \tau_n \). Then \( \tau \) is primitive if for any \( n \geq 1 \), there exists \( N > n \) such that \( M_{\tau_{\langle n,N \rangle}} > 0 \).

Observe that if \( \tau \) is primitive, then \( \langle \tau_{\langle 1,n \rangle} \rangle \) goes to infinity when \( n \) increases.

When \( \tau \) is primitive, we can use alternative definitions of limit points.

(i) There is a primitive directive \( \tau \) and a sequence \( a_n \in A_n \) of letters such that \( x = \text{the limit of} \ \tau_{\langle 0,n \rangle}(a_n^w) \)

(ii) There is a primitive directive \( \tau \) and a sequence \( a_n \in A_n \) of letters such that \( \{x\} = \bigcap_{n \geq 0} \tau_{\langle 0,n \rangle}(a_n) \) where \( [w] \) is the cylinder defined by \( w \).

(Exercise 7.20).

The following result generalizes the fact that every growing morphism has a power with a fixed point (Proposition 7.4.4).

**Proposition 7.4.3** Every primitive \( S \)-adic system has a limit point.

**Proof.** Let \( \tau = (\tau_n) \) be a primitive directive sequence. For \( a \in A_{n+1} \), denote by \( f_n(a) \) the first letter of \( \tau_n(a) \). Since \( A_0 \) is finite, there is a letter \( a \) in the decreasing sequence of sets \( \cap_{n \geq 0} f_{\langle 0,n \rangle}(A_n) \). By construction, there is a sequence \( (a_n) \) such that \( a_0 = a \) and \( a_n = f_n(a_{n+1}) \). Set \( u_n = \tau_{\langle 0,n \rangle}(a_n) \). By construction, every \( u_n \) is a prefix of \( u_{n+1} \) and since \( \tau \) is primitive, their lengths tend to infinity.

Let \( u^{(0)} \) be the unique one sided sequence having all \( u_n \) as prefixes. By shifting the sequences \( (\tau_n) \) and \( (a_n) \), we define a sequence \( u^{(n)} \) of sequences such that \( u^{(n)} = \tau_n(u^{(n+1)}) \). Thus \( u^{(0)} \) is a limit point of \( \tau \).

The following proposition generalizes the fact that a primitive substitution shift is minimal (Proposition 2.4.10).
Proposition 7.4.4 If \( \tau \) is primitive, it is a directive sequence and \( X(\tau) \) is minimal.

Proof. Since \( \langle \tau_{[n,N]} \rangle \to \infty \) with \( N \), the language \( \mathcal{L}^{(n)}(\tau) \) is extendable and thus \( \tau \) is a directive sequence.

Let us show that \( X(\tau) \) is uniformly recurrent. For this, let \( u \in \mathcal{L}(\tau) \). By definition of \( \mathcal{L}(\tau) \), there is an \( n > 1 \) and a letter \( a \in A_n \) such that \( u \) appears in \( \tau_{[1,n]}(a) \). Since \( \tau \) is primitive, there is an \( N > n \) such that for all \( b \in A_N \), the letter \( a \) appears in \( \tau_{[n,N]}(b) \). Let \( b, c \in A_N \) be such that \( bc \in \mathcal{L}^{(N)}(\tau) \). Then \( u \) appears in \( \tau_{[1,N]}(b) \) and in \( \tau_{[1,N]}(c) \) at a distance at most equal to \( 2|\tau_{[1,N]}| \) (see Figure 7.4.1). This shows that \( X(\tau) \) is uniformly recurrent.

\[ \end{proof} \]

A morphism \( \sigma : A^* \to B^* \) is said to be left proper (resp. right proper) when there exist a letter \( b \in B \) such that for all \( a \in A \), \( \sigma(a) \) starts with \( b \) (resp., ends with \( b \)). Thus it is proper if it is both left and right proper.

We say that sequence \( \tau = (\tau_n) \) of morphisms is left proper (resp. right proper, resp. proper) whenever each morphism \( \tau_n \) is left proper (resp. right proper, resp. proper). We also say that a shift is a left proper (resp. right proper, resp. primitive) \( S \)-adic shift if there exists a left proper (resp. right proper, resp. primitive) sequence of morphisms \( \tau \) such that \( X = X(\tau) \).

Let us give another way to define \( X(\tau) \) when \( \tau \) is primitive and proper. For a non erasing morphism \( \sigma : A^* \to B^* \), let \( \Omega(\sigma) \) be the closure of \( \cup_{k \in \mathbb{Z}} S^k \sigma(A^*) \).

Lemma 7.4.5 Let \( \tau = (\tau_n : A_{n+1}^* \to A_n^*)_{n \geq 1} \) be a primitive and proper sequence of morphisms. Then,

\[ X(\tau) = \cap_{n \geq 1} \Omega(\tau_{[1,n]}) \]

Proof. It is equivalent to prove that

\[ \mathcal{L}(\tau) = \cap_{n \geq 2} \text{Fac}(\tau_{[1,n]}(A_n^*)) \]

Consider first \( w \in \mathcal{L}(\tau) \). By definition of \( \mathcal{L}(\tau) \), there is an \( n \geq 2 \) and a letter \( a \in A_n \) such that \( w \) appears in \( \tau_{[1,n]}(a) \). Since \( \tau \) is primitive, there is for each \( N > n \) a letter \( b \in A_N \) such that \( a \) is a factor of \( \tau_{[n,N]}(b) \). This proves that \( w \in \text{Fac}(\tau_{[1,N]}(A_N)) \) for arbitrary large \( N \) and thus that \( w \in \cap_{n \geq 2} \text{Fac}(\tau_{[1,n]}(A_n^*)) \).
Conversely, let \( \ell_n \in A_n \) (resp. \( r_n \in A_n \)) be the first letter (resp. last letter) of all \( \tau_n(a) \) for \( a \in A_{n+1} \). Let \( w \in \cap_{n \geq 2} \text{Fac}(\tau_{[1,n]}(A_n^*)) \). Since \( \tau \) is primitive we may assume, telescoping if necessary the sequence \( \tau_n \), that \( \langle \tau_n \rangle \geq 2 \). For each \( n \geq 2 \), there is some \( w_n \in A_n^* \) such that \( w \) is a factor of \( \tau_{[1,n]}(w_n) \) and we can choose \( w_n \) of minimal length. Then \( |w| \geq 2^n(|w_n| - 2) \). Thus there is an \( n \geq 2 \) such that \( w_n \) has length at most 2. If \( |w_n| = 1 \) we obtain the conclusion \( w \in L(\tau) \). Assume that \( w_n \) has length 2. If \( w_n \neq r_n \ell_n \), then \( w_n \) is a factor of some \( \tau_{n+1}(a) \) for \( a \in A_{n+1} \) and we are done. Otherwise, consider any \( c \in A_{n+2} \) and \( d, e \) two consecutive letters of \( \tau_{n+1}(c) \). Then \( \tau_n \tau_{n+1}(c) \) has a factor \( r_n \ell_n \) and thus, we can choose \( w_{n+2} = c \).

Observe that the hypotheses that \( \tau \) is both primitive and proper cannot be dropped in the previous statement. Without these hypotheses, the inclusion \( X(\tau) \subset \cap_{n \in \mathbb{N}} \Omega(\tau_{[0,n]}) \) still holds (under the mild assumption that \( A_n \subset \text{Fac}(\tau_n(A_{n+1})) \), but not the reverse inclusion, as shown by the following examples.

**Example 7.4.6** Take for the directive sequence \( \tau \) the constant sequence equal to \( \tau \), defined by \( \tau(0) = 0010 \) and \( \tau(1) = 1 \) (this is the Chacon binary substitution). The directive sequence \( \tau \) is not primitive, and \( \cdots 111 \cdots \) belongs to \( \cap_{n \in \mathbb{N}} \Omega(\tau_{[0,n]}) \) but not to \( X(\tau) \).

**Example 7.4.7** In the case of the non-proper constant sequence given by \( \tau \) with \( \tau(0) = 0100 \) and \( \tau(1) = 101 \), the fixed point \( \tau^n(1 \cdot 1) \) belongs to \( \cap_{n \in \mathbb{N}} \Omega(\tau_{[0,n]}) \), but not to \( X(\tau) \), since 11 appears in no element of \( X(\tau) \).

With a left proper morphism \( \sigma : A^* \to B^* \) such that \( b \in B \) is the first letter of all images \( \sigma(a), a \in A \), we associate the right proper morphism \( \overline{\sigma} : A^* \to B^* \) defined by \( \overline{\sigma}(a) = \sigma(a)b \) for all \( a \in A \).

**Lemma 7.4.8** Let \((X,S)\) be an \( S \)-adic shift generated by the primitive and left proper directive sequence \( \tau = (\tau_n : A_{n+1}^* \to A_n^*)_{n \geq 1} \). Then \((X,S)\) is also generated by the primitive and proper directive sequence \( \tilde{\tau} = (\tilde{\tau}_n)_{n \geq 1} \), where for all \( n \), \( \tilde{\tau}_n = \tau_{2n-1} \tau_{2n} \). In particular, if \( \tau \) is unimodular, then so is \( \tilde{\tau} \).

**Proof.** Each morphism \( \tilde{\tau}_n \) is trivially proper. It is also clear that the unimodularity of \( \tau \) is preserved in this process. Now, let \( \sigma : A^* \to B^* \) be a proper morphism such that \( b \in B \) is the first letter of all images \( \sigma(a), a \in A \); for all \( x \in A^2 \), one has \( \overline{\sigma}(x) = S\sigma(x) \). Together with Lemma 7.4.5 this ends the proof.

**7.4.3 Unimodular \( S \)-adic shifts.**

We also say that sequence \( \tau = (\tau_n) \) of morphisms is unimodular whenever, for all \( n \geq 1 \), \( A_{n+1} = A_n \) and the matrix \( M_{\tau_n} \) has determinant of absolute value
7.4. S-ADIC REPRESENTATIONS

1. We say that a shift is a unimodular (resp. proper, resp. primitive) $S$-adic shift if there exists a unimodular (resp. proper, resp. primitive) sequence of morphisms $\tau$ such that $X = X(\tau)$.

**Lemma 7.4.9** All primitive unimodular proper $S$-adic shifts are aperiodic.

Proof. Let $\tau$ be a primitive unimodular proper directive sequence on the alphabet $A$ of cardinality $d \geq 2$. Suppose that the shift $X(\tau)$ is periodic, that is $X(\tau) = \{x, Tx, \ldots, T^{p-1}x\}$.

Since $\tau$ is primitive, there is some $n \geq 1$ such that $|\tau_{1,n}(a)| \geq p$.

Let $b \in A$ and set $\tau_{n+1}(b) = b_{0}b_{1}\cdots b_{k}$. Since the directive sequence $\tau$ is proper, $b_{0}b_{1}\cdots b_{k}b_{0}$ is also a word in $\mathcal{L}^{(n+1)}(\tau)$. Then $w = \tau_{1,n}(b_{0}b_{1}\cdots b_{k}b_{0})$ is a word which has both period $p$ (as a factor of $x$) and period $|\tau_{1,n+1}(b)|$. By Fine-Wilf Theorem (Exercise 2.11), we have $|\tau_{1,n+1}(1)| \equiv 0$ modulo $p$. But then every row of the matrix $M(\tau_{1,n+1})$ has a sum divisible by $p$ and thus its determinant is a multiple of $p$, which contradicts the unimodularity of $\tau$.

### 7.4.4 Recognizability and unimodular $S$-adic shifts.

We have seen in Section 2.4 the definition of recognizability of morphisms. Let us recall this definition here. Let $\varphi : A^{*} \to B^{*}$ be a nonerasing morphism. Let $(X,S)$ with $X \subset A^{2}$ be a shift space and let $Y$ be the closure of $\varphi(X)$ under the shift. Every $y \in Y$ has a representation as $y = S^{k}\varphi(x)$ with $x \in X$ and $0 \leq k < |\varphi(x)|$. We say that $\varphi$ is recognizable in the shift $X$ for the point $y$ if $y$ has only one such representation.

We say that $\varphi$ is recognizable in $X$ if it is recognizable in $X$ for every point $y \in Y$. We also say that $\varphi$ is recognizable in $X$ for aperiodic points if $\varphi$ is recognizable in $X$ for every aperiodic point $y \in Y$. Finally, we say that $\varphi$ is recognizable for aperiodic points if it is recognizable in the full shift for aperiodic points.

The notion of tower will be helpful in the proof of the following statement.

**Proposition 7.4.10** Let $\sigma : A^{*} \to B^{*}$ and $\tau : B^{*} \to C^{*}$ be morphisms. Let $X \subset A^{2}$ be a shift space and let $Y$ be the subshift generated by $\sigma(X)$. Then

1. $\tau \circ \sigma$ is recognizable on $X$ if and only if $\sigma$ is recognizable on $X$ and $\tau$ is recognizable on $Y$.

2. If $\sigma$ is recognizable in $X$ for aperiodic points and $\tau$ is recognizable in $Y$ for aperiodic points, then $\tau \circ \sigma$ is recognizable in $X$ for aperiodic points.

3. If $\tau \circ \sigma$ is recognizable in $X$ for aperiodic points, then $\tau$ is recognizable in $Y$ for aperiodic points.

Proof. Let $Z$ be the subshift of $C^{2}$ generated by $\tau(Y)$. Set $\rho = \tau \circ \sigma$. We have $\hat{\rho} = \hat{\tau} \circ \alpha$ (see Figure 7.4.2), where $\alpha : X^{\rho} \to Y^{\tau}$ is the following map. For each
(x, k) ∈ X^ρ, there is a unique pair (i, j) such that 

\[ k = |τ(b_0 \cdots b_{i-1})| + j \]

with \( j < |τ(b_i)| \) with \( σ(x_0) = b_0 \cdots b_{|σ(x_0)|} \). Then

\[ α(x, k) = (τ(x,i), j) \]

1. It is clear that \( α \) is injective if and only if \( \hat{σ} \) is injective. Since \( \hat{ρ} \) is surjective, it follows that \( \hat{σ} \) and \( \hat{τ} \) are injective.

2 and 3 follow easily since \( \hat{τ} \) sends periodic points to periodic points.

Recall (see Exercise 2.25) that a morphism \( ϕ : A^∗ \rightarrow B^∗ \) is called elementary if for every decomposition \( ϕ = β \circ α \) with \( α : A^∗ \rightarrow C^∗ \) and \( β : C^∗ \rightarrow B^∗ \), one has \( \text{Card}(C) ≥ \text{Card}(A) \).

**Proposition 7.4.11** Let \( ϕ : A^∗ \rightarrow B^∗ \) be a nonerasing morphism. If \( ϕ \) is elementary, the extension of \( ϕ \) to \( A^N \) is injective.

The proof is Exercise 2.25.

We will now prove the following result. It holds in particular when \( M(σ) \) is unimodular.

**Theorem 7.4.12** Let \( σ : A^∗ \rightarrow B^∗ \) be an elementary morphism. Then \( σ \) is recognizable for aperiodic points.

**Proof.** We proceed by induction on \( ||σ|| \).

If \( |σ(a)| = 1 \) for all \( a \in A \), then \( σ \) is a permutation of \( A \), whence recognizable in the full shift.

Assume now that \( σ \) is not recognizable on aperiodic points. Since \( M(σ) \) has rank \( \text{Card}(A) \), the morphism \( σ \) is elementary. Thus, by Proposition 7.4.11, \( σ \) is injective on \( A^N \). Thus there exist \( x, x' \in A^N \) and \( w \) with \( 0 < |w| < |σ(x_0)| \) such that \( σ(x) = wσ(x') \) for some proper suffix \( w \) of \( σ(a') \). Set \( σ(a') = vw \). We can then write \( σ = σ_1 \circ τ_1 \) with \( τ_1 : A^* \rightarrow A_1^* \) and \( σ_1 : A_1^* \rightarrow B^* \) and \( A_1 = A \cup \{a''\} \) where \( a'' \) is a new letter. We have \( τ_1(a') = a''τ(a) = a \) otherwise. Next \( σ_1(a') = v, σ_1(a'') = w \) and \( σ_1(a) = a \) otherwise. Note that \( ||σ_1|| = ||σ|| \). Since \( τ_1 \) is injective on \( A^N \), \( σ_1 \) is not injective on \( A_1^N \). By Proposition 7.3.11 again, \( σ_1 \) is not elementary. Thus \( σ_1 = σ_2 \circ τ_2 \) with \( τ_2 : A_1^* \rightarrow A_2^*, σ_2 : A_2^* \rightarrow B^* \) and
Card(A₂) < Card(A₁) = Card(A) + 1. On the other hand, since \( \sigma = (\tau_1 \circ \tau_2) \circ \sigma_2 \) is elementary, we have Card(A) ≤ Card(A₂). Thus

\[
\text{Card}(A) \leq \text{Card}(A₂) \leq \text{Card}(A)
\]

and consequently Card(A) = Card(A₂). Since \( \|\sigma_2\| < \|\sigma_1\| = \|\sigma\| \), we may apply the induction hypothesis to obtain that \( \sigma_2 \) is recognizable for aperiodic points. Since \( \tau_1 \) and \( \tau_2 \) are recognizable in the full shift, we obtain the conclusion by Proposition 7.4.10.

We will use the following corollary.

**Corollary 7.4.13** Let \( \sigma : A^* \rightarrow B^* \) be a morphism such that \( M(\sigma) \) has rank \( \text{Card}(A) \). Then \( \sigma \) is recognizable for aperiodic points.

This follows directly from Theorem 7.4.12. Indeed, a morphism \( \sigma : A^* \rightarrow B^* \) such that \( M(\sigma) \) has rank \( \text{Card}(A) \) is elementary.

**Example 7.4.14** The matrix of the morphism \( \sigma : a \rightarrow ab, b \rightarrow aa \) is

\[
M(\sigma) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}
\]

which has rank 2. The morphism is not recognizable for \( a^\infty \) but for every other point \( y \), the occurrence of a \( b \) is enough to determine the representation of \( y \) as \( y = S^k\sigma(x) \). For example, \( y_0 = b \) forces \( k = 1 \) and \( x_0 = a \). All the other values of \( x_n \) are then determined.

We use Corollary 7.4.13 to prove the following.

**Proposition 7.4.15** Let \( X \) be an \( S \)-adic shift with unimodular directive sequence \( \tau \). Then \( X \) is aperiodic and minimal if and only if \( \tau \) is primitive.

**Proof.** Recall that any \( S \)-adic subshift with a primitive directive sequence is minimal (Proposition 7.4.3) and that aperiodicity is proved in Lemma 7.4.9.

We only have to show that the condition is necessary. We assume that \( X \) is aperiodic and minimal. For all \( n \geq 1 \), \( (X^{(n)}(\tau), S) \) is trivially aperiodic. Let us show that it is minimal.

Assume by contradiction that for some \( n \geq 1 \), \( (X^{(n)}(\tau), S) \) is minimal, but not \( (X(\tau)^{(n+1)}, S) \). There exist \( u \in \mathcal{L}(X(\tau)^{(n+1)}) \) and \( x \in X^{(n+1)}(\tau) \) such that \( u \) does not occur in \( x \). By Corollary 7.4.13 \( \{\tau_n(v) \mid v \in \mathcal{L}(X^{(n+1)}(\tau)) \cap A^{|u|}\} \) is a finite clopen partition of \( \tau_n(X^{(n+1)}(\tau)) \). Thus, considering \( y = \tau_n(x), \) by minimality of \( (X^{(n)}(\tau), S), \) there exists \( k \geq 0 \) such that \( S^k y \) is in \( \tau_n([u]) \).

Take \( z \in [u] \) such that \( S^k z = \tau_n(z) \). Since \( y \) is aperiodic and since we also have \( S^k y = S^{k'} \tau_n(S^\ell x) \) for some \( \ell \in \mathbb{N} \) and \( 0 \leq k' < |\tau_n(x)| \), we obtain that \( \tau_n(z) = S^{k'} \tau_n(S^\ell x) \) with \( z \in [u], S^\ell x \not\in [u] \) and \( 0 \leq k' < |\tau_n(x)| \); this contradicts Corollary 7.4.13.
We now show that \( \lim_{n \to +\infty} \langle \tau_{[1,n]} \rangle = +\infty \). We again proceed by contradiction, assuming that \( \langle \tau_{[1,n]} \rangle \) is bounded. Then there exists \( N > 0 \) and a sequence \( (a_n)_{n \geq N} \) of letters in \( \mathcal{A} \) such that for all \( n \geq N \), \( \tau_n(a_{n+1}) = a_n \). We claim that there are arbitrary long words of the form \( a_k^N \) in \( \mathcal{L}(X^{(N)}(\tau)) \) which contradicts the fact that \( (X^{(N)}(\tau), S) \) is minimal and aperiodic. Since \( \tau \) is proper, for all \( n \geq N \) and all \( b \in \mathcal{A} \), \( \tau_n(b) \) starts and ends with \( a_n \). Since \( \langle \tau_{[1,n]} \rangle \) goes to infinity, there exists a sequence \( (b_n)_{n \geq N} \) of letters in \( \mathcal{A} \) such that \( |\tau_{|X,n}(b_n)| \) goes to infinity and for all \( n \geq N \), \( b_n \) occurs in \( \tau_n(b_{n+1}) \). This implies that there exists \( M \geq N \) such that for all \( n \geq M \), \( b_n \neq a_n \) and, consequently, that \( \tau_n(b_{n+1}) = a_n u_n \) for some word \( u_n \) containing \( b_n \). It is then easily seen that, for all \( k \geq 1 \), \( a_k^{b_n} \) is a prefix of \( \tau_{(M,M+k)}(b_{M+k}) \), which proves the claim.

We finally show that \( \tau \) is primitive. If not, there exist \( N > 1 \) and a sequence \( (a_n)_{n \geq N} \) of letters in \( \mathcal{A} \) such that for all \( n > N \), \( a_N \) does not occur in \( \tau_{|X,n}(a_n) \). Since \( \langle |\tau_{|X,n}(a_n)| \rangle \) goes to infinity, this shows that there are arbitrarily long words in \( \mathcal{L}(X^{(N)}(\tau)) \) in which \( a_N \) does not occur. Since \( \tau \) is unimodular, there is also a sequence \( (a'_n)_{n \geq N} \) of letters in \( \mathcal{A} \) such that \( a_N = a'_N \) and for all \( n \geq N \), \( a'_n \) occurs in \( \tau_n(a'_{n+1}) \). Again using the fact that \( |\tau_{|X,n}(a'_n)| \) goes to infinity, this shows that \( a_N \) belongs to \( \mathcal{L}(X^{(N)}(\tau)) \). We conclude that \( (X^{(N)}(\tau), S) \) is not minimal, a contradiction.

### 7.5 Dimension groups of unimodular \( S \)-adic shifts.

In this section we first prove a key result of this chapter, namely Theorem 7.5.1, which states that \( H(X, T) = C(X, Z)/\beta C(X, Z) \) is generated, as an additive group, by the classes of the characteristic functions of letter cylinders. We then deduce a simple expression for the dimension group of primitive unimodular proper \( S \)-adic subshifts.

#### 7.5.1 From letters to factors

We recall that \( \chi_U \) stands for the characteristic function of the set \( U \).

**Theorem 7.5.1** Let \((X, S)\) be a primitive unimodular proper \( S \)-adic shift. Any function \( f \in C(X, Z) \) is cohomologous to some integer linear combination of the form \( \sum_{a \in \mathcal{A}} \alpha_a \chi[|a|] \in C(X, Z) \). Moreover, the classes \( \langle \chi[|a|], a \in \mathcal{A} \rangle \) are \( \mathbb{Q} \)-independent.

**Proof.** Let \( \tau = (\tau_n : \mathcal{A}^* \to \mathcal{A}^*)_{n \geq 1} \) be a primitive unimodular proper directive sequence of \((X, S)\), hence \( X = X_{\tau} \). Using Proposition 7.4.10, all subshifts \((X_{\tau}^{(n)}), S)\) are minimal and aperiodic and \( \min_{a \in \mathcal{A}} |\tau_{[1,n]}(a)| \) goes to infinity when \( n \) increases.

Let us show that the group \( H(X, S) = C(X, Z)/\beta C(X, Z) \) is spanned by the set of classes of characteristic functions of letter cylinders \( \{\chi[|a|] \mid a \in \mathcal{A}\} \). From
Corollary 7.4.13 and using the fact that \((X, S)\) is minimal and aperiodic, one has, for all positive integer \(n\), that

\[ \mathcal{P}_n = \{ S^k \tau_{[1,n]}([a]) \mid 0 \leq k < |\tau_{[1,n]}(a)|, a \in \mathcal{A} \} \]

is a finite partition of \(X\) into clopen sets. This provides a family of nested Kakutani-Rohlin tower partitions.

We first claim that \(H(X, S)\) is spanned by the set of classes \(\cup_n \Omega_n\), where

\[ \Omega_n = \{ [\tau_{[1,n]}([a])] \mid a \in \mathcal{A} \} \quad n \geq 1. \]

In other words, \(H(X, S)\) is spanned by the set of classes of characteristic functions of \(X\). It suffices to check that, for all \(u^{-}u^{+} \in \mathcal{L}(X)\), the class \([\chi_{[u^{-}, u^{+}]}]\) is a linear integer combination of elements belonging to some \(\Omega_n\).

Let us check this assertion. Let \(u^{-}u^{+}\) be in \(\mathcal{L}(X)\). Since \(\min_{a \in \mathcal{A}} |\tau_{[1,n]}(a)|\) goes to infinity, there exists \(n\) such that \(|u^{-}|, |u^{+}| < \min_{a \in \mathcal{A}} |\tau_{[1,n]}(a)|\). The directive sequence \(\tau\) being proper, there exist words \(w, w'\) with respective lengths \(|w| = |u^{-}|\) and \(|w'| = |u^{+}|\) such that all images \(\tau_{[1,n]}(a)\) start with \(w\) and end with \(w'\).

Let \(x \in [u^{-}, u^{+}]\). Let \(a \in \mathcal{A}\) and \(k \in \mathbb{N}\), \(0 \leq k < |\tau_{[1,n]}(a)|\), such that \(x\) belongs to the atom \(S^k \tau_{[1,n]}([a])\). Observing that \(\tau_{[1,n]}([a])\) is included in \([w', \tau_{[1,n]}(a)]w\), this implies that the full atom \(S^k \tau_{[1,n]}([a])\) is included in \([u^{-}, u^{+}]\).

Consequently \([u^{-}, u^{+}]\) is a finite union of atoms in \(\mathcal{P}_n\). But each characteristic function of an atom of the form \(S^k \tau_{[1,n]}([a])\) is cohomologous to \(\chi_{\tau_{[1,n]}([a])}\). This thus proves the claim.

Now we claim that each element of \(\Omega_n\) is a linear integer combination of elements in \(\{ [\chi_{[a]}] \mid a \in \mathcal{A} \}\). More precisely, let us show that \(\chi_{\tau_{[1,n]}([b])}\) is cohomologous to

\[ \sum_{a \in \mathcal{A}} (M_{[1,n]}^{-1})_{b,a} \chi_{[a]}. \]

Let \(a \in \mathcal{A}\) and \(n \geq 1\). One has \([a] = \cup_{B \in \mathcal{P}_n} (B \cap [a])\) and thus \(\chi_{[a]}\) is cohomologous to the map

\[ \sum_{B \in \mathcal{A}} (M_{[1,n]}^{-1})_{a,b} \chi_{\tau_{[1,n]}([b])}, \]

by using the fact that the maps \(\chi_{S^k \tau_{[1,n]}([a])}\) are cohomologous to \(\chi_{\tau_{[1,n]}([a])}\). This means that for \(U = ([\chi_{[a]}])_{a \in \mathcal{A}} \in H(X, S)^{\mathcal{A}}\) and \(V = ([\chi_{\tau_{[1,n]}([a])}])_{a \in \mathcal{A}} \in H(X, S)^{\mathcal{A}}\), one has

\[ U = M_{[1,n]}^{-1} V \]

and as a consequence \(V = M_{[1,n]}^{-1} U\). This proves the claim and the first part of the theorem.

To show the independence, suppose that there exists some row vector \(\alpha = (\alpha_{a})_{a \in \mathcal{A}} \in \mathbb{Z}^{\mathcal{A}}\) such that \(\sum_{a} \alpha_{a} [\chi_{[a]}] = 0\). Hence there is some \(f \in C(X, \mathbb{Z})\) such that \(\sum_{a} \alpha_{a} \chi_{[a]} = f \circ S - f\). The morphisms of the directive sequence \(\tau\)
being proper, for all \( n \), there are letters \( a_n, b_n \) such that all images \( \tau_n(c) \), \( c \in A \), start with \( a_n \) and end with \( b_n \). From this, it is classical to check that \((\mathcal{P}_n)_n\) generates the topology of \( X \) (the proof is the same as in the proof of Lemma 7.2.4 that is concerned with the particular case \( \tau_{n+1} = \tau_n \) for all \( n \)).

We now fix some \( n \) for which \( f \) is constant on each atom of \((\mathcal{P}_n)_n\). Observe that for all \( x \in X \) and all \( k \in \mathbb{N} \), one has

\[
f(S^k x) - f(x) = \sum_{j=0}^{k-1} \alpha_{x(j)} = 0.\]

This holds for all \( c \), hence \( \alpha M_{\tau_{[1,n]}} = 0 \), which yields \( \alpha = 0 \), by invertibility of the matrix \( M_{\tau_{[1,n]}} \).

Observe that in the previous result, we can relax the assumption of minimality. Indeed, one checks that the same proof works if we assume that \((X, S)\) is aperiodic (recognizability then holds by Corollary 7.4.13 and that \( \min_{a \in A} \|\tau_{[1,n]}(a)\| \) goes to infinity.

We now derive two corollaries from Theorem 7.5.1 dealing respectively with invariant measures and with the image subgroup.

**Corollary 7.5.2** Let \((X, S)\) be a primitive unimodular proper \( S \)-adic shift over the alphabet \( A \) and let \( \mu, \mu' \in \mathcal{M}(X, S) \). If \( \mu \) and \( \mu' \) coincide on the letters, then they are equal, that is, if \( \mu([a]) = \mu'([a]) \) for all \( a \) in \( A \), then \( \mu(U) = \mu'(U) \), for any clopen subset \( U \) of \( X \).

**Corollary 7.5.3** Let \((X, S)\) be a primitive unimodular proper \( S \)-adic shift over the alphabet \( A \). The image subgroup of \((X, S)\) satisfies

\[
I(X, S) = \bigcap_{\mu \in \mathcal{M}(X, S)} \left\{ \sum_{a \in A} \mathbb{Z}\mu([a]) \right\}.
\]

### 7.5.2 An explicit description of the dimension group

Theorem 7.5.4 allows us to give a precise description of the dimension group of primitive unimodular proper \( S \)-adic shifts.

**Theorem 7.5.4** Let \((X, S)\) be a primitive unimodular proper \( S \)-adic shift over a \( d \)-letter alphabet. The linear map \( \Phi : H(X, S) \to \mathbb{Z}^d \) defined by \( \Phi([x[a]]) = e_a \), where \( \{e_a \mid a \in A\} \) is the canonical base of \( \mathbb{Z}^d \), defines an isomorphism of dimension groups from \( K^0(X, S) \) onto

\[
(\mathbb{Z}^d, \{ x \in \mathbb{Z}^d \mid \langle x, \mu \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X, S) \}) \cup \{0\}, 1 \),
\]

where the entries of \( 1 \) are equal to 1.
Proof. From Theorem 7.5.1, $\Phi$ is well defined and is a group isomorphism from $H(X,S)$ onto $\mathbb{Z}^d$. We obviously have $\Phi([1]) = \Phi(\sum_{a \in A} \chi[a]) = 1$ and it remains to show that

$$\Phi(H^+(X,S)) = \{ x \in \mathbb{Z}^d \mid \langle x, \mu \rangle > 0 \text{ for all } \mu \in \mathcal{M}(X,S) \cup \{0\} \}.$$ 

Any element of $H^+(X,S)$ is of the form $[f]$ for some $f \in C(X,\mathbb{N})$. From Theorem 7.5.1 there exists a unique vector $x = (x_a)_{a \in A}$ such that $[f] = \sum_{a \in A} x_a \chi[a]$. As $f$ is non-negative, we have, for any $\mu \in \mathcal{M}(X,S)$,

$$\langle \Phi([f]), \mu \rangle = \sum_{a \in A} x_a \mu([a]) = \int f d\mu \geq 0,$$

with equality if and only if $f = 0$ (in which case $x = 0$).

For the other inclusion, assume that $x = (x_a)_{a \in A} \in \mathbb{Z}^d$ satisfies $\langle x, \mu \rangle > 0$ for all $\mu \in \mathcal{M}(X,S)$ (the case $x = 0$ is trivial). We consider the function $f = \sum_{a \in A} x_a \chi[a]$. According to Proposition 4.3.6, the existence of $f' \in [f]$ such that $f'$ is non-negative is equivalent to the existence of a lower bound for ergodic sums. Assume by contradiction that there exists a point $x \in X$ such that the sequence $(\sum_{k=0}^n f \circ S^k(x))_{n \geq 0}$ is not bounded from below. Thus there is a an increasing sequence of positive integers $(n_i)_{i \geq 0}$ such that

$$\lim_{i \to +\infty} \sum_{k=0}^{n_i-1} f \circ S^k(x) = -\infty.$$

Extracting a subsequence $(m_i)_{i \geq 0}$ of $(n_i)_{i \geq 0}$ if necessary, there exists $\mu \in \mathcal{M}(X,S)$ satisfying

$$\langle x, \mu \rangle = \int f d\mu = \lim_{i \to +\infty} \frac{1}{m_i} \sum_{k=0}^{m_i-1} f \circ S^k(x) \leq 0,$$

which contradicts our hypothesis. The sequence $(\sum_{k=0}^n f \circ S^k(x))_{n \geq 0}$ is thus bounded from below and we conclude by using Lemma 4.3.6.

Remark 7.5.5 We cannot remove the hypothesis of being left or right proper in Theorem 7.5.4. Consider indeed the subshift $(X,S)$ defined by the primitive unimodular non-proper substitution $\tau$ defined over $\{a,b\}^*$ as $\tau: a \mapsto aab$, $b \mapsto ba$. The dimension group of $(X,S)$ is isomorphic to $(\mathbb{Z}^3, \{(a,1,1), (2,\lambda,1), (1,\lambda+2,1)\})$ where $\lambda = [2\lambda, \lambda + 1, 1]$ with $\lambda = (1 + \sqrt{5})/2$ (Exercise 7.22). Thus, although the matrix of $\tau$ is unimodular, the dimension group is $\mathbb{Z}^3$ and not $\mathbb{Z}^2$. 


7.6 Derivatives of substitutive sequences

In this section, we will prove a finiteness result characterizing substitutive shifts. It can be viewed as a counterpart of Schmerntzan-Carrol Theorem concerning interval exchange transformations (Theorem 9.3.1).

Let \((X,S)\) be a minimal shift space and let \(u \in \mathcal{L}(X)\). Let \(\varphi_u : \mathcal{B}^* \to \mathcal{A}^*\) be a coding morphism for the set \(\mathcal{R}_X^\prime(u)\) of left return words to \(u\). We normalize the alphabet \(\mathcal{B}_u\) by setting \(\mathcal{B}_u = \{0,1,\ldots,n-1\}\) with \(n = \text{Card}(\mathcal{R}_X^\prime(u))\).

Recall from Section 4.7 that the derivative shift of \(X\) with respect to the cylinder \(\mathcal{C}_u\) is the shift \(Y = \{y \in \mathcal{B}_u^Z \mid \varphi_u(y) \in X\}\). For \(x \in X\) and \(u \in \mathcal{L}(x)\), the derivative sequence of \(x\) with respect to \(u\) or \(u\)-derivative of \(x\), denoted \(D_u(x)\), is defined as follows. Let \(i \geq 0\) be the least integer such that \((T^i x)^+\) begins with \(u\), that is, such that \(T^i x \in [u]\). Then \(D_u(x)\) is the unique \(y \in Y\) such that \(T^i x = \varphi_u(y)\). In particular, when \(u\) is a prefix of \(x^+\), we have \(D_u(x) = y\) where \(y\) is the unique word of \(Y\) such that \(x = \varphi_u(y)\). Thus, by definition, we have in this case

\[ x = \varphi_u(D_u(x)). \]  

7.6.1 A finiteness condition

We will prove the following characterisation of primitive substitutive sequences by a finiteness condition.

**Theorem 7.6.1** Let \(x\) be a uniformly recurrent two-sided sequence. The following conditions are equivalent

(i) The sequence \(x\) is primitive substitutive.

(ii) The set of \(u\)-derivative sequences of \(x\) is finite, \(u\) being a word of \(\mathcal{L}(x)\).

(iii) The set of its \(u\)-derivative sequences is finite, \(u\) being a prefix of \(x^+\).

Note that, as a corollary of Theorem 7.6.1, we obtain that every shift of a substitutive sequence is substitutive. Indeed, if \(y = T^n x\) is a shift of a substitutive sequence \(x\), we have \(\mathcal{L}(x) = \mathcal{L}(y)\) and the set of \(u\)-derivatives of \(x\) and \(y\) are the same. Thus \(y\) is substitutive. A direct proof is given in Exercise 7.26.

Note also that a uniformly recurrent substitutive two-sided sequence is actually primitive substitutive (see Corollary 7.2.12). Thus we can replace condition (i) by

(i') The sequence \(x\) is substitutive.

Observe finally that in the definition of a substitutive sequence \(x\), one may relax all conditions on the pair \((\sigma,\phi)\) of morphisms defining \(x\) provided \(x\) is a two-sided infinite sequence (see Exercise 7.27).

We begin by considering the easy case where \(x\) is periodic.

**Proposition 7.6.2** Every periodic two-sided sequence is substitutive and has a finite number of derivatives.
7.6. DERIVATIVES OF SUBSTITUTIVE SEQUENCES

Proof. Let \( x = v^\infty \) (recall that \( w^\infty = \cdots vw \cdot vw \cdots \)) with \( v = a_0a_1 \cdots a_{n-1} \).

Let \( B = \{b_0, \ldots, b_{n-1} \} \) and let \( \sigma : B^* \to B^* \) be defined by \( \sigma(b_i) = b_jb_{j+1} \) with \( j = 2i \mod n \). Set \( w = b_0b_1 \cdots b_{n-1} \). Then \( \sigma(w) = w^2 \) and thus \( y = w^\infty \) is an admissible fixed point of \( \sigma \). We have \( x = \phi(y) \) where \( \phi : B^* \to A^* \) is the morphism defined by \( \phi(b_i) = a_i \) for \( 0 \leq i < n \). We conclude that \( x \) is substitutive.

Since \( X \) has only finitely many different cylinders \([u]\), the number of its derivatives is also finite.

7.6.2 Sufficiency of the condition

The implication (iii) \( \Rightarrow \) (i) will result of Proposition 7.6.3.

Proposition 7.6.3 Let \( x \) be an aperiodic uniformly recurrent two-sided sequence such that the set of derivatives \( D_u(x) \), for \( u \) prefix of \( x^+ \), is finite. Then \( x \) is primitive substitutive.

Proof. There exists a sequence of prefixes \((u_n)_{n \geq 1}\) of \( x^+ \) such that \( |u_n| < |u_{n+1}| \) and \( D_{u_n}(x) = D_{u_{n+1}}(x) \), for all \( n \geq 1 \). Clearly this implies, for all \( n \geq 0 \), that \( u_n \) is a prefix of \( u_{n+1} \). Take \( u = u_1 \).

We denote by \( \mathcal{R}'(u) \) the set \( \mathcal{R}'(u) \) where \( X \) is the shift generated by \( x \).

The sequence \( x \) being uniformly recurrent we can choose \( N \) so large that every factor of length \( N \) of \( x \) has an occurrence of each \( wu \) for \( w \in \mathcal{R}'(u) \). Since \( x \) is not periodic, the minimal length of a word in \( \mathcal{R}'(u_n) \) cannot be bounded independently of \( n \). Otherwise there exists a word \( v \) such that \( vu_n \) begins with \( u_n \) for all \( n \) large enough. As a consequence, \( |v| \) is a period of all \( u_n \) and thus of \( x^+ \). Since \( x \) is uniformly recurrent, this forces the period of all words in \( \mathcal{L}(x) \) to be bounded and thus \( x \) is periodic, a contradiction. Thus there exists some \( w = u_1 \) such that \( |wu| > N \) for all \( r \in \mathcal{R}'(w) \).

Let \( \varphi_u : B^*_u \to A^* \) denote a coding morphism for \( \mathcal{R}'(u) \). Since \( \mathcal{R}'(u) \) is a suffix code, the map \( \varphi_u \) is one to one. Since \( D_u(x) = D_w(x) \) we have \( B_u = B_w \).

Set \( B = B_u = B_w \). Since \( \mathcal{R}'(w) \subset \mathcal{R}'(u)^* \), we can define a morphism \( \tau : B^* \to B^* \) such that \( \varphi_u \circ \tau = \varphi_w \).

By the choice of \( u \) and \( w \), for every \( i, j \) in \( B \) the word \( \varphi_u(j)w \) appears in \( \varphi_w(i) \). By the dual of Lemma 7.3.3, this implies that \( j \) appears in \( \tau(i) \). This means that \( \tau \) is a primitive substitution. We have

\[ \varphi_u \circ \tau(D_u(x)) = \varphi_w(D_w(x)) = x. \]

Since \( \varphi_u \) is injective by Proposition 2.4.23 we obtain \( \tau(D_u(x)) = D_u(x) \). Hence \( D_u(x) \) is a fixed point of \( \tau \). Since \( x = \varphi_u(D_u(x)) \), Proposition 7.3.4 implies that there is a primitive substitution \( \zeta \) on an alphabet \( C \), an admissible fixed point \( z \) of \( \zeta \) and a map \( \theta : C \to A \) such that \( \theta(z) = x \). Thus \( x \) is primitive substitutive.
7.6.3 Necessity of the condition

We first prove the following preliminary result.

Proposition 7.6.4 Let \( y \) be a two-sided fixed point of a primitive substitution \( \sigma \). Let \( u \) be a non-empty prefix of \( y^+ \). The derived sequence \( D_u(y) \) is the fixed point of a primitive substitution \( \sigma_u : B_u^* \to B_u^* \) which satisfies

\[
\varphi_u \circ \sigma_u = \sigma \circ \varphi_u. \tag{7.6.2}
\]

Proof. For every \( i \in B_u \), the word \( u \) is a prefix of \( \sigma(u) \) and \( \varphi_u(i) \). Next

\[
\varphi_u(i)u \in \mathcal{L}(X) \Rightarrow \sigma(\varphi_u(i))u \in \mathcal{L}(X).
\]

hence \( \sigma(\varphi_u(i)) \) belongs to \( \mathcal{R}_X'(u)^+ \). We can therefore define a morphism \( \sigma_u : B_u \to B_u^+ \) by

\[
\varphi_u \circ \sigma_u = \sigma \circ \varphi_u.
\]

For all \( n \geq 1 \) we have \( \varphi_u \circ \sigma_u^n = \sigma^n \circ \varphi_u \). Let \( i, j \in B_u \). For \( n \) large enough, since \( y \)

is uniformly recurrent, the word \( \varphi_u(j)u \) appears in \( \sigma^n(\varphi_u(i)) \). By Lemma 7.3.3, this implies that \( j \) is a factor of \( \sigma_u^n(i) \). Thus \( \sigma_u \) is primitive. Moreover

\[
\varphi_u \circ \sigma_u(D_u(y)) = \sigma \circ \varphi_u(D_u(y)) = \sigma(y) = y = \varphi_u(D_u(y)). \tag{7.6.3}
\]

Since \( \varphi_u \) is injective by Proposition 2.4.23 it follows that \( \sigma_u(D_u(y)) = D_u(y) \). Thus \( D_u(y) \) is a fixed point of \( \sigma_u \).

The substitution \( \sigma_u : B_u^* \to B_u^* \) will be called a return substitution.

Example 7.6.5 If \( \sigma : A \to A^+ \) is the substitution defined by \( \sigma(a) = aba \) and \( \sigma(b) = aa \). Then \( \mathcal{R}(a) = \{a, ab\} \) and \( \varphi(a)(1) = a \) and \( \varphi(a)(2) = ab \). Then the return substitution \( \sigma_a \) is given by \( \sigma_a(1) = 21 \) and \( \sigma_a(2) = 2111 \).

Proposition 7.6.6 Let \( \phi : A^* \to B^* \) be a letter-to-letter morphism. Let \( y \in A^2 \), let \( x = \phi(y) \), and let \( u \) be a prefix of \( x^+ \). Then, there exists a prefix \( v \) of \( y^+ \) and a morphism \( \lambda_u : B_u^* \to B_u^* \) such that \( \varphi_u \circ \lambda_u = \phi \circ \varphi_v \) and \( \lambda_u(D_v(y)) = D_u(x) \).

Proof. Let \( v \) be the unique prefix of \( y^+ \) such that \( \phi(v) = u \). If \( w \) is a return word to \( v \) then \( \phi(w) \) is a concatenation of return words to \( u \). The morphism \( \varphi_u \) being one to one, we can define a morphism \( \lambda_u : B_u^* \to B_u^* \) by \( \varphi_u \circ \lambda_u = \phi \circ \varphi_v \). We have, as in Equation (7.6.3),

\[
\varphi_u \circ \lambda_u(D_v(y)) = \phi \circ \varphi_u(D_v(y)) = \phi(y) = x = \varphi_u(D_u(x)).
\]

This implies that \( \lambda_u(D_v(y)) = D_u(x) \).

Consequently to prove that (i) implies (iii) it suffices to prove that the sets \( \{\sigma_v \mid v \text{ prefix of } y^+\} \) and \( \{\lambda_u \mid u \text{ prefix of } x^+\} \) are finite.
Proposition 7.6.7 Let \( y \in B^\mathbb{Z} \) be a fixed point of a primitive substitution \( \sigma \), let \( \phi : B^* \to A^* \) be a letter-to-letter substitution and let \( x = \phi(y) \). The sets \( \{ \sigma_v \mid v \text{ prefix of } y^+ \} \) and \( \{ \lambda_u \mid u \text{ prefix of } x^+ \} \) are finite.

Proof. The periodic case is easy to check hence we suppose that \( y \) is non-periodic. By Proposition 7.3.4, the sequence \( y \) is linearly recurrent, say with constant \( K \). We start by proving that the set

\[
\{ \sigma_v : B_v^* \to B_v^* \mid v \text{ prefix of } y^+ \}
\]

is finite. For this, it suffices to prove that \( \text{Card}(B_v) \) and \( |\sigma_v(i)| \) are bounded independently of \( v \) and \( i \in B_v \). Set as usual \( |\sigma| = \sup\{|\sigma(a)| \mid a \in A\} \) and \( \langle \sigma \rangle = \inf\{|\sigma(a)| \mid a \in A\} \).

Let \( v \) be a non-empty prefix of \( y^+ \), let \( i \) be an element of \( B_v \) and let \( w = \phi_v(i) \) be a left return word to \( v \). Since \( y \) is not periodic, we have \( |w| \leq K|v| \) and thus \( |\sigma(w)| \leq |\sigma(K)|v| \). The length of each element of \( \mathcal{R}'(v) \) is larger than \( |v|/K \) by assertion 4 of Proposition 7.3.4. Now, since \( vw \in \mathcal{L}(y) \) and since \( vw \) begins with \( v \), we have that the word \( \sigma(w)v \) is in \( \mathcal{L}(y) \) and begins with \( v \). Thus we can write \( \sigma(w) = x_1x_2 \cdots x_k \) with \( k \geq 1 \) and \( x_i \in \mathcal{R}'(v) \) for \( 1 \leq i \leq k \). Since \( |x_i| \geq |v|/K \), we have

\[
|v| |\sigma(K)| \geq |\sigma(w)| \geq k|v|/K
\]

and we conclude that \( k \leq |\sigma(K)|^2 \). Therefore \( \sigma_v(i) = \sigma_{i_1i_2} \cdots i_i \) with \( x_j = \phi_v(i_j) \).

Thus we obtain finally that \( |\sigma_v(i)| \leq |\sigma(K)|^2 \). Moreover we know from assertion 5 of Proposition 7.3.4 that \( \text{Card}(B_v) \leq K(K + 1)^2 \). This ends the first part of the proof.

Let now \( u \) be a factor of \( x \) and \( v \) be a factor of \( y \) such that \( \phi(v) = u \). The length of a return word to \( u \) in \( x \) is bounded by the length of a return word to \( v \) in \( y \) and \( x \) is linearly recurrent with constant \( K \). Consequently, since \( x \) is non periodic, \( \langle \phi_u \rangle \geq |u|/K \) by Proposition 7.3.4 again. We have then for every prefix \( u \) of \( x^+ \),

\[
|\lambda_u(i)|(|1/K)||u| \leq |\lambda_u(i)||\phi_u| \leq |\phi_u(\lambda_u(i))| = |\phi(\phi_v(i))| = |\phi_v(i)| \leq |\phi_v| \leq K|v| = K|u|.
\]

Hence \( |\lambda_u(i)| \leq K^2 \). This completes the proof. \( \blacksquare \)

7.6.4 End of the proof

We now conclude the proof of the main result. Proof of Theorem 7.6.1. We have already proved that (iii) implies (i) (Proposition 7.6.3). We have also proved that (i) implies (iii) (Proposition 7.6.7).

Since (ii) clearly implies (iii), there only remains to prove that (i) implies (ii). We start with some notation. Let \( t \) be a word with prefix \( s \). By \( s^{-1}t \) we mean the word \( t \) such that \( t = sr \). In this way we have \( ss^{-1}t = t \).

Since the sequence \( x \) is primitive substitutive, there is a primitive substitution \( \sigma \), an admissible fixed point \( y \) of \( \sigma \) and a letter-to-letter morphism \( \phi \) such that \( x = \phi(y) \).
Since \( y \) is a fixed point of a primitive substitution, by Proposition 7.3.3 it is linearly recurrent, say with constant \( K \). It follows that \( x \) is linearly recurrent with the same constant \( K \).

Let \( u \) be a word of \( \mathcal{L}(x) \) and \( v \) be such that \( vu \) is a prefix of \( x \) and \( u \) has exactly one occurrence in \( vu \). Since \( v \) is a suffix of a word in \( \mathcal{R}'_X(u) \), we have \( |v| \leq K|u| \) and thus \( |vu| \leq (K + 1)|u| \).

If \( w \) is a left return word to \( vu \) then \( u \) is a prefix of \( v^{-1}wvu \) and \( v^{-1}wvu = (v^{-1}w)v \) is a word of \( \mathcal{L}(x) \). Hence \( v^{-1}wv \) is a concatenation of return words to \( u \). Thus, we can define \( \phi_{v,u} : \mathcal{B} \to \mathcal{B} \) by

\[
\varphi_u \circ \phi_{v,u}(i) = v^{-1}\varphi_{vu}(i)v,
\]

for all \( i \in B_{vu} \). We have

\[
\phi_{v,u}(\mathcal{D}_{vu}(x)) = \mathcal{D}_{u}(x).
\]

Indeed, by definition of \( v \), the integer \( i = |v| \) is the least integer \( i \geq 0 \) such that \( T^ix \) begins with \( u \). Thus, by definition of \( \mathcal{D}_{u}(x) \), we have

\[
v^{-1}x = \varphi_u(\mathcal{D}_{u}(x)).
\]

Now, by definition of the morphism \( \phi_{v,u} \), if \( \varphi_{vu}(y) = x \), we have using iteratively (7.6.4) on the prefixes of \( y \),

\[
\varphi_u \circ \phi_{v,u}(y) = v^{-1}\varphi_{vu}(y) = v^{-1}x.
\]

We conclude using Equations (7.6.5) and (7.6.7) and the equality \( y = \mathcal{D}_{vu}(x) \) that

\[
\varphi_u(\phi_{v,u}(y)) = \varphi_u(\mathcal{D}_{u}(x))
\]

whence (7.6.5) since \( \varphi_u \) is injective.

The set \( \{ \mathcal{D}_{u}(x); u = x_{[0,n]}, n \geq 0 \} \) being finite, it suffices to prove that the set

\[
H = \{ \phi_{v,u} : \mathcal{B} \to \mathcal{B}; vu = x_{[0,n]}, |vu| \leq (K + 1)|u|, n \geq 0 \}
\]

is finite to conclude. By Proposition 7.3.3 the length of every word in \( \mathcal{R}'_X(u) \) is at least \( |u|/K \). Thus for every \( i \in B_{vu} \), we have \( |\phi_{u}(\phi_{v,u}(i))| \geq |\phi_{v,u}(i)||u|/K \).

Therefore, or all \( i \in B_{vu} \) we have, using (7.6.4),

\[
|\phi_{v,u}(i)| \leq \frac{|\varphi_u(\phi_{v,u}(i))|}{|u|/K} = \frac{|\varphi_u(i)|}{|u|/K} \leq K^2.
\]

Moreover \( \text{Card}(B_s) \leq K(K + 1)^2 \) for all words \( s \in \mathcal{L}(x) \) hence \( H \) is finite.

**Example 7.6.8** Let \( \varphi : a \to ab, b \to a \) be the Fibonacci substitution and let \( x = \varphi^\omega(a) \) be the Fibonacci word. Let \( u_1, u_2, \ldots \) be the palindromic prefixes of \( x \). Since the directive word of \( x \) is \( (ab)^\omega \), we have by Equation (2.7.3) \( \varphi_{u_2n} = (L_{ab})^n \) and \( \varphi_{u_2n+1} = (L_{ab})^n L_a \). Let \( \overline{x} \) be the result of exchanging \( a, b \) in \( x \). We have \( x = L_a(\overline{x}) \) because \( x = \text{Pal}((ab)^\omega) = L_a(\text{Pal}(ba)^\omega)) = L_a(\overline{x}) \). Next \( x = L_{ab}(x) \)
by a similar computation (or also because \( L_{ab} = \varphi^2 \)). Thus \( D_{u_n}(x) = x \) and \( D_{u_{n+1}} = \bar{x} \).

If \( u \) is any factor of \( x \), we have \( R'_X(u) = v^{-1}R'_X(u_n)v \) where \( v \) is the shortest word such that \( vu \) is a prefix of some palindrome prefix \( u_n \). Consequently, \( D_u(x) = D_{u_n}(x) \). Thus the derivatives of \( x \) with respect to a factor of \( x \) are \( x \) and \( \bar{x} \).

### 7.6.5 A variant of the main result

We have considered in the first part of this section substitutive sequences. We now consider substitutive shifts.

First of all, we have the following result which shows that the class of minimal primitive substitutive shifts is closed under conjugacy.

**Theorem 7.6.9** A minimal shift is primitive substitutive if and only if it is conjugate to a primitive substitution shift.

**Proof.** Let \( \gamma : X \rightarrow Y \) be a conjugacy from a primitive substitution shift \( X = X(\sigma) \) where \( \sigma \) is primitive to a shift space \( Y \). Every conjugacy is a composition of a \( k \)-block code \( \gamma_k : X \rightarrow X(k) \) and a letter-to-letter morphism \( \phi : X(k) \rightarrow Y \). We have \( X(k) = X(\sigma_k) \) where \( \sigma_k \) is the \( k \)-block presentation of \( \sigma \), which is primitive by Proposition 2.4.34. Thus \( X(k) \) is a primitive substitution shift and consequently, \( Y \) is primitive substitutive.

Conversely, if \( Y \) is primitive substitutive, by Proposition 7.2.9, it is conjugate to a primitive substitution shift. ■

The following variant of Theorem 7.6.1 characterizes substitutive shifts. In the following statement, we consider two shifts as different if they cannot be identified by renaming the alphabet.

**Theorem 7.6.10** A minimal shift space \((X, S)\) is primitive substitutive if and only if there is a finite number of different derived systems on clopen sets \([u]\) for \( u \in L(X)\).

**Proof.** Consider \( X = X(\sigma, \phi) \) with \( \sigma : B^* \rightarrow B^* \) primitive and \( \phi : B^* \rightarrow A^* \) letter-to-letter. Let \( y \) be an admissible fixed point of \( \sigma \) and let \( x = \phi(y) \). Let \( uL(X) \), let \( U = [u] \) and let \((X_U, S_U)\) be the shift induced by \( X \) on \( U \). Since \( X \) is minimal, we have \( u \in L(x) \) and \( D_u(x) \in X_U \). Since \( X_U \) is also minimal, it is generated by \( D_u(x) \). By Theorem 7.6.1 since \( x \) is uniformly recurrent and primitive substitutive, there is a finite number of derived sequences \( D_u(x) \). Thus there is only a finite number of derived shifts \((X_U, T_U)\).

Conversely, suppose that there are a finite number of systems induced by \( X \) on clopen sets \( U = [u] \) for \( u \in L(X) \). If \( X \) is finite, then it is substitutive by Proposition 7.6.2. Otherwise, fix some \( x \in X \). Since \( X \) is minimal and infinite, \( x \) is aperiodic and uniformly recurrent. We argue as in the proof of Proposition 7.6.3. There exists an infinite sequence of prefixes \( u_n \) of \( x^+ \) with \( |u_n| < |u_{n+1}| \) such that \( X_{[u_n]} = X_{[u_{n+1}]} \). Set \( u = u_1 \). Since \( X \) is minimal, there
is an $N$ such that every word of $R'_X(\mu)u$ appears in every word of $L_N(X)$. Since $x$ is aperiodic, the words of $R'_X(\mu_n)$ cannot be of bounded length (the argument is the same as in the proof of Proposition 7.6.3) Thus there is some $w = u_l$ such that $|rw| > N$ for all $r \in R'_X(w)$. By hypothesis, we can identify $B_u$ and $B_w$ to an alphabet $B$ and set $Y = X_{[w]} = X_{[w]}$. Since $R'_X(w) \subset R'_X(u)$, we can define a morphism $\tau : B^* \rightarrow B^*$ such that $\varphi_u \circ \tau = \varphi_w$. By the choice of $u$ and $w$, the morphism $\tau$ is primitive. Then $\tau(Y) = Y$. Let $y \in B^*$ be an admissible fixed point of $\tau$ and let $z = \varphi_u(y)$. By Proposition 7.2.9 there is a primitive substitution $\zeta : C^* \rightarrow C^*$, an admissible fixed point $t$ of $\tau$ and a letter-to-letter morphism $\phi : C^* \rightarrow A^*$ such that $z = \phi(t)$. Thus $X$ is primitive substitutive.

Example 7.6.11 Consider again the Fibonacci shift $X = X(\varphi)$ as in Example 7.6.8. There are only two different shifts induced by $X$ on clopen sets $[u]$ for $u \in L(X)$, namely $X$ and its image $X$ by exchange of $a, b$.

7.7 Exercises

Section 7.1

7.1 Prove that the set $\mathbb{Z}(p_n)$ of $(p_n)$-adic integers is a compact topological group and actually a topological ring.

7.2 Show that the set of $x \in \mathbb{Z}(p_n)$ such that $x_n$ is eventually constant is a dense subgroup isomorphic to $\mathbb{Z}$.

7.3 Show that two odometers $(X, T)$ and $(X', T')$ are topologically conjugate if and only if the groups $X$ and $X'$ are isomorphic.

7.4 Let $(q_n)_{n \geq 1}$ be an increasing sequence of natural integers and let $Y = \{(y_n)_{n \geq 0} \mid 0 \leq y_n < q_{n+1}\}$ be the group of $(q_n)$-adic expansions. Set $p_1 = q_1$ and $p_{n+1} = p_nq_{n+1}$ for $n \geq 1$. Show that the map $\varphi : x \mapsto y$ defined by $y_0 = x_1$ and $y_n = (x_{n+1} - x_n)/p_n$ for $n \geq 1$ is homeomorphism from $\mathbb{Z}(p_n)$ onto $Y$.

7.5 Show that every integer $n$ has a unique factorial expansion

$$x = c_1 + c_22! + \ldots$$

with $0 \leq c_i \leq i$.

7.6 Show that $-1 = (\cdots 321)_i$.

7.7 Let $(G_n)_{n \geq 1}$ be a sequence of groups and let $\varphi_n : G_{n+1} \rightarrow G_n$ be a sequence of morphisms. Set $X = \prod_{n \geq 1} G_n$. The inverse limit of the sequence $(G_n, \varphi_n)$ is the set

$$X = \{(x_n)_{n \geq 1} \in X \mid \varphi_n(x_{n+1}) = x_n\}$$
Show that the group \( \mathbb{Z}_{(p_n)} \) is the inverse limit of the groups \( \mathbb{Z}/p_n\mathbb{Z} \) with \( \varphi_n(x_{n+1}) = x_{n+1} \mod p_n \).

**7.8** Show that the sequence of partitions \( \Psi(n) \) defined by Equation (7.1.1) is a refining sequence.

**7.9** A supernatural number is a formal product \( \prod_p p^{n_p} \) where for each prime number \( p \) we have \( n_p \in \mathbb{N} \cup \infty \). If \( (p_n)_{n \geq 1} \) is a sequence of integers with \( p_n | p_{n+1} \) for all \( n \geq 1 \) and \( (X, T) \) is the associated odometer, we associate to \( X \) the supernatural integer \( \sigma(X) = \prod_p p^{n_p} \) where \( p^{n_p} \) is the maximal power of \( p \) which divides some \( p_n \) (and thus all \( p_m \) for \( m \geq n \)). Show that two odometers \( (X, T) \) and \( (X', T') \) are topologically conjugate if and only if \( \sigma(X) = \sigma(X') \).

**7.10** A profinite group is an inverse limit of finite groups. Show that for every odometer \( (X, T) \), the group \( X \) is profinite.

**7.11** Let \( (X, T) \) be an invertible topological dynamical system which is expansive. Let \( \varepsilon \) be the expansivity constant. Let \( \gamma \) be a finite cover of \( X \) by sets \( C_1, C_2, \ldots, C_r \) of diameter at most \( \varepsilon \) (the diameter of a set \( C \), denoted \( \text{diam}(C) \), is the maximal distance of two points in the set).

Show that the diameter of the elements of the cover \( \bigvee_{j=-n}^{n} T^{-j} \gamma \) converges to 0 when \( j \to \infty \) (if \( \alpha, \beta \) are covers of \( X \), then \( \alpha \vee \beta \) is formed of the \( A \cap B \) for \( A \in \alpha \) and \( B \in \beta \) and \( T^{-j} \alpha \) by the \( T^{-j} A \) for \( A \in \alpha \)).

**7.12** Let \( (X, T) \) be an invertible topological dynamical system which is expansive with constant \( \varepsilon \). We aim to prove that \( (X, T) \) is conjugate to a shift space.

For this, let \( \{B_0, B_1, \ldots, B_{k-1}\} \) be a cover by open balls with radius \( \varepsilon \). Let \( C_0 = B_0 \) and for \( n \geq 1 \), let \( C_n = B_n \setminus (B_0 \cup \cdots \cup B_{n-1}) \).

1. Show that \( \gamma = \{C_0, \ldots, C_{k-1}\} \) is a closed cover of \( X \) with \( \text{diam}(C_i) < \varepsilon \) for each \( i \), \( C_i \cap C_j = \partial C_i \cap \partial C_j \) if \( i \neq j \) and \( \partial C_i \) having no interior (we denote \( \partial C = C \setminus \text{int}(C) \) where \( \text{int}(C) \) is the interior of \( C \)).

2. Set \( D = \bigcup_{i=0}^{k-1} \partial C_i \) and \( D_\infty = \bigcup_{n \in \mathbb{Z}} T^n D \). For \( x \in X \setminus D_\infty \), let \( y = \psi(x) \) be the sequence in \( A^\mathbb{Z} \) with \( A = \{0, \ldots, k-1\} \) defined by \( y_n = i \) if \( T^n(x) \in C_i \). Show that \( \psi \) extends to a conjugacy from \( X \) to a subshift of \( A^\mathbb{Z} \).

**Section 7.2**

**7.13** Suppose that \( (X, S) \) is a minimal substitution shift on the alphabet \( A \). Let \( x \in X \), \( \varphi : A^* \to B^* \) be a non erasing morphism and \( y = \varphi(x) \). Consider \( (Y, S) \) the subshift generated by \( y \). Prove that it is isomorphic to a primitive substitution shift.

**7.14** The Chacon ternary substitution is the primitive substitution \( \tau : 0 \to 0012, 1 \to 12, 2 \to 012 \). Show that \( w_n = \tau^n(0) \) satisfies the recurrence relation \( w_{n+1} = w_n w_n w_n \) where \( w_n \) is obtained from \( w_n \) by changing the initial letter
CHAPTER 7. SUBSTITUTION SHIFTS

0 into a 2. Deduce from this that the 1-block map \( \theta : 0,2 \to 0,1 \to 1 \) defines a conjugacy from the substitution shift \( X(\tau) \) defined by \( \tau \) to the substitution shift \( X(\sigma) \) defined by the Chacon binary substitution \( \sigma : 0 \to 0010, 1 \to 1 \).

7.15 Show that the factor complexity of the Chacon ternary shift is \( p_n(X) = 2n+1 \) (hint: show that the bispecial words are 0 and the words \( \alpha^n(012), \alpha^n(120) \) for \( n \geq 0 \) where \( \alpha(w) = 012 \tau(w) \) and where \( \tau \) is the ternary Chacon substitution).

7.16 Use Proposition 7.2.4 to prove Proposition 7.2.9. Develop the construction on the example of \( \tau : 0 \to 01, 1 \to 10 \) being the Thue-Morse morphism and \( \phi : 0 \to ab, b \to a \).

7.17 Let \( X \) be the Thue-Morse shift generated by \( \varphi : a \to ab, b \to ba \). Let \( f : \{aa, ab, ba, bb\} \to A_2 = \{x, y, z, t\} \) and let \( \varphi_2 : x \to yz, y \to yt, z \to zx, t \to zy \) be the 2-block presentation of \( \varphi \). Let \( B = \{u, v, w\} \) and let \( \phi : B^* \to A_2^* \) be a coding morphism for \( f(aK_X(a)) = f(\{aa, ab, abba\}) = \{x, yz, ytz\} \). Show that there is a morphism \( \tau \) such that \( \varphi_2 \circ \phi = \phi \circ \tau \) and derive a BV-representation of \( X \) with 3 vertices at each level \( n \geq 1 \).

Section 7.3

7.18 Show that if \((X, S)\) is LR with constant \( K \), then its \( k \)-th block presentation \((X^{(k)}, S)\) is LR with \( K \leq K(k-1) + 1 \). Conclude that the class of LR shifts is closed under conjugacy.

7.19 Consider the morphism \( \sigma : a \to abd, b \to bb, c \to c, d \to dc \). Show that the factor complexity \( p_n(X) \) of the shift \( X = X(\sigma) \) is not linear.

Section 7.4

7.20 Show that the following conditions are equivalent for a one-sided sequence \( x \).

(i) \( x \) is a limit point of a primitive \( S \)-adic system \( \tau \).

(ii) There is a primitive \( S \)-adic system \( \tau \) and a sequence \( (a_n) \) of letters \( a_n \in A_n \) such that \( x = \lim \tau(a_{[0,n])} \).

(iii) There is a primitive \( S \)-adic system \( \tau \) and a sequence \( (a_n) \) of letters \( a_n \in A_n \) such that \( \{x\} = \cap_n [\tau(a_{[0,n]}(a_n))] \) where \( [w] \) denotes the cylinder \( \{y \in A_0^N | y_{[0,|w|]} = w\} \).

7.21 Show that for every sequence \( x \in A_0^N \), there are morphisms \((\sigma_a)_{a \in A} : (A \cup \#)^* \to (A \cup \#)^* \) and \( \phi : (A \cup \#)^* \to A^* \) where \( \# \) is a letter not in \( A \) such that \( x = \lim \phi \circ \sigma_0 \circ \sigma_1 \cdots \sigma_n(\#) \) and thus that \((\varphi, \sigma_0, \sigma_1, \ldots)\) is an \( S \)-adic representation of \( x \).
7.22 Let \((X, S)\) be the shift generated by the substitution \(\tau : a \to aab, b \to ba\). Show that the dimension group of \((X, S)\) is isomorphic to 
\[(\mathbb{Z}^3, \{x \in \mathbb{Z}^3 : \langle x, v \rangle > 0\} \cup \{0\}, (1, 2, 1))\]
where \(v = [2\lambda, \lambda + 1, 1]\) with \(\lambda = (1 + \sqrt{5})/2\).

7.23 Let \(\varphi : A^* \to B^*\) be a nonerasing morphism. Set \(U = \varphi(A)\). For \(u \in B^*\), set 
\[M_\varphi(u) = \{v \in U^* \mid uv \in U^*u\}\]
The set \(M_\varphi(u)\) is for every \(u \in B^*\) a submonoid of \(U^*\).

Prove that a morphism \(\varphi : A^* \to B^*\) is recognizable in a subshift \(X\) of \(A^\mathbb{Z}\) for aperiodic points, then for every \(u \in \mathcal{L}(X)\) which is not in \(U^*\), the period of words in \(M_\varphi(u) \cap \mathcal{L}(X)\) is bounded.

Prove that the converse is true if \(\varphi(A)\) is a prefix code and \(\varphi\) is injective on \(A\).

7.24 A morphism \(\varphi : A^* \to B^*\) is left permutative if every word \(\varphi(a)\) for \(a \in A\) begins with a distinct letter. In particular, \(\varphi\) is injective on \(A\) and \(\varphi(A)\) is a prefix code.

Prove that if \(\varphi : A^* \to B^*\) is left permutative, then it is recognizable at aperiodic points. Hint: use Exercise 7.23

Section 7.6

7.25 Show that the sequence 001\(^\omega\) is substitutive but not purely substitutive.

7.26 Show that every shift of a substitutive sequence is substitutive. Hint: consider a \(k\)-th higher block presentation of the shift generated by the sequence.

7.27 Let \(\tau : B^* \to B^*\) be a morphism prolongable on \(a \in B\) and let \(y = \tau^\omega(a)\). Let \(\phi : B^* \to A^*\) be a morphism such that \(x = \phi(y)\) is an infinite word. Show that the sequence \(x\) is substitutive. Hint: adapt the proof of Proposition 7.2.9 to the case where \(\tau\) is not primitive and \(\phi\) is possibly erasing.

7.28 Let \(\sigma : \{0, 1, 2\}^* \to \{0, 1, 2\}^*\) be the substitution

\[0 \to 01222, 1 \to 10222, 2 \to \varepsilon\]

and let \(x = \sigma^\omega(0)\). Show that \(x\) is not the fixed point of a non-erasing substitution. Hint: show that erasing the letter 2 in \(x\) gives the Thue-Morse sequence.
7.8 Solutions

Section 7.1

7.1 Each \(\mathbb{Z}/p_n\mathbb{Z}\) is a compact topological group for the discrete topology. The direct product of compact topological groups is a compact topological group for the product topology by Tychonov theorem. Since \(\mathbb{Z}_{(p_n)}\) is a closed subgroup of the direct product of the \(\mathbb{Z}/p_n\mathbb{Z}\), the result follows. Each \(\mathbb{Z}/p_n\mathbb{Z}\) is also a finite ring and \(\mathbb{Z}_{(p_n)}\) is a subring of their direct product which is also a topological ring.

7.2 Let \(G\) be the set of \(x = (x_n) \in \mathbb{Z}_{(p_n)}\) such that \(x_n\) is eventually constant. The map \(\varphi: G \to \mathbb{Z}\) such that \(\varphi(x)\) is the value of all \(x_n\) for \(n\) large enough is an injective morphism from \(G\) into \(\mathbb{Z}\). Since the sequence \((p_n)\) is strictly increasing, it is onto. It is clear that \(G\) is dense in \(\mathbb{Z}_{(p_n)}\).

7.3 Assume first that \(\varphi: X \to X'\) is a topological conjugacy from \((X,T)\) onto \((X',T')\). Set \(\alpha(x) = \varphi(x) - \varphi(0)\). Then \(\alpha\) is another conjugacy from \((X,T)\) onto \((X',T')\). It satisfies \(\alpha(0) = 0\) and \(\alpha(1) = \alpha(T0) = T\alpha(0) = T0 = 1\). Since \(\mathbb{Z}\) is dense in \(X\) and \(X'\), it is an isomorphism. Conversely, assume that \(\varphi: X \to X'\) is a group isomorphism. Since \(\varphi(1)\) generates \(\mathbb{Z}\), we have \(\varphi(1) = 1\) or \(-1\). In the first case, \(\varphi\) is a conjugacy. In the second case, the map \(\psi(x) = -\varphi(x)\) is a conjugacy.

7.4 The inverse of \(\varphi\) is the map from \(X\) to \(\mathbb{Z}_{(p_n)}\) defined by

\[
x_n = y_0 + y_1p_1 + \ldots + y_{n-1}p_{n-1}.
\]

7.5 Set \(x_n = x \mod (n+1)!\) with \(x_0 = 0\). Then

\[
c_n = (x_n - x_{n-1})/n!
\]

with \(n \geq 1\) gives the unique factorial expansion of \(x\).

7.6 This holds because \(1 + 2.2! + \ldots + n.n! = (n+1)! - 1\), as one may verify by induction on \(n\).

7.7 The verification is easy.

7.8 Condition (KR1) is satisfied since \(\cap_{n \geq 1} B(n) = \{0\}\). Condition (KR2) is also satisfied because \(B(n+1) \subset B(n)\) and for every \(j \leq p_{n+1}\),

\[
B^j(n+1) = \cup_{k \equiv j \mod p_n} B^k B(n).
\]

Finally, since

\[
\cap_{n \geq 1} T^{x_n} B(n) = \{x\}
\]
7.8. SOLUTIONS

condition (KR3) is also satisfied.

**7.9** If \((X, T)\) and \((X', T')\) are conjugate, then \(\sigma(X) = \sigma(X')\). Indeed, for every prime \(p\), \(p^n\) divides some \(p_n\) if and only if the group \(X\) has elements of order \(p^n\).

Conversely, let \(\sigma = \prod p^n\) be a supernatural number. Consider the odometer \((Y, S)\) corresponding to the sequence \((q_1, q_2, \ldots)\) defined by

\[
q_n = \prod_{p^n \leq q} p^n \times \prod_{p^n \leq q'} p^n.
\]

By construction, \(\sigma(Y) = \sigma\).

Let now \((X, T)\) be an arbitrary odometer corresponding to \((p_1, p_2, \ldots)\) such that \(\sigma(X) = \sigma\). Note that for each \(x \in X\) and each integer \(q \geq 1\) the sequence \((x \mod q)_{n \geq 1}\) is nondecreasing and eventually equal to an integer denoted \(\max(x \mod q)\).

Consider the continuous map \(y \mapsto x\) from \(Y\) to \(X\) defined by

\[
x_n = \max(y \mod p_n).
\]

Its inverse is the map \(x \mapsto y\) from \(X\) to \(Y\) where \(y_n\) is the unique integer \(< q_n\) such that

\[
y_n = \max x \mod p^n
\]

for all \(p < n\) with \(m = n\) if \(n_p = \infty\) and \(m = n_p\) otherwise. The existence and uniqueness of \(y_n\) are guaranteed by the Chinese Remainder Theorem. This shows that \((X, T)\) and \((Y, S)\) are topologically conjugate. Thus all odometers such that \(\sigma(X) = \sigma(Y)\) are conjugate.

**7.10** We have seen in Exercise 7.7 that the odometer \((X, T)\) associated with the sequence \((p_1, p_2, \ldots)\) the inverse limit of the finite groups \(\mathbb{Z}/p_n\mathbb{Z}\).

**7.11** Arguing by contradiction, assume that there is an \(\varepsilon_0 > 0\), a strictly increasing sequence \(n_i\) and a sequence \(x_i, y_i\) of points such that \(d(x_i, y_i) \geq \varepsilon_0\) and \(x_i, y_i \in \bigcap_{j=-n_i}^{n} T^{-j}C_{i,j}\) where \(C_{i,j} \in \gamma\). Choose a subsequence \(i_k\) such that \(x_k \rightarrow x\) and \(y_k \rightarrow y\). Then \(d(x, y) \geq \varepsilon_0\). For every \(j \in \mathbb{Z}\), there is an \(\ell_j\) such that \(x_{i_k}, y_{i_k} \in T^{-j}C_{\ell_j}\) for an infinity of \(k\). Thus \(x, y \in T^{-j}C_{\ell_j}\) and therefore \(d(x, y) \leq \varepsilon\). This implies \(x = y\), a contradiction.

**7.12** 1. It is clear that \(\gamma\) is a closed cover of \(X\) by sets with radius at most \(\varepsilon\). Next, for \(i < j\), we have \(C_i \cap C_j = C_i \cap \partial C_j\) because \(\text{int}(C_j) = B_j \setminus \{B_{0} \cup \ldots \cup B_{j-1}\}\) and thus \(C_i \cap C_j = C_i \cap \partial C_j = \partial C_i \cap \partial C_j\) because \(\partial C_i \cap \text{int}(C_j) \subset B_i \setminus \{B_{0} \cup \ldots \cup B_{j-1}\} = \emptyset\).

By Baire Category Theorem, the set \(X \setminus D_\infty\) is dense. The map \(\psi : X \setminus D_\infty\) is injective. Indeed, suppose that \(\psi(x) = \psi(y)\). For every \(\alpha > 0\), by Exercise 7.11 we can choose \(N\) such that \(\text{diam}(\bigcup_{n \leq N} T^n\gamma) < \alpha\). Then \(\psi(x)_n = \psi(y)_n\) implies that \(x, y\) are in the same element of \(\bigcup_{n \leq N} T^n\gamma\) and thus \(d(x, y) < \alpha\). This shows that \(x = y\). The extension of \(\psi\) to \(X\) is then a conjugacy from \(X\) to the closure of \(\psi(X)\) which is a subshift of \(A^2\).
Section 7.2

7.13 Let $\sigma$ be a substitution such that $X = X(\sigma)$. Replacing if necessary $\sigma$ by one of its powers, let $x$ be an admissible fixed point of $\sigma$. Set $r = x_{-1}$ and $\ell = x_0$. Let $\psi : B \to R_X(r \cdot \ell)$ be a coding morphism and let $\tau : B^* \to B^*$ be the substitution such that $\psi \circ \tau = \sigma \circ \psi$. By Proposition 7.2.11 the morphism $\tau$ is primitive. Thus by Proposition 7.2.9 applied to the substitution $\tau$ and the morphism $\phi = \varphi \circ \psi$, the shift space $Y$ is conjugate to a primitive substitution shift $X(\zeta)$ (see Figure 7.8.1).

\[
\begin{array}{c}
X(\sigma) \xrightarrow{\psi} X(\tau) \xrightarrow{\gamma} X(\zeta) \\
\phi \quad \phi \\
Y \quad \theta
\end{array}
\]

Figure 7.8.1: The conjugacy from $X(\zeta)$ onto $Y$.

7.14 Set $w_n = 0t_n$ and thus $w'_n = 2t_n$ for $n \geq 0$. We have then

\[
w_{n+1} = \tau(w_n) = 0012\tau(t_n) = 0\tau(2t_n) = 0\tau(w'_n)
\]

showing that $t_{n+1} = \tau(w'_n)$ for $n \geq 0$. Thus

\[
w_{n+1} = \tau(w_{n-1}w_{n-1}1w'_{n-1}) = w_n w_n 12\tau(w'_{n-1}) = w_n w_n 12t_n = w_n w_n 1w'_n.
\]

The map $\theta$ sends the infinite word $\tau^w(0)$ to $\sigma^w(0)$ and thus maps $(X(\tau), S)$ to $(X(\sigma), S)$. Its inverse is the map which replaces 0 by 2 when there is a 1 just before. Thus $\theta$ is a conjugacy.

7.15 The bispecial words of length at most 3 are $\varepsilon$, 0, 012 and 120. Their extension graphs are shown in Figure 7.8.2.

Let now $\alpha$ be the map defined by $\alpha(x) = 012\tau(x)$. A bispecial word $y$ of length at least 4 begins and ends with 012 (see the trees of left and right special words in Figure 7.8.3).
Figure 7.8.3: The trees of left and right special words.

Thus \( y = 012\tau(x) = \alpha(x) \) where \( x \) is a bispecial word.

Let us verify that if the extension graph of \( x \) is the graph \( E(012) \) (see Figure 7.8.2), the same holds for the extension graph of \( y = \alpha(x) \). Indeed, since \( 0x0 \in L(X) \), the word \( \tau(0x0) = 0012\tau(x)0012 = 0y0012 \) is also in \( L(X) \) and thus \( (0, 0) \in E(y) \). Since \( 2x0 \in L(X) \) and since a letter 2 is always preceded by a letter 1, we have \( b cx a \in L(X) \). Thus \( \tau(12x0) = 12y0012 \in L(X) \) and thus \( (2, 0) \in E(y) \). The proof of the other cases is similar. The same property holds for a word \( x \) with the extension graph on the right of Figure 7.8.2.

We conclude that \( b_n(x) = 0 \) for every \( n \geq 0 \). Indeed, this is true for \( n = 0 \), since \( m(\varepsilon) = m(0) = 0 \) and for \( n = 3 \) since \( m(012) = 1 \), \( m(120) = -1 \). Let \( n \geq 4 \) be such that there are bispecial words of length \( n \). These bispecial words are of the form \( \alpha_k(012) \), \( \alpha_k(120) \) (note that these two words have the same length) and thus

\[
b_n = m(\alpha_k(012)) + m(\alpha_k(120)) = m(012) + m(120) = 1 - 1 = 0.
\]

**7.16** Let \( M \) be the composition matrix of \( \tau \) and let \( v \) be the vector \((|\phi(b)|)|_{b \in B}\). Let \( B = (V, E, \leq) \) be the stationary diagram with matrices \((v, M, M', \ldots)\). Let \( P, Q \) be the nonnegative matrices, with \( P \) a partition matrix, such that \( M = PQ \) defined by Proposition 7.2.14. Let \( M' = QP \) and \( w = [1 \ 1 \ \ldots \ 1]^t \). Let \( B' = (V', E', \leq') \) be the ordered Bratteli diagram with matrices \((w, M', M', \ldots)\) with the order induced from \((V, E, \leq)\).

Let \( C \) be the index of the columns of \( P \) identified with \( \{b_p \mid 1 \leq p \leq |\phi(b)|\} \). Let \( \zeta \) be the morphism read on \( B' \). Let \( \gamma : B \to C^* \) be the morphism defined by \( \gamma(b) = b_1 \cdots b_{|\phi(b)|} \). Let finally \( \theta : C \to A \) be defined by \( \theta(b_p) = \phi(b_p) \).

It is clear that \( \zeta \circ \gamma = \gamma \circ \tau \) since both are equal to the morphism read on \( PQP \) (with the order induced by \( B \)). The equality \( \phi = \theta \circ \gamma \) holds by definition of \( \gamma \) and \( \theta \). This implies assertion 1.

Assertion 2 is clear since \( \phi \) is injective from \( X(\tau) \) to \( X \).

Assume finally that \( \tau \) is eventually proper. We can also assume that \( X(\tau) \) is aperiodic since otherwise the result is trivial. By Proposition 7.2.14 the system \((X_E, T_E)\) is isomorphic to \((X(\tau), S)\). Since \( B \) is properly ordered, \( B' \) is also properly ordered and thus \( \zeta \) is eventually proper by Proposition 7.2.2.

Consider \( \tau : 0 \to 01, 1 \to 10 \) and \( \phi : 0 \to 01, 1 \to 0 \). The matrices
Figure 7.8.4: The diagrams $B$ and $B'$.

$M, P, Q, M'$ are

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad M' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The morphisms $\gamma, \zeta$ and $\theta$ are

$$\gamma : 0 \to ab, 1 \to c$$

$$\zeta : a \to ab, b \to c, c \to cab$$

$$\theta : a \to a, b \to b, c \to a$$

7.17

Set $\tau : u \to v, v \to wu, w \to wvu$. Then it is easy to verify that $\varphi_2 \circ \phi = \phi \circ \tau$ (note that the existence of such $\tau$ is guaranteed for every primitive morphism $\varphi$, see the proof of Proposition 5.6.3). The substitution $\tau$ is eventually proper and aperiodic and $\phi(B)$ is circular. Thus, by Proposition 7.2.14 a BV-representation of $X$ is shown in Figure 7.8.5.

Figure 7.8.5: A BV-representation of the Thue-Morse shift.
Section 7.3

7.18 Let \( f : \mathcal{L}_k(X) \to A_k \) be a bijection extended as usual to a map \( f : \mathcal{L}_{n+k-1}(X) \to \mathcal{L}_n(X^{(k)}) \). For every \( w \in \mathcal{L}_n(X^{(k)}) \), we have

\[
R_{X^{(k)}}(w) = f(R_X(u)) \quad \text{where} \quad w = f(u).
\]

Thus we have for every \( x \in R_X(u) \)

\[
|f(x)| = |x| - k + 1 \leq K|u| - k + 1 \leq K(n + k - 1) - k + 1 \leq K(k - 1) + 1.
\]

7.19 Let \( x = \sigma^w(a) \). We have \( x = a \prod_{i \geq 0} b^i dc^i \) as one may verify by computing \( \sigma(x) \). For \( 0 \leq i \leq j \leq (n - 2)/2 \), the word \( b^i dc^j bw \in \mathcal{L}(x) \) for some \( w \in \{b, c, d\}^{n-i-j+2} \). This shows that \( p_n(x) \) grows like \( n^2 \).

Section 7.4

7.20 For a primitive \( S \)-adic system, we have \( \langle \tau[0, n) \rangle \to \infty \). Thus (ii) and (iii) are equivalent with the same system \( \tau \).

(i) \( \Rightarrow \) (iii) Let \( a_n \) be the first letter of \( w^{(n)} \). Since \( \tau[0, n)(a_n) \) is a prefix of \( x \) with length which tends to \( \infty \), this proves (iii).

(iii) \( \Rightarrow \) (i) Set \( w^{(0)} = x \). Let \( y_n \) be in \( \tau[1, n)(a_n) \) for each \( n \geq 1 \). Let \( y_n \) be a subsequence of the \( y_i \) converging to \( y \in A_1^\infty \). We replace \( \tau \) by its telescoping with respect to the subsequence \( n_i \) and the sequence \( (a_n) \) by the sequence \( (a_n) \). Set \( w^{(1)} = y \). Then \( x = \tau_0(w^{(1)}) \) and \( \{w^{(1)}\} = \cap_{n \geq 1} \tau[1, n)(a_n) \). Continuing in this way, we build a sequence \( w^{(n)} \) with the required properties.

7.21 Define \( \sigma_a \) by

\[
\sigma_a(b) = \begin{cases} 
  a^\# & \text{if } b = a \\
  b & \text{otherwise}
\end{cases}
\]

7.22 The set \( \mathcal{L}_2(X) \) is \( \{aa, ab, ba, bb\} \) put in bijection with \( \{x, y, z, t\} \). The morphism \( \tau_2 \) is \( x \to xyz, y \to ytz, z \to zx, t \to zy \). The composition matrices \( M \) and \( M_2 \) are

\[
M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}
\]

The graph \( \Gamma_2(X) \) is the complete graph on two vertices and thus a choice for the matrix \( P \) with rows a basis of its cycles is

\[
P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}
\]
The corresponding matrix $N$ such that $PM_2 = NP$ is shown on the right. A left eigenvector for the maximal eigenvalue $\lambda^2 = (3 + \sqrt{5})/2$ is $[2 \lambda \quad \lambda + 1 \quad 1]$. We conclude by Proposition 7.6.6 that the dimension group has the form indicated (note that $N$ is the matrix of Example 3.5.3).

**7.23** Assume that $M_\varphi(u) \cap \mathcal{L}(X)$ contains words with arbitrary large period. Then there is an aperiodic point in $X$ of the form $\cdots v_{-1}v_0v_1 \cdots = \cdots w_{-1}w_0w_1 \cdots$ where $w_i = w_iu$, $v_i \in M_\varphi(u)$, $w_i \in U^*$ and $u \in \mathcal{L}(X) \setminus U^*$. Thus $\varphi$ is not recognizable in $X$ for aperiodic points.

Assume now that $\varphi(B)$ is a prefix code and that $\varphi$ is injective on $B$. This implies that if $\varphi(x) = \varphi(x')$, then $x^+ = x'^+$.

Suppose that $\varphi$ is not recognizable in $X$ at an aperiodic point $y = \varphi(x)$ and let $(x', k) \neq (x, 0)$ be such that $y = S^k \varphi(x')$ with $0 \leq k < |\varphi(x_0)|$. We cannot have $x = x'$ since $y$ is aperiodic. Thus $k \neq 0$. For an infinity of $n < 0$ there is an $m < 0$ and $k_n$ with $0 \leq k_n < |\varphi(x_n)|$ such that $\varphi(S^n x) = S^{k_n} \varphi(S^m x')$. By the preceding remark, we have $k_n \neq 0$. Let $u_n$ be the word of length $k_n$ such that $\varphi(S^n x) = u_n \varphi(S^m x')$. By shifting $y$ is necessary, we can assume that all $u_n$ are equal to $u$ (see Figure 7.8.6).

Then all words $v_m = \varphi(x_m \cdots x_m)$ are in $M_\varphi(u)$, which thus contains words of arbitrary large period (see Figure 7.8.6).

**7.24** Let $P$ be the set of prefixes of the words of $\overline{U} = \varphi(A)$. Since $\varphi$ is left permutative, there is for every $p \in P$ exactly one $a \in A$ such that $pa \in P \cup U$.

Assume that $M_\varphi(u)$ contains words arbitrary large period. Let $v, w \in M_\varphi(u)$ be two words with distinct periods. Replacing if necessary $v, w$ by some power, we may assume that none is a prefix of the other. Let $r$ be the longest common prefix of $v, w$ and let $p \in P$ be such that $r = xp$ with $x \in U^*$. Then $p$ has two right extensions by distinct letters, a contradiction. Thus $\varphi$ is recognizable at aperiodic points by Exercise 7.23.

**Section 7.6**

**7.25** Set $x = 001^\omega$. Let $\varphi: 0 \to 01, 1 \to 1$ and $\phi: 0 \to 00, 1 \to 1$. Then $\varphi^\omega(0) = 01^\omega$ and $\phi \circ \varphi^\omega(0) = x$. Thus $001^\omega$ is substitutive. It cannot be purely substitutive because if $\psi(x) = x$, we have $\psi(0) = 001^n$ with $n \geq 0$ and thus $\psi(x)$ begins with $001^n 001^n$. 


7.26 Let \( \varphi : A^* \to A^* \) be a substitution and let \( y \) be a fixed point of \( \varphi \). Let \( \phi : A^* \to B^* \) be a letter-to-letter morphism and let \( x = \phi(y) \). Let \( X = X(\varphi) \) be the shift generated by \( y \). For \( k \geq 1 \), let \( f : L_k(X) \to A_k \) be a bijection from \( L_k(X) \) onto an alphabet \( A_k \) and let \( \gamma_k : X \to A_k^* \) be the corresponding higher block code. The shift \( X^{(k)} \) is the substitution shift generated by the \( k \)-th block presentation \( \varphi_k \) of \( \varphi \) and \( z = \gamma_k(y) \) is a fixed point of \( \varphi_k \). Define \( \theta : A_k \to B \) by

\[
\theta(\gamma_k(0)) = (n+1) \mod k.
\]

Then \( \theta(z) = \phi(T^{k-1}x) \) which shows that the sequence \( T^{k-1}x \) is substitutive.

7.27 We first prove that we can modify the pair \((\tau, \phi)\) in such a way that

\[
|\phi \circ \tau(b)| \geq \phi(b) \tag{7.8.1}
\]

for every \( b \in B \) and with strict inequality when \( b = a \).

Since \( \lim |\tau^n(a)| = \infty \), there are \( 1 \leq j < k \) such that

\[
|\phi \circ \tau^j(b)| \leq |\phi \circ \tau^k(b)|
\]

for every \( b \in B \) with strict inequality if \( b = a \). Set \( \tau' = \tau^{k-j} \), \( \phi' = \phi \circ \tau^j \). Then

\[
|\phi' \circ \tau'(b)| = |\phi \circ \tau^j \circ \tau k - j(b)| = |\phi \circ \tau^k(b)|
\]

\[
\geq |\phi \circ \tau^j(b)| = |\phi'(b)|
\]

for every \( b \in B \) with strict inequality when \( b = a \).

We now assume that \((\tau, \phi)\) satisfies Equation (7.8.1). Proceeding as in the proof of Proposition 7.2.9 we define an alphabet \( C = \{b_p \mid b \in B, 1 \leq p \leq \phi(b)\} \), a map \( \theta : C \to A \) by \( \theta(b_p) = (\phi(b))_p \) and a map \( \gamma : B \to C^+ \) by \( \gamma(b) = b_1 b_2 \cdots b_{\phi(b)} \). In this way, we have \( \theta \circ \gamma = \phi \).

Finally, we define the substitution \( \zeta \) essentially as in the proof of Proposition 7.2.9 (with the difference that this time the inequality \( |\tau(b)| \geq |\phi(b)| \) is replaced by the weaker inequality (7.8.1)). For every \( b \in B \), we have

\[
|\gamma \circ \tau(b)| = |\phi \circ \tau(b)| \geq |\phi(b)|
\]

by (7.8.1). Thus, we can define words \( w_1, w_2, \ldots, w_{\phi(b)} \in C^* \) such that

\[
\gamma \circ \tau(b) = w_1 w_2 \cdots w_{\phi(b)}
\]

with \( |w_1| > 1 \) when \( b = a \). Then we define the morphism \( \zeta : C^* \to C^* \) by

\[
\zeta(b_p) = w_p
\]

We then have by construction \( \zeta \circ \gamma = \gamma \circ \tau \) and thus \( x = \theta \circ \zeta^\omega(a_1) \).

7.28 Let \( \chi : 0 \to 0, 1 \to 1, 2 \to \varepsilon \) be the morphism erasing 0. Let \( \mu : 0 \to 01, 1 \to 10 \) be the Thue-Morse morphism and let \( t = \mu^\omega(0) \). Since \( \mu \circ \chi = \chi \circ \sigma \), we have \( t = \chi(x) \).
Let $\tau$ be a non-erasing substitution such that $x = \tau^n(0)$. Then $\chi(\tau(2)^3) = \chi(\tau(2))^3$ is a factor of $t$ which is a cube and thus $\chi(\tau(2)) = \varepsilon$. Since the factors of $x$ in $2^*$ are $\varepsilon, 2, 22, 222$, this forces $\tau(2) = 2$. Now $\tau(0)$ is a prefix of $x$ and thus $\tau(0) = 01u$ for some $u \in \{0,1,2\}^*$. Next, since $\tau(1222) = \tau(1)222$ cannot end with $2^4$, we have $\tau(1) = ya$ with $y \in \{0,1,2\}^*$ and $a \in \{0,1\}$. Finally, $\tau(10) = ya01u$ has a factor of length 3 in $\{0,1\}^*$ while the factors of $x$ in $\{0,1\}^*$ are of length at most 2, a contradiction.

### 7.9 Notes

Odometers, also called solenoids [Katok and Hasselblatt, 1995] or adding machines [Brown, 1976], are a classical object in dynamical systems theory.

The ring $\mathbb{Z}_p$ of $p$-adic integers is contained in a field, called the field of $p$-adic numbers. A classical reference to $p$-adic numbers and $p$-adic analysis is [Koblitz, 1984]. The factorial representation of integers (Exercise 7.5) is described in [Knuth, 1998]. Supernatural numbers (Exercise 7.9), also called generalized natural numbers or Steinitz numbers are used to define orders of profinite groups. The fact that they give a complete invariant for odometers (Exercise 7.9) is a result proved by [Glimm, 1960] in the context of uniformly hyperfinite algebras of UHF-algebras (see Chapter 10).

The notion of expansive system is classical in topological dynamics (see [Katok and Hasselblatt, 1995] for example). Exercises 7.11 and 7.12 is from [Walters, 1982, Theorem 5.24] (see also [Kurka, 2003]).

The equal path number property and Theorem 7.1.5 are from [Gjerd and Johansen, 2000].

#### 7.9.1 Substitution shifts

In [Vershik and Livshits, 1992] the authors showed that when $\sigma$ is a primitive substitution then the subshift it generates can be represented (in a measure-theoretic sense) by an ordered Bratteli diagram $B$ where $\sigma$ is the substitution we read on $B$.

Theorem 7.2.1 was first proven in [Forrest, 1997]. The proofs given in that paper are mostly of existential nature and do not state a method to compute effectively the BV-representation associated with substitution systems. Another proof was given in [Durand et al., 1999] that provides such an algorithm. Proposition 7.2.3 is from [Durand et al., 1999].

Proposition 7.2.4 is [Forrest, 1997, lemma 15]. As mentioned in this paper, the proof uses an important technique called state-splitting or symbol splitting. We actually use in the proof of Proposition 7.2.4 an output split. See [Lind and Marcus, 1995] for a systematic presentation of state splitting.

Proposition 7.2.5 is also from [Durand et al., 1999]. It is a modification of an unpublished result of Rauzy.

For more details about the Chacon substitution, see [Ferenczi, 1995] or [Fogg, 2002].
7.9. NOTES

7.9.2 Linearly recurrent shifts

Linearly recurrent shifts, also called linearly repetitive shifts were introduced in Durand et al. (1999).

Theorem 7.3.2 is from Damanik and Lenz (2006). The fact that the equivalent conditions of Theorem 7.3.2 are also equivalent to unique ergodicity is from Durand (2000a).

Proposition 7.3.4 is proved in Durand et al. (1999). The corollary asserting that the factor complexity of a primitive substitution shift is at most linear can be found in Michel (1976) (see also Pansiot (1984)). Exercise 7.4.4 is from Allouche and Shallit (2003; Example 10.4.1). The factor complexity of substitutional shifts has been extensively studied. By a result of Ehrenfeucht et al. (1975), one always has \( p_n(X) = O(n^2) \). See Allouche and Shallit (2003) for a survey of this question.

Theorem 7.3.6 is from Durand (2003). Theorem 7.3.7 is from Downarowicz and Maass (2008).

The notion of equicontinuity is classical in analysis. The condition defining an equicontinuous dynamical system is equivalent to the uniform continuity of the family \( T^n \) of maps from \( X \) to itself. Theorem 7.3.7 is proved in Downarowicz and Maass (2008).

7.9.3 \( S \)-adic shifts

The notion of \( S \)-adic shift was introduced in Ferenczi (1996), using a terminology initiated by Vershik and coined out by Bernard Host. For more information, see Fogg (2002), Berthé et al. (2019b), Berthé and Delecroix (2014). See also Thuswaldner (2020) for a recent survey on \( S \)-adic systems.

Proposition 7.4.4 is from Durand (2000b, Lemma 7). Lemma 7.4.8 is a weaker version of Durand and Leroy (2012, Corollary 2.3). Theorem 7.4.12 is Berthé et al. (2019a, Theorem 3.1). Proposition 7.4.3 is from Arnoux et al. (2014).

The original reference for the Theorem of Fine-Wilf (Exercise 2.11) is Fine and Wilf (1965).

Proposition 7.4.11 is Berstel et al. (2009, Exercise 5.1.5) where it is proved as a variant of a result called the Defect Theorem (see Lothaire (1997)).

Theorem 7.4.12 is from Berthé et al. (2019b, Theorem 5.1). Exercise 7.24 is from Berthé et al. (2019b, Lemma 3.3).

Theorem 7.5.1 is from Berthé et al. (2020).

Corollary 7.5.2 extends a statement initially proved for interval exchanges by Ferenczi and Zamboni (2008). In Corollaries 7.5.2 and 7.5.3 the assumption of being proper can be dropped. The proof then uses the measure-theoretical Bratteli-Vershik representation of primitive unimodular \( S \)-adic subshift given in Berthé et al. (2019b, Theorem 6.5).

Note that one recovers, with the description of dimension groups of primitive unimodular proper \( S \)-adic shifts of Theorem 7.5.4, the results obtained

7.9.4 Derivatives of substitutive sequences

Substitutive sequences have been considered early by Cobham (1968) who has proved the statement that every infinite sequence \( x = \phi(\tau^\omega(a)) \) is substitutive, whatever be the morphisms \( \phi : B^* \to A^* \) and \( \tau : B^* \to B^* \) (Exercise 7.27). This result was proved independently by Pansiot (1983) (see the presentation in Allouche and Shallit (2003)). We follow the proof given in Cassaigne and Nicolas (2003). The effective computability of the representation as a substitutive sequence was proved by Honkala (2009) and Durand (2013).

Theorem 7.6.1 is from Durand (1998). It is closely related with the results of Holton and Zamboni (1999) who proved independently that conditions (i) and (ii) are equivalent. Theorem 7.6.10 is from (Holton and Zamboni, 1999, Theorem 1.3)
Chapter 8

Dendric shifts

In this chapter, we define the important class of dendric shifts, which are defined by a condition on the possible extensions of a word in their language. We prove a striking property of the sets of return words, namely that for every minimal dendric shift, the set of return words forms a basis of the free group (Theorem 8.1.14). We show that they have a finite $S$-adic representation (Theorem 8.1.40). We illustrate these results on the class of Sturmian shifts (Section 8.2) which are a particular case of interval exchange shifts considered in the next chapter. We next present the class of specular shifts (Section 8.3) which is build to generalize the class of linear involutions, itself a natural generalization of interval exchanges, and also presented in the next chapter.

8.1 Dendric shifts

Let $X$ be a shift space on the alphabet $A$. We will assume in this chapter that $A \subset L(X)$ and $n \geq 1$, we denote

\[ L_X(w) = \{ a \in A \mid aw \in L(X) \} \]
\[ R_X(w) = \{ b \in A \mid wb \in L(X) \} \]
\[ E_X(w) = \{ (a, b) \in L_X(w) \times R_X(w) \mid awb \in L(X) \} \]

The extension graph of $w$, denoted $E_X(w)$, is the undirected bipartite graph whose set of vertices is the disjoint union of $L_X(w)$ and $R_X(w)$ and whose edges are the elements of $E_X(w)$.

When the context is clear, we denote $L(w), R(w), E(w)$ and $E(w)$ instead of $L_X(w), R_X(w), E_X(w)$ and $E_X(w)$.

When in need to distinguish the disjoint copies of $L(w)$ and $R(w)$ forming the vertices of the extension graph $E(w)$, we denote them by $1 \otimes L(w)$ and $R(w) \otimes 1$. A path in an undirected graph is reduced if it does not contain successive equal edges. For any $w \in L(X)$, since any vertex of $L_X(w)$ is connected to at least one vertex of $R_X(w)$, the bipartite graph $E_X(w)$ is a tree if and only if there is a unique reduced path between every pair of vertices of $L_X(w)$ (resp. $R_X(w)$).
CHAPTER 8. DENDRIC SHIFTS

The shift $X$ is said to be eventually dendric with threshold $m \geq 0$ if $E_X(w)$ is a tree for every word $w \in \mathcal{L}_{\geq m}(X)$. It is said to be dendric if we can choose $m = 0$. Thus, a shift $X$ is dendric if and only if $E_X(w)$ is a tree for every word $w \in \mathcal{L}(X)$.

When $X$ is a dendric shift (resp. eventually dendric shift), we also say that $\mathcal{L}(X)$ is a dendric set (resp. eventually dendric set).

An important observation is that, in any shift space, for a word $w \in \mathcal{L}(X)$ which is not bispecial, the graph $E(w)$ is always a tree. Indeed, if $w$ is not left-special, all vertices of $R(w)$ are connected to the unique vertex of $L(w)$ and thus $E(w)$ is a tree. Also, if $w$ is bispecial and such that $E(w)$ is a tree, then $w$ is neutral. Indeed, in a tree $m(w) = e(w) - \ell(w) - r(w) + 1 = 0$ by a well known property of trees. We begin with an example of a non minimal dendric shift.

**Example 8.1.1** Let $X$ be the shift space such that $\mathcal{L}(X) = a^*ba^*$ (we denote $a^* = \{a^n \mid n \geq 0\}$). The bispecial words are the words in $a^*$. Their extension graph is the tree represented in Figure 8.1.1. Thus $X$ is a dendric shift.

![Figure 8.1.1: The graph $E(a)$.](image)

An important example of minimal dendric shifts is formed by strict episturmian shifts (also called Arnoux-Rauzy shifts).

**Proposition 8.1.2** Every Arnoux-Rauzy shift is dendric. In particular, every Sturmian shift is dendric.

**Proof.** Let $X$ be an Arnoux-Rauzy shift. For every bispecial word $w \in \mathcal{L}(X)$, there is exactly one letter $\ell$ such that $\ell w$ is right-special and one letter $r$ such that $wr$ is left-special. Thus the extension graph $E(w)$ has exactly two vertices $\ell, r$ which have degree more than one, with $\ell \in L(w)$ and $r \in R(w)$. Any vertex distinct of $\ell, r$ is connected to either $\ell$ or $r$ by an edge but not to both and $\ell, r$ are connected by an edge. Thus $E(w)$ is a tree. ■

**Example 8.1.3** Let $X$ be the Fibonacci shift, which is generated by the morphism $a \mapsto ab, b \mapsto a$. It is a Sturmian shift (Example 2.5.1). The graph $E(a)$ is shown in Figure 8.1.2.

![Figure 8.1.2: The graph $E(a)$.](image)
A shift space $X$ is said to be a *eventually dendric of characteristic* $c$ if

1. for any $w \in \mathcal{L}_{\geq 1}(X)$, the extension graph $\mathcal{E}(w)$ is a tree and
2. the graph $\mathcal{E}(\varepsilon)$ is a disjoint union of $c$ trees.

Eventually dendric shifts of characteristic $c \geq 1$ are eventually dendric. Indeed, since the extension graphs of all nonempty words are trees, the shift space is eventually dendric with threshold 1. It is a dendric shift if $c = 1$.

**Example 8.1.4** Let $X$ be the shift generated by the morphism $a \mapsto ab, b \mapsto cda, c \mapsto cd, d \mapsto abc$. It is dendric of characteristic 2 (Exercise 8.1). The extension graph $\mathcal{E}(\varepsilon)$ is shown in Figure 8.1.3.

![Figure 8.1.3: The extension graph $\mathcal{E}(\varepsilon)$.](image)

**Example 8.1.5** Let $X$ be the *Tribonacci shift*, which is the substitution shift generated by the *Tribonacci substitution* $\sigma: a \mapsto ab, b \mapsto ac, c \mapsto a$. It is an Arnoux-Rauzy shift (see Example 2.5.2) and thus a dendric shift.

The following statement shows that eventually dendric shifts have at most linear complexity. Recall from Chapter 2 that we denote $p_n(X) = \text{Card}(\mathcal{L}_n(X))$ and that $s_n(X) = p_{n+1}(X) - p_n(X)$.

**Proposition 8.1.6** Let $X$ be an eventually dendric shift on the alphabet $A$. Then $X$ has at most linear complexity, that is, $p_n(X) \leq Kn$ for some constant $K$. If $X$ is dendric, then

$$p_n(X) = (\text{Card}(A) - 1)n + 1 \quad (8.1.1)$$

**Proof.** Let $b_n(X) = s_{n+1}(X) - s_n(X)$. Since $X$ is eventually dendric, there is $n \geq 1$ such that the extension graph of every word in $\mathcal{L}_{\geq n}(X)$ is a tree. Then $b_p(X) = 0$ for every $p \geq n$. Indeed, by Proposition 2.2.15 we have $b_p(X) = \sum_{w \in \mathcal{L}_p(X)} m(w)$. Since all words of length $p$ in $\mathcal{L}(X)$ are neutral, the conclusion follows. Thus $s_p(X) = s_{p+1}(X)$ for every $p \geq n$, whence our conclusion.

If $X$ is dendric, then $b_n(X) = 0$ for all $n \geq 1$ and thus $s_n(X)$ is constant. Since $s_1(X) = \text{Card}(A) - 1$ by the assumption $A \subseteq \mathcal{L}(X)$, this implies $p_n(X) = (\text{Card}(A) - 1)n + 1$. \[\blacksquare\]
Corollary 8.1.7 The dendric shifts on two letters are the Sturmian shifts.

Proof. We have already seen that a Sturmian shift is dendric (Proposition 8.1.2). Conversely, if X is a dendric shift on two letters, its factor complexity is \( p_n(X) = n + 1 \) by Proposition 8.1.6 and thus X is Sturmian.

On more than two letters, the class of dendric shifts is larger than the class of Arnoux-Rauzy shifts since it contains, as we shall see in Chapter 9, the class of interval exchange shifts.

The converse of Proposition 8.1.6 is not true, as shown by the following example.

Example 8.1.8 The Chacon ternary shift is the substitution shift \( X \) on the alphabet \( A = \{0, 1, 2\} \) generated by the Chacon ternary substitution \( \tau : 0 \to 0012, 1 \to 12, 2 \to 012 \). Its complexity is \( p_n(X) = 2n + 1 \) (see Exercise 7.15). It is not eventually dendric (Exercise 8.2).

8.1.1 Generalized extension graphs

We will need to consider extension graphs which correspond to extensions by words instead of letters. Let \( X \) be a shift space. For \( w \in \mathcal{L}(X) \), and \( U, V \subset \mathcal{L}(X) \), let \( L_U(w) = \{ \ell \in U \mid \ell w \in \mathcal{L}(X) \} \), let \( R_V(w) = \{ r \in V \mid wr \in \mathcal{L}(X) \} \) and let \( E_{U,V}(w) = \{ (\ell, r) \in U \times V \mid \ell wr \in \mathcal{L}(X) \} \). The generalized extension graph of \( w \) relative to \( U, V \) is the following undirected graph \( E_{U,V}(w) \). The set of vertices is made of two disjoint copies of \( L_U(w) \) and \( R_V(w) \). The edges are the elements of \( E_{U,V}(w) \). The extension graph \( E(w) \) defined previously corresponds to the case where \( U, V = A \).

Example 8.1.9 Let \( X \) be the Fibonacci shift. Let \( w = a, U = \{ aa, ba, b \} \) and let \( V = \{ aa, ab, b \} \). The graph \( E_{U,V}(w) \) is represented in Figure 8.1.4.

![Figure 8.1.4: The graph \( E_{U,V}(w) \).](image)

The following property shows that in a dendric shift, not only the extension graphs but, under appropriate hypotheses, all generalized extension graphs are acyclic.

Proposition 8.1.10 Let \( X \) be a shift space and \( n \geq 1 \) be such that for every \( w \in \mathcal{L}_{\geq n}(X) \), the graph \( E(w) \) is acyclic. Then, for any \( w \in \mathcal{L}_{\geq n}(X) \), any finite suffix code \( U \) and any finite prefix code \( V \), the generalized extension graph \( E_{U,V}(w) \) is acyclic.
The proof uses the following lemma.

**Lemma 8.1.11** Let \( X \) be a shift space. Let \( w \in \mathcal{L}(X) \) and let \( U, V, T \subset \mathcal{L}(X) \). Let \( \ell \in \mathcal{L}(X) \backslash U \) be such that \( \ell w \in \mathcal{L}(X) \). Set \( U' = (U \setminus T\ell) \cup \ell \). If the graphs \( E_{U',V}(w) \) and \( E_{T,V}(\ell w) \) are acyclic then \( E_{U,V}(w) \) is acyclic.

**Proof.** Assume that \( E_{U,V}(w) \) contains a cycle \( C \). If the cycle does not use any of the vertices in \( U' \), it defines a cycle in the graph \( E_{T,V}(\ell w) \) obtained by replacing each vertex \( t\ell \) for \( t \in T \) by a vertex \( t \). Since \( E_{T,V}(\ell w) \) is acyclic, this is impossible. If it uses a vertex of \( U' \) it defines a cycle of the graph \( E_{U',V}(w) \) obtained by replacing each possible vertex \( t\ell \) by \( \ell \) (and suppressing the possible identical successive edges created by the identification). This is impossible since \( E_{U,V}(w) \) is acyclic. Thus \( E_{U,V}(w) \) is acyclic.

**Proof of Proposition 8.1.10** We show by induction on the sum of the lengths of the words in \( U, V \) that for any \( w \in \mathcal{L}_{\geq n}(X) \), the graph \( E_{U,V}(w) \) is acyclic.

Let \( w \in \mathcal{L}_{\geq n}(X) \). We may assume that \( U = L_U(w) \) and \( V = R_V(w) \) and also that \( U, V \neq \emptyset \). If \( U, V \subset A \), the property is true.

Otherwise, assume for example that \( U \) contains words of length at least 2. Let \( u \in U \) be of maximal length. Set \( u = a\ell \) with \( a \in A \). Let \( T = \{ b \in A \mid b\ell \in U \} \). Then \( U' = (U \setminus T\ell) \cup \ell \) is a suffix code and \( \ell w \in \mathcal{L}(X) \) since \( U = U(w) \).

By induction hypothesis, the graphs \( E_{U',V}(w) \) and \( E_{T,V}(\ell w) \) are acyclic. By lemma 8.1.11 the graph \( E_{U,V}(w) \) is acyclic.

We prove now a similar statement concerning connectivity. For \( w \in \mathcal{L}(X) \), we say that a suffix code \( U \subset \mathcal{L}(X) \) is \( (X,w) \)-maximal if every word \( v \) such that \( vw \in \mathcal{L}(X) \) is comparable with a word in \( U \) for the suffix order. For \( w = \varepsilon \), we say \( X \)-maximal instead of \( (X,\varepsilon) \)-maximal. Thus a suffix code is \( X \)-maximal if it is not strictly contained in any suffix code \( U' \subset \mathcal{L}(X) \). The same definitions hold symmetrically for prefix codes. Thus a prefix code \( V \subset \mathcal{L}(X) \) is \( (X,w) \)-maximal if every word \( v \) such that \( wv \in \mathcal{L}(X) \) is comparable for the prefix order with a word of \( V \).

For example, when \( X \) is recurrent, the set \( \mathcal{R}_X(w) \) is an \( (X,w) \)-maximal prefix code and \( \mathcal{R}_X'(w) \) is an \( (X,w) \)-maximal suffix code.

**Proposition 8.1.12** Let \( X \) be an eventually dendric shift with threshold \( n \). For any \( w \in \mathcal{L}_{\geq n}(X) \), any finite \( (X,w) \)-maximal suffix code \( U \subset \mathcal{L}(X) \) and any finite \( (X,w) \)-maximal prefix code \( V \subset \mathcal{L}(X) \), the generalized extension graph \( E_{U,V}(w) \) is a tree.

For a shift space \( X \), two finite sets \( U, V \subset \mathcal{L}(X) \) and \( w \in \mathcal{L}(X) \), denote \( \ell_U(w) = \text{Card}(L_U(w)) \), \( r_V(w) = \text{Card}(R_V(w)) \) and \( e_{U,V}(w) = \text{Card}(E_{U,V}(w)) \). Next, we define

\[
m_{U,V}(w) = e_{U,V}(w) - \ell_{U,V}(w) - r_{U,V}(w) + 1.
\]

Thus, for \( U = V = A \), the integer \( m_{U,V}(w) \) is the multiplicity \( m(w) \) of \( w \).
Lemma 8.1.13 Let $X$ be a shift space and $n \geq 0$ be such that $m(w) = 0$ for every $w \in \mathcal{L}_{\geq n+1}(X)$. Then for every $w \in \mathcal{L}_{\geq n}(X)$, every finite $(X, w)$-maximal suffix code $U$ and every finite $(X, w)$-maximal prefix code $V$, we have $m_{U,V}(w) = m(w)$.

Proof. We use an induction on the sum of the lengths of the words in $U$ and in $V$. We may assume that $Uw, wV \subset \mathcal{L}(X)$.

If $U, V$ contain only words of length 1, since $U$ (resp. $V$) is an $(X, w)$-maximal suffix (resp. prefix) code, we have $U = L(w)$ and $V = R(w)$ and there is nothing to prove. Assume next that one of them, say $V$, contains words of length at least 2. Let $p$ be a nonempty proper prefix of $V$. Set $V' = (V \setminus pA) \cup \{p\}$. If $wp \not\in \mathcal{L}(X)$, then $m_{U,V}(w) = m_{U,V'}(w)$ and the conclusion follows by induction hypothesis. Thus we may assume that $wp \in \mathcal{L}(X)$. Then

$$m_{U,V'}(w) - m_{U,V}(w) = e_{U,A}(wp) - \ell_U(wp) - r_A(wp) + 1 = m_{U,A}(wp).$$

By induction hypothesis, we have $m_{U,V}(w) = m(w)$. But $m_{U,A}(wp) = 0$ since $|wp| \geq n + 1$, whence the conclusion.

Proof of Proposition 8.1.12. Let $w \in \mathcal{L}_{\geq n}(X)$. By Proposition 8.1.10, the graph $\mathcal{E}_{U,V}(w)$ is acyclic. Since, by Lemma 8.1.13, we have $m_{U,V}(w) = 0$, it follows that $\mathcal{E}_{U,V}(w)$ is a tree.

8.1.2 Return Theorem

We will now prove the following result (called the Return Theorem). Recall that we assume in this chapter that $A \subset \mathcal{L}(X)$ for a shift space $X$ on the alphabet $A$.

Theorem 8.1.14 Let $X$ be a minimal dendric shift on the alphabet $A$. For every $u \in \mathcal{L}(X)$, the set $\mathcal{R}_X(u)$ is a basis of the free group on $A$.

Note that that, in the particular case of an episturmian shift $X$, the property results directly from Equation (2.7.3). Indeed, the set of return words $\mathcal{R}_X(u)$ is for every $u \in \mathcal{L}(X)$ conjugate to a set of the form $\alpha(A)$ where $\alpha$ is an automorphism of the free group on $A$.

A shift space $X$ is neutral if every word $u$ in $\mathcal{L}(X)$ is neutral. A dendric shift is of course neutral but the converse is false (see Exercise 8.3). The first step of the proof is the following statement. It shows in particular that, under the hypotheses below, the cardinality of sets of return words in constant. We had already met this property in the case of strict episturmian shifts (see Equation (2.7.3)).

Theorem 8.1.15 If $X$ is a recurrent neutral shift such that $A \subset \mathcal{L}(X)$, then for every $u \in \mathcal{L}(X)$, one has $\text{Card}(\mathcal{R}_X(u)) = \text{Card}(A)$.
Note the surprising consequence that every recurrent neutral shift is minimal. Indeed, if \( X \) is recurrent, all sets of return words are finite by Theorem 8.1.15. Thus the shift is minimal. Thus, we could weaken the hypothesis in Theorem 8.1.14 to require \( X \) to be only recurrent.

The proof of Theorem 8.1.15 uses the following lemma.

**Lemma 8.1.16** Let \( X \) be a neutral shift. For every \( v \in \mathcal{L}(X) \), set \( \rho(v) = r_X(v) - 1 \). Then one has
\[
\sum_{a \in \mathcal{L}(v)} \rho(av) = \rho(v).
\] (8.1.2)

**Proof.** Since \( v \) neutral, we have \( e_X(v) - \ell_X(v) - r_X(v) + 1 = 0 \). Thus
\[
\sum_{a \in \mathcal{L}(v)} \rho(av) = \sum_{a \in \mathcal{L}(v)} (r_X(av) - 1) = e_X(v) - \ell_X(v) = r_X(v) - 1 = \rho(v).
\]

**Proof of Theorem 8.1.15.** Let \( U \) be the set of proper prefixes of \( uR_X(u) \) which are not proper prefixes of \( u \). We claim that \( U \) is an \( X \)-maximal suffix code. Indeed, assume first that \( v, v' \in U \) with \( v \) is a proper suffix of \( v' \). Then \( u \) is a proper prefix of \( v \) and thus \( u \) appears as a factor of \( v' \) otherwise than a suffix (see Figure 8.1.5), a contradiction.

![Figure 8.1.5: The set \( U \) is a suffix code.](image)

Now, since \( X \) is recurrent, every long enough word in \( \mathcal{L}(X) \) has a factor equal to \( u \). Thus it has a suffix which begins with \( u \) and has no other factor equal to \( u \). This suffix is in \( U \). This proves the claim concerning \( U \).

Set, as in Lemma 8.1.16, \( \rho(v) = r_X(v) - 1 \) for every \( v \in \mathcal{L}(X) \).

Consider first the tree formed by the set \( P \) of prefixes of \( uR_X(u) \). The children of \( p \in P \) are the \( pa \in P \) for \( a \in A \). Since \( uR_X(u) \) is a prefix code, the leaves are the elements of \( uR_X(u) \). The internal nodes are the elements of \( Q = P \setminus uR_X(u) \). As in any finite tree, the number of leaves minus 1 is equal to the sum over the internal nodes \( v \) of the integers \( d(v) - 1 \), where \( d(v) \) is the number of children of \( v \). For \( v \in Q \), since \( R_X(u) \) is an \((X,u)\)-maximal prefix code, we have
\[
d(v) = \begin{cases} r_X(v) & \text{if } v \in U \\ 1 & \text{otherwise.} \end{cases}
\]
Thus
\[
\text{Card}(uR_X(u)) - 1 = \sum_{v \in Q} (d(v) - 1) = \sum_{v \in U} \rho(v).
\]
(8.1.3)

Consider now the tree formed by the set \( S \) of suffixes of \( U \). The root is \( \varepsilon \) and the children of a word \( v \in S \) are the words \( av \in S \) with \( a \in A \). Since \( U \) is an \( X \)-maximal suffix code, the leaves of the tree \( S \) are the elements of \( U \) and the children of \( v \in S \setminus U \) are all the \( av \) for \( a \in L_X(v) \). Since for every internal node of \( S \), the sum of the \( \rho(av) \) taken over the children of \( v \) is equal to \( \rho(v) \) (by Lemma 8.1.16), we have
\[
\sum_{v \in U} \rho(v) = \rho(\varepsilon) = \text{Card}(A) - 1
\]
(8.1.4)

with the last equality resulting from the hypothesis \( A \subset L(X) \). Comparing (8.1.3) and (8.1.4), we obtain the desired equality.

We illustrate the proof with the following example.

**Example 8.1.17** Let \( X \) be the Fibonacci shift and let \( u = aa \). We have \( R_X(u) = \{ baa, babaa \} \) and thus \( U = \{ aa, aab, aaba, aabab, aababa \} \). The tree \( P \) is represented in Figure 8.1.6 on the left. The tree \( S \) is represented on the right with the value of \( \rho \) indicated on the leaves. The unique leaf of \( S \) with a nonzero value of \( \rho \) is the unique right-special word which belongs to \( U \), namely \( aaba \).

We now come to the second part of the proof of Theorem 8.1.14.

In a graph \( G = (V,E) \) labeled by an alphabet \( A \), we consider for every edge \( e \) from \( v \) to \( w \) with label \( a \), an inverse edge \( e^{-1} \) which goes from \( w \) to \( v \) and is labeled \( a^{-1} \). A generalized path in \( G \) is a sequence formed of consecutive edges or their inverses. The label of a generalized path is the reduced word which is the reduction of the label of the path. Thus it is an element of the free group on \( A \).

**Lemma 8.1.18** Let \( X \) be a dendric shift such that \( A \subset L(X) \). For every \( n \geq 1 \), the group defined by the Rauzy graph \( \Gamma_n(X) \) with respect to one of its vertices is the free group on \( A \).
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Proof. We will show that a sequence of Stallings foldings reduces any Rauzy graph $\Gamma_{n+1}(X)$ to $\Gamma_n(X)$.

Consider two vertices $ax, bx$ of $\Gamma_{n+1}(X)$ differing only by the first letter. Since the extension graph of $x$ is a tree, there is a path $a_0, b_1, \ldots, a_{k-1}, b_k, a_k$ in $\mathcal{E}(x)$ such that $a = a_0$ and $b = a_k$. The successive Stallings foldings at $xb_1, \ldots, xb_k$ identify the vertices $a_0x, \ldots, a_kx$. In this way, $\Gamma_{n+1}(X)$ is mapped onto $\Gamma_n(X)$.

Thus the groups defined by the Rauzy graphs $\Gamma_n(X), \Gamma_{n-1}(X), \ldots, \Gamma_1(X)$ are all identical. Since $A \subset \mathcal{L}(X)$, the graph $\Gamma_1(X)$ defines the free group on $A$, and thus the same is true for $\Gamma_n(X)$.

We illustrate the proof of Lemma 8.1.18 with the following example.

Example 8.1.19 Consider the Fibonacci shift and the Rauzy graphs $\Gamma_n(X)$ for $n = 1, 2, 3$ represented in Figure 8.1.7. Since there are edges labeled $b$ in $\Gamma_3(X)$ from $aa$ and $ba$ to $ab$, a Stallings folding merges $aa$ and $ba$. The result is $\Gamma_2(X)$. Similarly, since there are edges labeled $a$ from the vertices $a$ and $b$ to vertex $a$ in $\Gamma_2(X)$, we merge the vertices $a$ and $b$. The result is $\Gamma_1(X)$.

Lemma 8.1.20 Let $G$ be a labeled strongly connected graph and $x$ be a vertex. The group defined by $G$ with respect to $x$ is generated by the set $S$ of labels of paths from $x$ to $x$ in $G$.

Proof. Consider a generalized path $\pi$ from $x$ to $x$ labeled $y$. We have to prove that $y$ belongs to the subgroup $(S)$ generated by $S$. We use an induction on the number $r$ of inverse edges used in the path $\pi$. If $r = 0$, then $y$ is in $S$. Otherwise, we can write $y = ua^{-1}v$ where $x \xrightarrow{u} p \xrightarrow{a^{-1}} q \xrightarrow{v} x$ is a factorization of the path $\pi$. Since $G$ is strongly connected, there are (ordinary) paths $p \xrightarrow{t} x$ and $x \xrightarrow{w} q$. Then

$$y = utt^{-1}a^{-1}w^{-1}wv = (ut)(wat)^{-1}wv.$$

By definition we have $wat \in S$ (see Figure 8.1.8) and by induction hypothesis, we have $ut, wv \in (S)$. This shows that $y \in (S)$ and concludes the proof.
Proposition 8.1.21 Let $X$ be a minimal shift space and let $u \in L(X)$. There exists an $n \geq 1$ with the following property. Let $x \in L_n(X)$ be a word ending with $u$ and let $S$ be the set of labels of paths from $x$ to itself in $\Gamma_{n+1}(X)$. Then $S$ is contained in $R_X(u)^*$.

Proof. Let $n$ be the maximal length of the words in $uR_X(u)$. Consider $y \in S$. Since $y$ is the label of a path from $x$ to $x$ in $\Gamma_{n+1}(X)$, the word $xy$ ends with $x$. Thus there is a unique factorisation $y = y_1y_2 \cdots y_k$ in nonempty words $y_i$ where for each $i$ with $1 \leq i \leq k$, the word $uy_i$ ends with $u$ and has no other occurrence of $u$ except as a prefix or as a suffix. But, by the choice of $n$, the prefix of length $n$ of $uy_i$ has a factor $u$ other than as a prefix and thus $|uy_i| \leq n$. Now since $uy_i$ is the label of a path of length at most $n$ in $\Gamma_n(X)$, it is in $L(X)$. This implies that $y_i$ is in $R_X(u)$ and proves the claim.

We are now ready for the proof of Theorem 8.1.14.

Proof of Theorem 8.1.14. Let $n \geq 1$ be such that the property of Proposition 8.1.21 holds for $x \in L_n(X)$.

The inclusion $S \subset R_X(u)^*$ implies the inclusion $\langle S \rangle \subset \langle R_X(u) \rangle$. But, by Lemma 8.1.20 $S$ generates the group defined by $\Gamma_{n+1}(X)$. By Lemma 8.1.18 this group is the whole free group on $A$. Thus $R_X(u)$ generates the free group on $A$. Since any generating set of $F(A)$ having $\text{Card}(A)$ elements is a basis, and since $R_X(u)$ has $\text{Card}(A)$ elements by Theorem 8.1.15, this implies our conclusion.

Example 8.1.22 Let $A = \{u, v, w\}$, and let $X = X(\sigma)$ be the shift generated by the substitution $\sigma : u \rightarrow vuwwv, v \rightarrow vuww, w \rightarrow vuwv$. The shift $X$ is minimal since $\sigma$ is primitive. It is also dendric. Consider indeed the morphism $\phi : u \rightarrow aa, v \rightarrow ab, w \rightarrow ba$. Then we have $\phi \circ \sigma = \varphi^3 \circ \phi$ where $\varphi$ is the Fibonacci morphism. Indeed, we have

$$\phi \circ \sigma(u) = \phi(vuwwv) = abaabaab = \varphi^3(aa)$$

and similarly for $v, w$. This shows that $X$ is obtained by reading the Fibonacci shift with nonoverlapping blocks of length 2 and thus that $X$ is dendric (this is actually a particular case of Theorem 8.1.25). We have

$$R_X(u) = \{wwvu, wvru, vvuv\}$$
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which is a basis of the free group on \{u, v, w\}. Note that \(\phi(R_X(u))\) is a basis of a subgroup of index 2 of the free group on \{a, b\}.

8.1.3 Derivatives of minimal dendric shifts

Let \(X\) be a minimal shift space and let \(u \in \mathcal{L}(X)\). Let \(\varphi\) be a bijection from an alphabet \(B\) onto the set \(R_X(u)\) extended as usual to a morphism from \(B^*\) into \(A^*\). The shift space \(Y = \varphi^{-1}(X)\) is called the derivative of \(X\) with respect to \(u\). Actually, \(Y\) is the derivative system of \(X\) on the clopen set \([u]\) (see Section 4.7).

We will use (in the proof of Theorem 8.1.40) the following closure property of the family of minimal dendric shifts.

**Theorem 8.1.23** Any derivative of a minimal dendric shift is a minimal dendric shift on the same number of letters.

**Proof.** Let \(X\) be a minimal dendric shift on the alphabet \(A\) containing \(A\), let \(u \in \mathcal{L}(X)\) and let \(\varphi\) be a bijection from an alphabet \(B\) onto \(U = R_X(u)\). By Theorem 8.1.15, the set \(R_X(u)\) has \(\text{Card}(A)\) elements. Thus we may choose \(B = A\).

Set \(Y = \varphi^{-1}(X)\). Since \(Y\) is an induced system, it is minimal.

Consider \(y \in \mathcal{L}(Y)\) and set \(x = \varphi(y)\). Let \(\varphi'\) be the bijection from \(A\) onto \(U' = R'_X(u)\) such that \(u\varphi(b) = \varphi'(b)u\) for every \(b \in B\). For \(a, b \in B\), we have

\[(a, b) \in \mathcal{E}(y) \iff (\varphi'(a), \varphi(b)) \in \mathcal{E}_{U', U}(ux),\]

where \(\mathcal{E}_{U', U}(ux)\) denotes the generalized extension graph of \(ux\) relative to \(U', U\).

Indeed,

\[ab \in \mathcal{L}(Y) \iff u\varphi(a)x\varphi(b) \in \mathcal{L}(X) \iff \varphi'(a)ux\varphi(b) \in \mathcal{L}(X).\]

The set \(U'\) is a \((X, u)\)-maximal suffix code and the set \(U\) is a \((X, u)\)-maximal prefix code. By Proposition 8.1.12 the generalized extension graph \(\mathcal{E}_{U', U}(ux)\) is a tree. Thus the graph \(\mathcal{E}(y)\) is a tree. This shows that \(Y\) is a dendric shift. ■

**Example 8.1.24** Let \(X\) be the Tribonacci shift (see Example 8.1.5). It is the shift generated by the substitution \(\sigma\) defined by \(\sigma(a) = ab, \sigma(b) = ac, \sigma(c) = a\). We have \(R_X(a) = \{a, ba, ca\}\). Let \(\varphi : A \to R_X(a)\) be the morphism defined by \(\varphi(a) = a, \varphi(b) = ba, \varphi(c) = ca\) and let \(\varphi' : A \to R'_X(a)\) be such that \(a\varphi(x) = \varphi'(x)a\) for all \(x \in A\). We have \(\sigma = \varphi' \circ \pi\) where \(\pi\) is the circular permutation \(\pi = (abc)\). Let \(x = \varphi^\omega(a)\). Set \(z = \varphi'^{-1}(x)\). Since \(\varphi' \pi(x) = x\), we have \(z = \pi(x)\). Thus the derivative of \(X\) with respect to \(a\) is the shift \(\pi(X)\).

8.1.4 Bifix codes in dendric shifts

A **bifix code** on the alphabet \(A\) is a set \(U\) of words on \(A\) which is both a prefix code and a suffix code. For example, for every \(n \geq 1\), a set of words of length \(n\) is a bifix code.
Let $X$ be a shift space and let $U \subset \mathcal{L}(X)$ be a finite bifix code which is an $X$-maximal prefix and suffix code. Let $f$ be a coding morphism for $U$. Then $f^{-1}(\mathcal{L}(X))$ is factorial and extendable. The shift space $Y$ such that $\mathcal{L}(Y) = f^{-1}(\mathcal{L}(X))$ is called a decoding of $X$ by $U$. We denote $Y = f^{-1}(X)$.

The following result expresses a closure property of the family of dendric shifts.

**Theorem 8.1.25** The decoding of a dendric shift by a finite bifix code which is an $X$-maximal prefix and suffix code is a dendric shift.

**Proof.** Let $U \subset \mathcal{L}(X)$ be a finite bifix code which is an $X$-maximal prefix and suffix code. Let $f : B^* \to A^*$ be a coding morphism for $U$ and let $Y = f^{-1}(X)$. For $w \in \mathcal{L}(Y)$ and $a, b \in B$, we have

$$(a, b) \in E_Y(w) \iff (f(a), f(b)) \in E_{U, U}(f(w))$$

and thus $E_Y(w)$ is a tree by Proposition 8.1.12. This shows that $Y$ is dendric.

**Example 8.1.26** Let $X$ be the Fibonacci shift on $\{a, b\}$ and consider the bifix code $U = \{aa, ab, ba\}$. The corresponding decoding of $X$ is the shift of Example 8.1.22.

We will prove the following result.

**Theorem 8.1.27** Let $X$ be a dendric shift on the alphabet $A$. A finite bifix code $U \subset \mathcal{L}(X)$ which is a basis of the free group on $A$ is equal to $A$.

The following example shows that the hypothesis that $X$ is dendric is necessary in Theorem 8.1.27.

**Example 8.1.28** Let $A = \{a, b, c\}$ and $U = \{ab, acb, acc\}$. The set $U$ is a bifix code. It is also a basis of the free group on $A$. Indeed, we have $accb = (acb)(ab)^{-1}(acb)$ and $b = (acc)^{-1}(accb)$. Next, $a = (ab)b^{-1}$ and $c = a^{-1}(acb)b^{-1}$. Thus $a, b, c$ belong to the group generated by $U$. Observe that we can verify directly that no dendric shift $X$ can be such that $U \subset \mathcal{L}(X)$. Indeed, this would force $ab, cb, cc, ac \in \mathcal{L}(X)$ and thus the extension graph of $\epsilon$ to contain a cycle (see Figure 8.1.11).

To prove Theorem 8.1.27 we introduce the following notion. Let $U$ be a bifix code and let $P$ (resp. $S$) be the set of proper prefixes (resp. suffixes) of the words of $U$. The incidence graph of $U$ is the following undirected graph. Its set of vertices is the disjoint union of $P$ and $S$. The edges are $(\epsilon, \epsilon)$ and the pairs $(p, s) \in P \times S$ such that $ps \in U$.

**Example 8.1.29** Let $X$ be the Fibonacci shift and let $U = \mathcal{L}_3(X)$. The incidence graph of $U$ is represented in Figure 8.1.9 (in each of the three parts, the vertices on the left are in $P$ and those on the right in $S$).


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![Diagram](image)

**Figure 8.1.9:** The incidence graph of \( U = \mathcal{L}_3(X) \).

**Proposition 8.1.30** Let \( X \) be a dendric shift and let \( U \subset \mathcal{L}(X) \) be a bifix code. Let \( P \) (resp. \( S \)) be the set of proper prefixes (resp. proper suffixes) of \( U \) and let \( G \) be the incidence graph of \( U \). Then the following assertions hold.

(i) The graph \( G \) is acyclic.

(ii) The intersection of \( P' = P \setminus \{ \varepsilon \} \) (resp. \( S' = S \setminus \{ \varepsilon \} \)) with each connected component of \( G \) is a suffix (resp. prefix) code.

(iii) For every reduced path \((v_1, u_1, \ldots, u_n, v_{n+1})\) in \( G \) with \( u_1, \ldots, u_n \in P' \) and \( v_1, \ldots, v_{n+1} \) in \( S' \), the longest common prefix of \( v_1, v_{n+1} \) is a proper prefix of all \( v_1, \ldots, v_n, v_{n+1} \).

(iv) Symmetrically, for every reduced path \((u_1, v_1, \ldots, v_n, u_{n+1})\) in \( G \) with \( u_1, \ldots, u_n+1 \in P' \) and \( v_1, \ldots, v_n \in S' \), the longest common suffix of \( u_1, u_{n+1} \) is a proper suffix of \( u_1, u_2, \ldots, u_{n+1} \).

**Proof.** Assertions (iii) and (iv) imply Assertions (i) and (ii). Indeed, assume that (iii) holds. Consider a reduced path \((v_1, u_1, \ldots, u_n, v_{n+1})\) in \( G \) with \( u_1, \ldots, u_n \in P' \) and \( v_1, \ldots, v_{n+1} \) in \( S' \). If \( v_1 = v_{n+1} \), then \( v_1 \) is a prefix of all \( v_i \) and in particular of \( v_2 \), a contradiction since \( U \) is a bifix code. Thus \( G \) is acyclic and (i) holds. Next, if \( v_1, v_{n+1} \) are comparable for the prefix order, their longest common prefix is one of them, a contradiction with (iii) again. The assertion on \( P' \) is proved in an analogous way using assertion (iv).

We prove simultaneously (iii) and (iv) by induction on \( n \geq 1 \).

The assertions hold for \( n = 1 \). Indeed, if \( v_1v_1, u_1v_2 \in U \) and if \( v_1 \in \mathcal{L}(X) \) is a prefix of \( v_2 \in S' \), then \( u_1v_1 \) is a prefix of \( u_1v_2 \), a contradiction with the hypothesis that \( U \) is a prefix code. The same holds symmetrically for \( u_1v_1, u_2v_1 \in U \) since \( U \) is a suffix code.

Let \( n \geq 2 \) and assume that the assertions hold for any path of length at most \( 2n - 2 \). We treat the case of a path \((v_1, u_1, \ldots, u_n, v_{n+1})\) in \( G \) with \( u_1, \ldots, u_n \in P' \) and \( v_1, \ldots, v_{n+1} \) in \( S' \). The other case is symmetric.

Let \( p \) be the longest common prefix of \( v_1 \) and \( v_{n+1} \). We may assume that \( p \) is nonempty since otherwise the statement is obviously true. Any two elements of the set \( U = \{ u_1, \ldots, u_n \} \) are connected by a path of length at most \( 2n - 2 \) (using elements of \( \{ v_2, \ldots, v_n \} \)). Thus, by induction hypothesis, \( U \) is a suffix code. Similarly, any two elements of the set \( V = \{ v_1, \ldots, v_n \} \) are connected by a path of length at most \( 2n - 2 \) (using elements of \( \{ u_1, \ldots, u_{n-1} \} \)). Thus \( V \) is a
prefix code. We cannot have \( v_1 = p \) since otherwise, using the fact that \( u_n p \) is a prefix of \( u_n v_{n+1} \) and thus in \( S' \), the generalized extension graph \( \mathcal{E}_{U,V}(\varepsilon) \) would have the cycle \((p, u_1, v_2, \ldots, u_n, p)\), a contradiction since \( \mathcal{E}_{U,V}(\varepsilon) \) is acyclic by Proposition 8.1.10. Similarly, we cannot have \( v_{n+1} = p \).

Set \( W = p^{-1} V \) and \( V' = (V \setminus p W) \cup \{p\} \). Since \( V \) is a prefix code and since \( p \) is a proper prefix of \( V \), the set \( V' \) is a prefix code. Suppose that \( p \) is not a proper prefix of all \( v_2, \ldots, v_n \). Then there exist \( i, j \) with \( 1 \leq i < j \leq n + 1 \) such that \( p \) is a proper prefix of \( v_i, v_j \) but not of any \( v_{i+1}, \ldots, v_{j-1} \). Then \( v_{i+1}, \ldots, v_{j-1} \in V' \) and there is the cycle \((p, u_i, u_{i+1}, \ldots, u_{j-1}, u_{j-1}, p)\) in the graph \( \mathcal{E}_{U,V'}(\varepsilon) \). This is in contradiction with Proposition 8.1.10 because, \( V' \) being a prefix code, \( \mathcal{E}_{U,V'}(\varepsilon) \) is acyclic. Thus \( p \) is a proper prefix of all \( v_2, \ldots, v_n \).

Let \( X \) be a dendric shift and let \( U \subseteq \mathcal{L}(X) \) be a bifix code. Let \( P \) be the set of proper prefixes of the words of \( U \). Let \( \theta_U \) be the equivalence on \( P \) defined by \( p \equiv q \mod \theta_U \) if \( p, q \) are in the same connected component of the incidence graph of \( U \). Note that, since \( U \) is bifix, the class of \( \varepsilon \) is reduced to \( \varepsilon \). The coset graph of \( U \) is the following labeled graph. The set vertices is the set \( R \) of classes of \( \theta_U \). There is an edge labeled \( a \) from the class of \( p \) to the class of \( q \) in each of the following cases

(i) \( q = pa \),

(ii) \( q = \varepsilon \) and \( pa \in U \).

Example 8.1.31 Let \( X \) be the Fibonacci shift. The coset graph of \( \{a, bab\} \) is shown in Figure 8.1.10 on the left and the coset graph of \( \{a, bab, baab\} \) on the right.

![Figure 8.1.10: The coset graphs of \( \{a, bab\} \) and of \( \{a, bab, baab\} \).](image)

A simple path from a vertex \( v \) to itself in a graph is a path which is not a concatenation of two nonempty paths from \( v \) to itself.

Proposition 8.1.32 Let \( X \) be a dendric shift and let \( U \subseteq \mathcal{L}(X) \) be a finite bifix code. Let \( P \) be the set of proper prefixes of \( U \) and let \( H = \langle U \rangle \) be the subgroup generated by \( U \). Let also \( C \) be the coset graph of \( U \).

1. For every \( p, q \in P \), \( p \equiv q \mod \theta_U \) implies \( Hp = Hq \).

2. If \( p \equiv p' \mod \theta_U \) and if \( p \xrightarrow{\alpha} q \), \( p' \xrightarrow{\alpha} q' \) are edges in \( C \), then \( q \equiv q' \mod \theta_U \).
3. Every $u \in U$ is the label of a simple path from $\varepsilon$ to itself in $C$.

4. The graph $C$ is the Stallings graph of the subgroup $(U)$ generated by $U$.

Proof. Let $C = (R, E)$ be the coset graph of $U$ and let $G$ be its incidence graph.

1. The first assertion is clear since $\theta_U$ is the equivalence on $P$ generated by the pairs $p, q$ such that there is an $s$ with $ps, qs \in X$ and thus $p, q \in Hs^{-1}$.

2. We assume that $q = pa$ and $q' = p'a$. The other cases are similar. Let $s, s'$ be such that $qs, q's' \in U$. Let $p = u_0, v_1, u_1, \ldots, v_n, u_n = p'$ be a path from $p$ to $p'$ in the incidence graph $G$. Set $v_0 = as$ and $v_{n+1} = as'$. Then $(v_0, u_0, \ldots, u_n, v_{n+1})$ is a path in $G$. But since the letter $a$ is a common prefix of $v_0$ and $v_{n+1}$, by Proposition 8.1.30 it is also a common prefix of all $v_i, v_i$. Set $v_i = av_i'$ for $0 \leq i \leq n + 1$. Then $(u_0a, v_1', u_1a, \ldots, v_n', u_n)$ is a path from $q$ to $q'$ in the coset graph $C$ and thus $q \equiv q' \mod \theta_U$.

3. This follows from the fact that $C$ can be obtained by Stallings foldings from the graph on $P$ with edges $p \xrightarrow{a} q$ if either $pa = q$ or $q = \varepsilon$ and $pa \in U$.

4. Let $K$ be the group defined by the coset graph $C$. Let us show that $K$ is equal to $H$. By construction, we have $X \subset K$ and thus $H \subset K$. The converse follows easily from Assertion 1.

Let us finally show that $C$ is Stallings reduced. Assume that $p, q \in P$ are such that there are edges with the same label $a$ from the class $\bar{p}, \bar{q}$ of $p, q$ to the same vertex $\bar{r}$. Let $v$ be the label of a path from $\bar{r}$ to $\varepsilon$ which does not pass by $\varepsilon$ before. Then $pav, qav \in X$ and thus $p \equiv q \mod \theta_U$, which implies that $\bar{p} = \bar{q}$. This shows that $C$ is Stallings reduced.

We are now ready to prove Theorem 8.1.27

Proof of Theorem 8.1.27. Let $X$ be a dendric shift on the alphabet $A$. Let $U \subset C(X)$ be a bifix code which is a basis of the free group on $A$. By Proposition 8.1.32 the coset graph $C$ of $U$ has only one vertex $\varepsilon$ and loops $\varepsilon \xrightarrow{a} \varepsilon$ for every $a \in A$. Since, by Proposition 8.1.32 again, every word of $U$ is the label of a simple path from $\varepsilon$ to itself in the coset graph of $U$, we have $U \subset A$ and thus $U = A$.

8.1.5 Tame automorphisms

An automorphism $\alpha$ of the free group on $A$ is positive if $\alpha(a) \in A^+$ for every $a \in A$. We say that a positive automorphism of the free group on $A$ is tame if it belongs to the submonoid generated by the permutations of $A$ and the automorphisms $\alpha_{a,b}, \tilde{\alpha}_{a,b}$ defined for $a, b \in A$ with $a \neq b$ by

$$\alpha_{a,b}(c) = \begin{cases} ab & \text{if } c = a, \\ c & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\alpha}_{a,b}(c) = \begin{cases} ba & \text{if } c = a, \\ c & \text{otherwise}. \end{cases}$$

Thus $\alpha_{a,b}$ places a letter $b$ after each $a$ and $\tilde{\alpha}_{a,b}$ places a letter $b$ before each $a$. The above automorphisms and the permutations of $A$ are called the elementary positive automorphisms on $A$. The monoid of positive automorphisms is not
finitely generated as soon as the alphabet has at least three generators (see the Notes Section).

A basis $U$ of the free group is positive if $U \subset A^+$. A positive basis $U$ of the free group is tame if there exists a tame automorphism $\alpha$ such that $U = \alpha(A)$.

**Example 8.1.33** The set $U = \{ba, cba, cca\}$ is a tame basis of the free group on $\{a, b, c\}$. Indeed, one has the following sequence of elementary automorphisms.

\[
(b, c, a) \xrightarrow{\alpha_{cb}} (b, cb, a) \xrightarrow{\alpha^{2}_c} (b, cb, cca) \xrightarrow{\alpha_{ba}} (ba, cba, cca).
\]

The fact that $U$ is a basis can be checked directly since $(cba)(ba)^{-1} = c$, $c^{-2}(cca) = a$ and finally $(ba)a^{-1} = b$.

The following result will play a key role in the proof of the main result of this section (Theorem 8.1.37).

**Proposition 8.1.34** A set $U \subset A^+$ is a tame basis of the free group on $A$ if and only if $U = A$ or there is a tame basis $V$ of the free group on $A$ and $u, v \in V$ such that $U = (V \setminus u) \cup uv$ or $U = (V \setminus v) \cup uv$.

**Proof.** Assume first that $U$ is a tame basis of the free group on $A$. Then $U = \alpha(A)$ where $\alpha$ is a tame automorphism of $\langle A \rangle$. Then $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ where the $\alpha_i$ are elementary positive automorphisms. We use an induction on $n$. If $n = 0$, then $U = A$. If $\alpha_n$ is a permutation of $A$, then $U = \alpha_1 \alpha_2 \cdots \alpha_{n-1}(A)$ and the result holds by induction hypothesis. Otherwise, set $\beta = \alpha_1 \cdots \alpha_{n-1}$ and $V = \beta(A)$. By induction hypothesis, $V$ is tame. If $\alpha_n = \alpha_{a,b}$, set $u = \beta(a)$ and $v = \beta(b) = \alpha(b)$. Then

\[
U = \alpha(A \setminus a) \cup \alpha(a) = \beta(A \setminus a) \cup \beta(ab) = (V \setminus u) \cup uv
\]

and thus the condition is satisfied. The case were $\alpha_n = \alpha_{a,b}$ is symmetrical.

Conversely, assume that $V$ is a tame basis and that $u, v \in V$ are such that $U = (V \setminus u) \cup uv$. Then, there is a tame automorphism $\beta$ of $F(A)$ such that $V = \beta(A)$. Set $a = \beta^{-1}(u)$ and $b = \beta^{-1}(v)$. Then $U = \beta \circ \alpha_{a,b}(A)$ and thus $U$ is a tame basis. 

We note the following corollary.

**Corollary 8.1.35** A tame basis of the free group which is a bifix code is the alphabet.

**Proof.** Assume that $U$ is a tame basis which is not the alphabet. By Proposition 8.1.34 there is a tame basis $V$ and $u, v \in V$ such that $U = (V \setminus v) \cup uv$ or $U = (V \setminus u) \cup uv$. In the first case, $U$ is not prefix. In the second one, it is not suffix.
Example 8.1.36 The set $U = \{ab, acb, acc\}$ is a basis of the free group on \{a, b, c\} (see Example 8.1.28). The set $U$ is bifix and thus it is not a tame basis by Corollary 8.1.35.

The following result is a remarkable consequence of Theorem 8.1.27.

**Theorem 8.1.37** Any basis of the free group included in the language of a minimal dendric shift is tame.

**Proof.** Let $X$ be a minimal dendric shift. Let $U \subset L(X)$ be a basis of the free group on $A$. We use an induction on the sum $\lambda(U)$ of the lengths of the words of $U$. If $U$ is bifix, by Theorem 8.1.27, we have $U = A$. Next assume, for example that $U$ is not prefix. Then there are nonempty words $u, v$ such that $u, uv \in U$. Let $V = (U \setminus uv) \cup v$. Then $V$ is a basis of the free group and $\lambda(V) < \lambda(U)$. By induction hypothesis, $V$ is tame. Since $U = (V \setminus v) \cup uv$, $U$ is tame by Proposition 8.1.34. 

Example 8.1.38 The set $U = \{ab, acb, acc\}$ is a basis of the free group which is not tame (see Example 8.1.36). Accordingly, the extension graph $\mathcal{E}(\varepsilon)$ relative to the set of factors of $U$ is not a tree (see Figure 8.1.11).

![Figure 8.1.11: The graph $\mathcal{E}(\varepsilon)$.](image)

8.1.6 $S$-adic representation of dendric shifts

We now study the $S$-adic representations of dendric shifts. We first recall a general construction allowing to build $S$-adic representations of any minimal aperiodic shift (Proposition 8.1.39) which is based on return words. Using Theorem 8.1.37, we show that this construction actually provides $S_e$-representations of minimal dendric shifts (Theorem 8.1.40), where $S_e$ is the set of elementary positive automorphisms of the free group on $A$.

Let $S$ be a set of morphisms and $\tau = (\tau_n)_{n \geq 1}$ be a directive sequence of morphisms in $S$ with $\tau_n : A^*_{n+1} \rightarrow A^*_n$ and $A_1 = A$. When $\tau$ is primitive and proper, we have by Lemma 7.4.5

$$\mathcal{L}(\tau) = \bigcap_{n \geq 1} \text{Fac}(\tau_{[0,n]}(A^*_{n+1}))$$

(8.1.5)

where $\text{Fac}(L)$ denotes the set of factors of the words in $L$.

The next proposition provides a general construction to get a primitive proper $S$-adic representation of any aperiodic minimal shift space $X$. 


Proposition 8.1.39 An aperiodic shift $X$ is minimal if and only if it has a primitive proper $S$-adic representation for some (possibly infinite) set $S$ of morphisms.

Proof. The direct implication follows from Proposition 7.4.15.

Let us prove the converse. Since $X$ is aperiodic, we have $p_{n+1} > p_n(X)$ for all $n ≥ 1$ by Theorem 2.2.12. Thus there is for each $n ≥ 1$, a right-special word $u_n$ of length $n$ in $L(X)$ such that $u_n$ is a suffix of $u_{n+1}$. By assumption, $X$ is minimal so that $R_X(u_{n+1})$ is finite for all $n$. Since $u_n$ is right-special, $R_X(u_{n+1})$ has cardinality at least 2 for all $n$. For all $n$, let $A_n = \{0, \ldots, \text{Card}(R_X(u_n)) - 1\}$ and let $\alpha_n : A_n^* → A^*$ be a coding morphism for $R_X(u_n)$. The word $u_n$ being suffix of $u_{n+1}$, we have $\alpha_n(A_{n+1}) ⊂ \alpha_n(A_n^+)$.

Indeed, for such sets, the set $R_X(u_n)$ is a prefix of words of length $n$. For all $n$, let $R_X(u_n) = \alpha_0 \circ \tau_0 \circ \tau_1 \circ \cdots \circ \tau_{n-1}(A_n)$ and $L(X) = \bigcap_{n ∈ \mathbb{N}} \text{Fac}(\alpha_0 \circ \tau_0 \circ \cdots \circ \tau_n(A_n^+))$. Without loss of generality, we can suppose that $u_0 = ε$ and $A_0 = A$. In this case we get $\alpha_0 = id$ and the shift space $X$ thus has an $S$-adic representation with $S = \{\tau_n \mid n ∈ \mathbb{N}\}$.

By construction, each morphism $\sigma_n$ is right proper. By Lemma 7.4.8 we can modify the morphisms $\sigma_n$ to make them proper.

Finally, by Proposition 7.4.15 again, the $S$-adic representation is primitive.

Even for minimal shifts with linear factor complexity, the set of morphisms $S = \{\tau_n \mid n ≥ 1\}$ considered in Proposition 8.1.39 is usually infinite as well as the sequence of alphabets $(A_n)_{n ≥ 1}$ is usually unbounded. For dendric shifts, the next theorem significantly improves the only if part of Proposition 8.1.39. Indeed, for such sets, the set $S$ can be replaced by the set $S_e$ of elementary positive automorphisms. In particular, $A_n$ is equal to $A$ for all $n$.

Theorem 8.1.40 Every minimal dendric shift has

1. a primitive proper unimodular $S$-adic representation and also

2. a primitive $S_e$-adic representation.

Proof. For any non-ultimately periodic sequence $(u_n)_{n ≥ 0}$ of words of $L(X)$ such that $u_0 = ε$ and $u_n$ is suffix of $u_{n+1}$, the sequence of morphisms $(\tau_n)_{n ≥ 0}$ built in the proof of Proposition 8.1.39 is a primitive proper $S$-adic representation of $X$ with $S = \{\tau_n \mid n ≥ 0\}$. By Theorem 8.1.14 the set $R_X(u_1)$ is a basis of the free group on $A$. This implies that the matrix $M(\tau_n)$ is unimodular. This proves the first assertion.

To prove assertion 2, all we need to do is to consider such a sequence $(u_n)_{n ≥ 0}$ such that $\tau_n$ is tame for all $n$.

Let $u_1 = a^{(0)}$ be a letter in $A$. Since $X$ is dendric, the set $R_X(u_1)$ has $\text{Card}(A)$ elements by Theorem 8.1.15. Let $\tau_0 : A → R_X(u_1)$ be a bijection. By Theorem 8.1.14 again, since the set $R_X(u_1)$ is a basis of the free group on $A$, by Theorem 8.1.37 it is a tame basis. Thus the morphism $\tau_0 : A^* → A^*$ is
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a tame automorphism. Let \( a^{(1)} \in A \) be a letter and set \( u_2 = \tau_0(a^{(1)}) \). Thus \( u_2 \in R_X(u_1) \) and \( u_1 \) is a suffix of \( u_2 \). By Theorem 8.1.23 the derived shift \( X^{(1)} = \tau_0^{-1}(X) \) is a minimal dendric shift on the alphabet \( A \). We thus reiterate the process with \( a^{(1)} \) and we conclude by induction with \( u_n = \tau_0 \cdots \tau_{n-2}(a^{(n-1)}) \) for all \( n \geq 2 \).

We illustrate Theorem 8.1.40 by the following example.

**Example 8.1.41** Let \( A = \{a, b, c\} \), let \( \sigma \) be the substitution defined by \( \sigma(a) = ac, \sigma(b) = bac, \sigma(c) = cbac \) and let \( X \) be the substitution shift generated by \( \sigma \). It can be shown that \( X \) is dendric (Exercise 8.12). We have \( \sigma = \alpha_{a,c} \alpha_{b,a} \alpha_{c,b} \).

Thus \( S \) has the \( S \)-adic representation \( (\sigma_n)_{n \geq 0} \) given by the periodic sequence

\[
\sigma_3 = \alpha_{a,c}, \quad \sigma_4 = \alpha_{b,a}, \quad \sigma_5 = \alpha_{c,b}.
\]

The converse of Theorem 8.1.40 is not true, as shown by Example 8.1.42 below (see also Exercise 8.14).

**Example 8.1.42** Let \( A = \{a, b, c\} \) and let \( \sigma : a \mapsto ac, b \mapsto bac, c \mapsto cb \). The substitution shift generated by \( \sigma \) (it is generated by the fixed point \( \sigma(\omega) \)) is not dendric since \( bb, bc, cb, cc \in L(X) \) and thus \( E(\varepsilon) \) has a cycle, although \( \sigma \) is a tame automorphism since \( \sigma = \alpha_{a,c} \alpha_{c,b} \).

8.1.7 Dimension groups of dendric shifts.

Recall that \( \mathcal{M}(X, S) \) denotes the set of invariant measures on a shift space \((X, S)\).

**Theorem 8.1.43** The dimension group of a minimal dendric shift \((X, S)\) on the alphabet \( A \) is \((G, G^+, 1_G)\) with \( G = \mathbb{Z}^A \), \( G^+ = \{x \in \mathbb{Z}^d \mid \langle x, \mu \rangle > 0, \mu \in \mathcal{M}(X, S)\} \cup 0 \) and \( 1_G = 1 \) where 1 is the vector with all components equal to 1.

**Proof.** By Theorem 8.1.40, \( X \) has a primitive proper and unimodular \( S \)-adic representation. Thus the form of the dimension group is given by Theorem 7.5.4. \( \square \)

**Example 8.1.44** Consider again the dendric shift \( X \) generated by the unimodular substitution \( \sigma : a \mapsto ac, b \mapsto bac, c \mapsto cbac \) of Example 8.1.41. Since every word in the image of \( a \) ends with \( ac \), the morphism \( \tau : x \mapsto cx(x)c^{-1} \) (where \( c^{-1} \) is the inverse of \( c \) in the free group) is proper and \( X = X(\tau) \). Since \( \tau \) is proper, the shift \( X \) has a stationary BV-representation with matrix \( M(\sigma) = M(\tau) \).

This implies by Theorem 6.3.4 that the dimension group of \((X, S)\) is the group of the matrix \( M(\sigma) \). The matrix \( M(\sigma) \) is

\[
M(\sigma) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.
\]
The dominant eigenvalue is the largest root of \( \lambda^3 - 4\lambda^2 + 5\lambda - 1 = 0 \). The vector \( w = [\lambda \quad \lambda - 1 \quad (\lambda - 1)^2] \) is a corresponding eigenvector. The map \( x \mapsto (x, w) \) sends \( \mathbb{K}^3(X, S) \) onto \( \mathbb{Z}[\lambda] \). The image of the unit vector \( 1_M \) is \( \lambda^2 \) which a unit of \( \mathbb{Z}[\lambda] \). Thus the dimension group of \( X \) is isomorphic to \( \mathbb{Z}[\lambda] \).

## 8.2 Sturmian shifts

We illustrate the preceding results on the family of Sturmian shifts. We have seen that Sturmian shifts are dendric (Proposition 8.1.2). In the particular case of dendric shifts, all the general results concerning dendric shifts can be formulated more precisely. We have already seen that for the Return Theorem. We give below the complete description of the \( S \)-adic representations of Sturmian shifts.

### 8.2.1 BV-representation of Sturmian shifts

We define the morphisms \( \rho_n, \gamma_n, n \geq 1 \) from \( \{0, 1\} \) to \( \{0, 1\}^* \) by

\[
\rho_n(0) = 01^n+1, \quad \rho_n(1) = 01^n \quad \text{and} \quad \gamma_n(0) = 10^n+1, \quad \gamma_n(1) = 10^n.
\]

The morphisms \( \rho_n, \gamma_n \) are related as follows to the elementary automorphisms \( L_0, L_1 \) introduced in Section 2.5. We have for every \( n \geq 1 \) and every \( u \in \{0, 1\}^* \),

\[
1^n\rho_n(u) = L_{1^n0}(u)1^n, \quad 0^n\gamma_n(u) = L_{0^n1}(u)0^n,
\]

as one verifies easily for \( u = 0, 1 \), which implies the identities for all \( u \).

The following result will be used to give a KR-representation of Sturmian shifts.

**Proposition 8.2.1** Let \( X \) be a Sturmian shift of slope \( \alpha = [0, 1 + d_1, d_2, \ldots] \). There is a Sturmian shift space \( Y \) and an \( n \geq 1 \) such that either \( X = \rho_n(Y) \) or \( X = \gamma_n(Y) \). More precisely, if \( d_1 > 0 \), then \( n = d_1 \), \( X = \gamma_n(Y) \) and \( Y \) is the Sturmian shift of slope \( [0, d_2, d_3, \ldots] \). If \( d_1 = 0 \), then \( n = d_2 \), \( X = \rho_n(Y) \) and \( Y \) is the Sturmian shift of slope \( [0, d_3, d_4, \ldots] \).

**Proof.** Let \( X^+ \) be the one-sided shift space associated to \( X \) and let \( x \) be the standard Sturmian word which belongs to \( X^+ \). By Theorem 2.5.3, we have \( x = \text{Pal}(\Delta) \) where \( \Delta = 0^{d_1}1^{d_2} \cdots \) is the directive word of \( x \). Assume that \( d_1 > 0 \) and set \( \Delta = 0^n1\Delta' \) with \( n = d_1 \). Then, by Justin Formula 2.5.3, we have \( x = L_{0^n1}(\text{Pal}(\Delta')) \). We have seen that \( L_{0^n1}(u)0^n = 0^n\gamma_n(u) \) for every \( u \in \{0, 1\}^* \). Thus \( x = 0^n x' \), where \( x' \) is a concatenation of words in \( \{10^n+1, 10^n\} \). The word \( y = \text{Pal}(\Delta') \) is Sturmian and \( x' = \gamma_n(y) \). This shows that \( X^+ = \gamma_n(Y^+) \) where \( Y^+ \) is the one-sided shift generated by \( y \). Finally, we obtain \( X = \gamma_n(Y) \) where \( Y \) is the two-sided shift associated to \( Y^+ \). The case where \( \Delta \) begins with \( 1 \) is analogous. 

\[\square\]
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Example 8.2.2 Let \( X = X(\sigma) \) with \( \sigma : 0 \rightarrow 01, 1 \rightarrow 01 \). Actually, \( X \) is the Sturmian shift of slope \( \alpha = \sqrt{2} - 1 \). Indeed, we have \( \alpha = [0, 2, 2, \ldots] \) and thus the directive word of \( c_\alpha \) is \( 011001100 \cdot \cdot \cdot = \omega \). The standard word with slope \( \alpha \) is thus the fixed point of the morphism \( L_{0110} : 0 \rightarrow 01010, 1 \rightarrow 0101001 \) which is \( \sigma^2 \). We have \( \sigma(0)0 = 0\gamma_1(1) \) and \( \sigma(1)0 = 0\gamma_1(0) \). Thus \( X = \gamma_1(Y) \) where \( Y \) is obtained from \( X \) by exchanging 0 and 1.

Let \((X, S)\) be a Sturmian shift. For \( a \in \{0, 1\} \), recall that \([a] = \{(x_i)_{i \in \mathbb{Z}} \in X \mid x_0 = a\}\).

Theorem 8.2.3 Let \( X \) be a Sturmian shift on \( \{0, 1\} \). There exists a sequence \((\zeta_n)_{n \in \mathbb{N}}\) taking values in \( \{\rho_1, \gamma_1, \rho_2, \gamma_2, \ldots\} \) such that \((P(n))_n\) is a refining sequence of KR-partitions with \( P(1) = \{[0], [1]\} \) and, for \( n \geq 2 \),

\[ P(n) = \{S^k \zeta_1 \cdot \cdot \cdot \zeta_n([a]) \mid 0 \leq k < |\zeta_1 \cdot \cdot \cdot \zeta_{n-1}(a)|, a \in \{0, 1\}\} . \]

Proof. We apply iteratively Proposition 8.2.1 to obtain the sequence \((\zeta_n)\). To see that condition (KR1) (the intersection of the bases is reduced to a point) is satisfied, note that, if for example \( \zeta_n = \rho_i \), then

\[ \zeta_1 \cdot \cdot \cdot \zeta_n([a]) \subseteq [\zeta_1 \cdot \cdot \cdot \zeta_{n-1}(1).\zeta_1 \cdot \cdot \cdot \zeta_n(a) \cdot \zeta_1 \cdot \cdot \cdot \zeta_{n-1}(0)] . \]

Let \((X, S)\) be a Sturmian shift and \((P(n))_n\) be the sequence of partitions given by Corollary 8.2.3. With such a sequence is associated an ordered Bratteli-Vershik diagram \( B = (V, E, \leq) \) which can be described as follows. For all \( n \geq 1 \), \( V_n \) consists of two vertices, the substitution read on \( E_{n+1} \) is \( \zeta_n \), with \( E(1) \) consisting of a simple hat. We have thus proved the following statement.

Corollary 8.2.4 A shift space is Sturmian if and only if it has a BV-representation with simple hat, with two vertices at every level and such that the substitution read on \( E_{n+1} \) is some \( \rho_i \) or \( \gamma_i \).

Example 8.2.5 Let \( X = X(\sigma) \) with \( \sigma \) as in Example 8.2.2. The sequence \((\zeta_n)\) can be chosen to be \( \zeta_n = \tau \) where \( \tau : 0 \rightarrow 10, 1 \rightarrow 100 \) (the composition of \( \gamma_1 \) with the exchange of 0, 1). The corresponding Bratteli diagram is shown in Figure 8.2.1.

8.2.2 Linearly recurrent Sturmian shifts

We obtain as a corollary of Theorem 8.2.3 the following result.

Corollary 8.2.6 A Sturmian shift \( X \) of slope \( \alpha = [a_0, a_1, \ldots] \) is linearly recurrent if and only if the coefficients \( a_i \) are bounded.
Figure 8.2.1: The BV-representation of the Sturmian shift $X$.

Proof. Assume first that the coefficients are not bounded. By Exercise 2.40, the shift $X$ is not linearly recurrent.

Conversely, if the coefficients $a_i$ are bounded, the sequence $(\zeta_n)$ has a finite number of terms and $X$ is linearly recurrent by Theorem 8.2.3.

We can add one more equivalent condition in Corollary 8.2.6, namely: $L(X)$ is $K$-power free for some $K \geq 2$ (Exercise 8.15).

We note the following additional result.

**Theorem 8.2.7** Let $X$ be a Sturmian shift with slope $\alpha$. Then $X$ is a substitutive shift if and only if $\alpha$ is quadratic.

Proof. Assume first that $X$ is Sturmian with a slope $\alpha$ which is quadratic. By Lagrange Theorem (see Appendix C), the continued fraction expansion of $\alpha = [0, 1 + d_1, d_2, \ldots]$ is eventually periodic. The directive word $x = 0^{d_1}1^{d-2} \ldots$ of the standard word $s = c_\alpha$ is eventually periodic. Set $x = uy$ with $y = u^\omega$. Then $s = L_u(y)$ and $y$ is a fixed point of $L_v$. This shows that $s$ is substitutive and thus that $X$ is substitutive.

Conversely,

We complete

8.2.3 Derivatives of Sturmian shifts

We have seen that

8.3 Specular shifts

We end this chapter with the description of a family of shifts which generalizes dendric shifts and is build as an abstract model for the transformations called linear involutions described in the next chapter.

8.3.1 Specular groups

We begin with the definition of a class of groups which generalizes free groups. We consider an alphabet $A$ with an involution $\theta : A \to A$, possibly with some fixed points. We also consider the group $G_\theta$ generated by $A$ with the relations
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\[ a\theta(a) = 1 \text{ for every } a \in A. \] Thus \( \theta(a) = a^{-1} \) for \( a \in A \). The set \( A \) is called a natural set of generators of \( G_\theta \).

When \( \theta \) has no fixed point, we can set \( A = B \cup B^{-1} \) by choosing a set of representatives of the orbits of \( \theta \) for the set \( B \). The group \( G_\theta \) is then the free group on \( B \), denoted \( F_B \).

In general, the group \( G_\theta \) is a free product of a free group and a finite number of copies of \( \mathbb{Z}/2\mathbb{Z} \), that is \( G_\theta = \mathbb{Z}^* \ast (\mathbb{Z}/2\mathbb{Z})^j \) where \( i \) is the number of orbits of \( \theta \) with two elements and \( j \) the number of its fixed points. Such a group will be called a specular group of type \((i,j)\). These groups are very close to free groups, as we will see. The integer \( \text{Card}(A) = 2i + j \) is called the symmetric rank of the specular group \( \mathbb{Z}^* \ast (\mathbb{Z}/2\mathbb{Z})^j \). Two specular groups are isomorphic if and only if they have the same type. Indeed, the commutative image of a group of type \((i,j)\) is \( \mathbb{Z}^i \times (\mathbb{Z}/2\mathbb{Z})^j \) and the uniqueness of \( i,j \) follows from the fundamental theorem of finitely generated Abelian groups (see Appendix D).

**Example 8.3.1** Let \( A = \{a, b, c, d\} \) and let \( \theta \) be the involution which exchanges \( b, d \) and fixes \( a, c \). Then \( G_\theta = \mathbb{Z}^* \ast (\mathbb{Z}/2\mathbb{Z})^2 \) is a specular group of symmetric rank 4.

The Cayley graph of a specular group \( G_\theta \) with respect to the set of natural generators \( A \) is a regular tree where each vertex has degree \( \text{Card}(A) \). The specular groups are actually characterized by this property.

By the Kurosh Subgroup Theorem, any subgroup of a free product \( G_1 \ast G_2 \ast \cdots \ast G_n \) is itself a free product of a free group and of groups conjugate to subgroups of the \( G_i \) (see Appendix D). Thus, we have, replacing the Nielsen-Schreier Theorem of free groups, the following result.

**Theorem 8.3.2** Any subgroup of a specular group is specular.

It also follows from the Kurosh Theorem that the elements of order 2 in a specular group \( G_\theta \) are the conjugates of the \( j \) fixed points of \( \theta \) and this number is thus the number of conjugacy classes of elements of order 2. Indeed, an element of order 2 generates a subgroup conjugate to one the subgroups generated by the letters.

A word on the alphabet \( A \) is \( \theta \)-reduced (or simply reduced) if it has no factor of the form \( a\theta(a) \) for \( a \in A \). It is clear that any element of a specular group is represented by a unique reduced word.

A subset of a group \( G \) is called symmetric if it is closed under taking inverses. A set \( X \) in a specular group \( G \) is called a monoidal basis of \( G \) if it is symmetric, if the monoid that it generates is \( G \) and if any product \( x_1x_2\cdots x_m \) of elements of \( X \) such that \( x_kx_{k+1} \neq 1 \) for \( 1 \leq k \leq m - 1 \) is distinct of 1. The alphabet \( A \) is a monoidal basis of \( G_\theta \) and the symmetric rank of a specular group is the cardinality of any monoidal basis (two monoidal bases have the same cardinality since the type is invariant by isomorphism).

Let \( H \) be a subgroup of a specular group \( G \). Let \( Q \) be a set of reduced words on \( A \) which is a prefix-closed set of representatives of the right cosets \( Hg \) of \( H \).
Such a set is traditionally called a **Schreier transversal** for $H$ (the proof of its existence is classical in the free group and it is the same in any specular group).

Let 

$$
U = \{ paq^{-1} \mid a \in A, p, q \in Q, pa \notin Q, pa \in Hq \}. \tag{8.3.1}
$$

Each word $x$ of $U$ has a unique factorization $paq^{-1}$ with $p, q \in Q$ and $a \in A$. The letter $a$ is called the **central part** of $x$. The set $U$ is a monoidal basis of $H$, called the **Schreier basis** relative to $Q$ (the proof is the same as in the free group, see Appendix D).

One can deduce directly Theorem 8.3.2 from these properties of $U$. Indeed, let $\varphi : B \to U$ be a bijection from a set $B$ onto $U$ which extends to a morphism from $B^*$ onto $H$. Let $\sigma : B \to B$ be the involution sending each $b$ to $c$ where $\varphi(c) = \varphi(b)^{-1}$. Since the central parts never cancel, if a nonempty word $w \in B^*$ is $\sigma$-reduced then $\varphi(w) \neq 1$. This shows that $H$ is isomorphic to the group $G_\sigma$. Thus $H$ is specular.

If $H$ is a subgroup of index $n$ of a specular group $G$ of symmetric rank $r$, the symmetric rank $s$ of $H$ is

$$
 s = n(r - 2) + 2. \tag{8.3.2}
$$

This formula replaces Schreier’s Formula (which corresponds to the case $j = 0$). It can be proved as follows. Let $Q$ be a Schreier transversal for $H$ and let $U$ be the corresponding Schreier basis. The number of elements of $U$ is $nr - 2(n - 1)$. Indeed, this is the number of pairs $(p, a) \in Q \times A$ minus the $2(n - 1)$ pairs $(p, a)$ such that $pa \in Q$ with $pa$ reduced or $pa \in Q$ with $pa$ not reduced. This gives Formula (8.3.2).

**Example 8.3.3** Let $G$ be the specular group of Example 8.3.1. Let $H$ be the subgroup formed by the elements represented by a reduced word of even length. The set $Q = \{ 1, a \}$ is a prefix-closed set of representatives of the two cosets of $H$. The representation of $G$ by permutations on the cosets of $H$ is represented in Figure 8.3.1. The monoidal basis corresponding to Formula

$$
\begin{align*}
& a, b, c, d \\
& 1 & 2 \\
& a, b, c, d
\end{align*}
$$

Figure 8.3.1: The representation of $G$ by permutations on the cosets of $H$.

(8.3.1) is $U = \{ ab, ac, ad, ba, ca, da \}$. The symmetric rank of $H$ is 6, in agreement with Formula (8.3.2) and $H$ is a free group of rank 3.

**Example 8.3.4** Let again $G$ be the specular group of Example 8.3.1. Consider now the subgroup $K$ stabilizing 1 in the representation of $G$ by permutations on the set $\{ 1, 2 \}$ of Figure 8.3.2. We choose $Q = \{ 1, b \}$. The set $U$ corresponding to Formula (8.3.1) is $U = \{ a, bad, bb, bcd, c, dd \}$. The group $K$ is isomorphic to $\mathbb{Z} \ast (\mathbb{Z} / 2\mathbb{Z})^4$. 
Any specular group $G = G_\theta$ has a free subgroup of index 2. Indeed, let $H$ be the subgroup formed of the reduced words of even length. It has clearly index 2. It is free because it does not contain any element of order 2 (such an element is conjugate of a fixed point of $\theta$ and thus is of odd length).

We will need two more properties of specular groups. Both are well-known to hold for free groups (see Appendix D).

A group $G$ is called residually finite if for every element $g \neq 1$ of $G$, there is a morphism $\varphi$ from $G$ onto a finite group such that $\varphi(g) \neq 1$.

**Proposition 8.3.5** Any specular group is residually finite.

**Proof.** Let $K$ be a free subgroup of index 2 in the specular group $G$. Let $g \neq 1$ be in $G$. If $g \notin K$, then the image of $g$ in $G/K$ is nontrivial. Assume $g \in K$. Since $K$ is free, it is residually finite. Let $N$ be a normal subgroup of finite index of $K$ such that $g \notin N$. Consider the representation of $G$ on the right cosets of $N$. Since $g \notin N$, the image of $g$ in this finite group is nontrivial. ■

A group $G$ is said to be Hopfian if every surjective morphism from $G$ onto $G$ is also injective. By a result of Malcev, any finitely generated residually finite group is Hopfian. We thus deduce from Proposition 8.3.5 that any specular group is Hopfian. As a consequence, we have the following result, which will be used later.

**Proposition 8.3.6** Let $G$ be a specular group of type $(i, j)$ and let $U \subset G$ be a symmetric set with $2i + j$ elements. If $U$ generates $G$, it is a monoidal basis of $G$.

**Proof.** Let $A$ be a set of natural generators of $G$. Considering the commutative image of $G$, we obtain that $U$ contains $j$ elements of order 2. Thus there is a bijection $\varphi$ from $A$ onto $X$ such that $\varphi(a^{-1}) = \varphi(a)^{-1}$ for every $a \in A$. The map $\varphi$ extends to a morphism from $G$ to $G$ which is surjective since $U$ generates $G$. Then $\varphi$ being surjective, it also injective since $G$ is Hopfian, and thus $U$ is a monoidal basis of $G$. ■

### 8.3.2 Specular shifts

We assume given an involution $\theta$ on the alphabet $A$ generating the specular group $G_\theta$. 
A symmetric, factorial and extendable set $S$ of reduced words on the alphabet $A$ is called a *laminary set* on $A$ relative to $\theta$. Thus the elements of a laminary set $S$ are elements of the specular group $G_\theta$ and the set $S$ is contained in $G_\theta$.

A *specular shift* is a shift space $X$ such that $L(X)$ is a laminary set on $A$ which is also dendric of characteristic 2. Thus, in a specular shift, the extension graph of every nonempty word is a tree and the extension graph of the empty word is a union of two disjoint trees. If $X$ is a specular shift, we also say that $L(X)$ is a *specular set*.

The following is a very simple example of a specular shift.

**Example 8.3.7** Let $A = \{a, b\}$ and let $\theta$ be the identity on $A$. Then the periodic shift formed of the infinite repetitions of $ab$ is a specular shift.

As a second example, we find every dendric shift giving rise to a specular shift (that we may consider as degenerate).

**Example 8.3.8** Let $A = B \cup B^{-1}$ be a symmetric alphabet and let $\theta : b \to b^{-1}$. For every dendric shift $Y$ on the alphabet $B$, the set $L(Y) \cup L(Y)^{-1}$ is a laminary set which is dendric of characteristic 2. Thus the shift $X$ such that $L(X) = L$ is specular.

The next example is more interesting.

**Example 8.3.9** Let $A = \{a, b, c, d\}$ and let $X = X(\sigma)$ where $\sigma : A^* \to A^*$ is defined by

$$\sigma(a) = ab, \quad \sigma(b) = cda, \quad \sigma(c) = cd, \quad \sigma(d) = abc.$$  

We have already seen that $X$ is eventually dendric of characteristic 2 (Example 8.3.14). We will see later (Example 8.3.15) that $X$ is specular relative to the involution $\theta = (bd)$.

The following result shows in particular that in a specular shift the two trees forming $\mathcal{E}(\varepsilon)$ are isomorphic since they are exchanged by the bijection $(a, b) \to (b^{-1}, a^{-1})$. To distinguish the disjoint copies if $L(w)$ and $R(w)$ forming the vertices of the extension graph $\mathcal{E}(w)$, we denote them by $1 \otimes L(w)$ and $R(w) \otimes 1$.

**Proposition 8.3.10** Let $X$ be a specular shift. Let $T_0, T_1$ be the two trees such that $\mathcal{E}_X(\varepsilon) = T_0 \cup T_1$. For any $a, b \in A$ and $i = 0, 1$, one has $(1 \otimes a, b \otimes 1) \in T_i$ if and only if $(1 \otimes b^{-1}, a^{-1} \otimes 1) \in T_{1-i}$.

**Proof.** Assume that $(1 \otimes a, b \otimes 1)$ and $(1 \otimes b^{-1}, a^{-1} \otimes 1)$ are both in $T_0$. Since $T_0$ is a tree, there is a path from $1 \otimes a$ to $a^{-1} \otimes 1$. We may assume that this path is reduced, that is, does not use consecutively twice the same edge. Since this path is of odd length, it has the form $(u_0, v_1, u_1, \ldots, u_p, v_p)$ with $u_0 = 1 \otimes a$ and $v_p = a^{-1} \otimes 1$. Since $L(X)$ is symmetric, we also have a reduced path $(v_p^{-1}, u_p^{-1}, \ldots, u_1^{-1}, u_0^{-1})$ which is in $\mathcal{E}(\varepsilon)$ (for $u_i = 1 \otimes a_i$, we
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denote $v_i^{-1} = v_i^{-1} \otimes 1$ and similarly for $v_i^{-1}$ and thus in $T_0$ since $T_0, T_1$ are disjoint. Since $v_p^{-1} = u_0$, these two paths have the same origin and end. But if a path of odd length is its own inverse, its central edge has the form $(x, y)$ with $x = y^{-1}$, as one verifies easily by induction on the length of the path. This is a contradiction with the fact that the words of $\mathcal{L}(X)$ are reduced. Thus the two paths are distinct. This implies that $\mathcal{E}(\varepsilon)$ has a cycle, a contradiction.  

We say that a laminary set $S$ is orientable if there exist two factorial sets $S_+, S_-$ such that $S = S_+ \cup S_-$ with $S_+ \cap S_- = \{\varepsilon\}$ and for any $x \in S$, one has $x \in S_-$ if and only if $x^{-1} \in S_+$ (where $x^{-1}$ is the inverse of $x$ in $G_\theta$).

The following result shows in particular that for any dendric shift $X$ on the alphabet $B$, the set $\mathcal{L}(X) \cup (\mathcal{L}(X)^{-1})$ is a specular set on the alphabet $A = B \cup B^{-1}$.

**Theorem 8.3.11** Let $X$ be a specular shift on the alphabet $A$. Then, $\mathcal{L}(X)$ is orientable if and only if there is a partition $A = A_+ \cup A_-$ of the alphabet $A$ and a dendric shift $Y$ on the alphabet $B = A_+ \cup A_-$ such that $\mathcal{L}(X) = \mathcal{L}(Y) \cup (\mathcal{L}(Y)^{-1})$.

**Proof.** Let $X$ be a specular shift on the alphabet $A$ which is orientable. Let $(S_+, S_-)$ be the corresponding pair of subsets of $S = \mathcal{L}(X)$. The sets $S_+, S_-$ are biextendable, since $S$ is. Set $A_+ = A \cap S_+$ and $A_- = A \cap S_-$. Then $A = A_+ \cup A_-$ is a partition of $A$ and, since $S_+, S_-$ are factorial, we have $S_+ \subset A_+$ and $S_- \subset A_-$. Let $T_0, T_1$ be the two trees such that $\mathcal{E}(\varepsilon) = T_0 \cup T_1$. Assume that a vertex of $T_0$ is in $A_+$. Then all vertices of $T_0$ are in $A_+$ and all vertices of $T_1$ are in $A_-$. Moreover, $\mathcal{E}_{S_+}(\varepsilon) = T_0$ and $\mathcal{E}_{S_-}(\varepsilon) = T_1$. Thus $S_+ = \mathcal{L}(Y)$ with $Y$ a dendric shift and $S_- = \mathcal{L}(Y)^{-1}$.

The following result follows easily from Proposition 2.2.10.

**Proposition 8.3.12** The factor complexity of a specular shift containing the alphabet $A$ is $p_n = n(k - 2) + 2$ for $n \geq 1$ with $k = \text{Card}(A)$.

8.3.3 Doubling maps

We now introduce a construction which allows one to build specular shifts.

A transducer is a labeled graph with vertices in a set $Q$ and edges labeled in $\Sigma \times A$. The set $Q$ is called the set of states, the set $\Sigma$ is called the input alphabet and $A$ is called the output alphabet. The graph obtained by erasing the output letters is called the input automaton. Similarly, the output automaton is obtained by erasing the input letters.

Let $A$ be a transducer with set of states $Q = \{0, 1\}$ on the input alphabet $\Sigma$ and the output alphabet $A$. We assume that

1. Every letter of $\Sigma$ acts on $Q$ as a permutation.

2. the output labels of the edges are all distinct.
We define two maps \( \delta_0, \delta_1 : \Sigma^* \to A^* \) corresponding to the choice of 0 and 1 respectively as initial vertices. Thus \( \delta_0(u) = v \) (resp. \( \delta_1(u) = v \)) if the path starting at state 0 (resp. 1) with input label \( u \) has output \( v \). The pair \( \delta = (\delta_0, \delta_1) \) is called a doubling map and the transducer \( A \) a doubling transducer. The image of a set \( T \in a \) on the alphabet \( \Sigma \) by the doubling map \( \delta \) is the set \( S = \delta_0(T) \cup \delta_1(T) \).

If \( A \) is a doubling transducer, we define an involution \( \theta_A \) as follows. For any \( a \in A \), let \( (i, \alpha, a, j) \) be the edge with input label \( \alpha \) and output label \( a \). We define \( \theta_A(a) \) as the output label of the edge starting at \( 1 - j \) with input label \( \alpha \). Thus, \( \theta_A(a) = \delta_1(\alpha) = a \) if \( i + j = 1 \) and \( \theta_A(a) = \delta_{1-i}(\alpha) \neq a \) if \( i = j \).

Recall that the reversal of a word \( w = a_1a_2 \cdots a_n \) is the word \( \hat{w} = a_n \cdots a_2a_1 \). A set \( S \) of words is closed under reversal if \( w \in S \) implies \( \hat{w} \in S \) for every \( w \in S \).

**Theorem 8.3.13** For any dendric shift \( X \) on the alphabet \( \Sigma \), such that \( \mathcal{L}(X) \) is closed under reversal and any doubling map \( \delta \), the image of \( \mathcal{L}(X) \) by \( \delta \) is a specular set relative to the involution \( \theta_A \).

*Proof.* Set \( T = \mathcal{L}(X) \) and \( S = \delta_0(T) \cup \delta_1(T) \). The set \( S \) is clearly biextendable. Assume that \( x = \delta_i(y) \) for \( i \in \{0, 1\} \) and \( y \in T \). Let \( j \) be the end of the path starting at \( i \) and with input label \( y \). Since each letter acts on the two elements of \( Q \) as the identity or as a transposition, there is a path labeled \( \hat{y} \) from \( j \) to \( i \) and also a path labeled \( \hat{y} \) from \( 1 - j \) to \( 1 - i \). Thus \( x^{-1} = \delta_{1-j}(\hat{y}) \). Since \( T \) is closed under reversal, \( x^{-1} \in \delta_{1-j}(T) \). This shows that \( S \) is symmetric and that it is laminary.

Next, for any nonempty word \( x = \delta_i(y) \), the graph \( \mathcal{E}_S(x) \) is isomorphic to the graph \( \mathcal{E}_T(y) \). Indeed, let \( j \) be the end of the path with origin \( i \) and input label \( y \). For \( a, b \in A \), one has \( axb \in S \) if and only if \( cyd \in T \) where \( c \) (resp. \( d \) is the input label of the edge with output label \( a \) (resp. \( b \) ending in \( i \) (resp. with origin \( j \)).

Finally, consider the map \( \pi \) from \( S \cap A^2 \) onto \( \{0, 1\} \) which assigns to \( ab \in S \cap A^2 \) the state \( i \) which is the end of the edge of \( A \) with output label \( a \) (and the origin of the edge with output label \( b \)). Set \( S_i = \pi^{-1}(i) \). We have a partition \( S \cap A^2 = S_0 \cup S_1 \) such that each \( S_i \) is isomorphic to \( \mathcal{E}_T(\varepsilon) \) and moreover \( ab \in S_i \) if and only if \( (ab)^{-1} \in S_{1-i} \). Thus \( S \) is specular.

We now give several examples of specular shifts obtained by a doubling map. The first one is obtained by doubling the Fibonacci shift.

**Example 8.3.14** Let \( \Sigma = \{\alpha, \beta\} \) and let \( X \) be the Fibonacci shift. Let \( \delta \) be the doubling map given by the transducer of Figure 8.3.3 on the left.

Then \( \theta_A \) is the involution \( \theta \) of Example 8.3.1 and the image of \( \mathcal{L}(X) \) by \( \delta \) is a specular set \( S \) on the alphabet \( A = \{a, b, c, d\} \). The graph \( \mathcal{E}_S(\varepsilon) \) is represented in Figure 8.3.3 on the right.

Note that \( S \) is the set of factors of the fixed point \( g^2(a) \) of the morphism

\[
g : a \mapsto abca, \quad b \mapsto cda, \quad c \mapsto cdac, \quad d \mapsto abc.
\]
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The morphism $g$ is obtained by applying the doubling map to the cube $f^3$ of the Fibonacci morphism $f$ in such a way that $g^0(a) = \delta_0(f^0(a))$.

In the next example (due to Julien Cassaigne), the specular set is obtained using a morphism of smaller size.

Example 8.3.15 Let $A = \{a, b, c, d\}$. Let $T$ be the set of factors of the fixed point $x = f^\omega(\alpha)$ of the morphism $f : \alpha \mapsto \alpha\beta, \beta \mapsto \alpha\beta\alpha$. It is a Sturmian set. Indeed, $x$ is the characteristic sequence of slope $-1 + \sqrt{2}$ (see Section 2.7). The sequence $s_n = f^n(\alpha)$ satisfies $s_n = s_{n-1}s_{n-2}$ for $n \geq 2$. The image $S$ of $T$ by the doubling automaton of Figure 8.3.3 is the set of factors of the fixed point $\sigma^\omega(a)$ of the morphism $\sigma$ from $A^*$ into itself defined by

$$
\sigma(a) = ab, \quad \sigma(b) = cda, \quad \sigma(c) = cd, \quad \sigma(d) = abc.
$$

Thus the set $S$ is the same as that of Example 8.1.4.

8.3.4 Odd and even words

We introduce a notion which plays, as we shall see, an important role in the study of specular shifts. Let $X$ be a specular shift on the alphabet $A$ with $A \subset L(X)$. Any letter $a \in A$ occurs exactly twice as a vertex of $E(\varepsilon)$, one as an element of $L(\varepsilon)$ and one as an element of $R(\varepsilon)$. A letter $a \in A$ is said to be even if its two occurrences appear in the same tree. Otherwise, it is said to be odd. Observe that if a specular shift $X$ is recurrent, there is at least one odd letter.

Example 8.3.16 Let $X$ be the shift of period $ab$ as in Example 8.3.7. Then $a$ and $b$ are odd.

A word $w \in S$ is said to be even if it has an even number of odd letters. Otherwise it is said to be odd. The set of even words has the form $U^* \cap S$ where $U \subset S$ is a bifix code, called the even code. The set $U$ is the set of even words without a nonempty even prefix (or suffix).

Proposition 8.3.17 Let $X$ be a minimal specular shift. The even code is a finite bifix code which is an $X$-maximal prefix and suffix code.

Proof. The even code $U$ is bifix by definition. To prove that it is an $X$-maximal prefix code, let us verify that any $w \in S$ is comparable for the prefix order with
an element of the even code $X$. If $w$ is even, it is in $U^*$. Otherwise, since $S$ is recurrent, there is a word $u$ such that $wuw \in S$. If $u$ is even, then $wuw$ is even and thus $wuw \in U^*$. Otherwise $wu$ is even and thus $wu \in U^*$. This shows that $U$ is $X$-maximal. A word of the form $awb$ with $a, b$ odd and $w$ even cannot be an internal factor of $U$. Indeed, if $pawbq$ is even, either $p, q$ are even and then $p, awb, q$ are in $U^*$ or $p, q$ are odd and $pa, w, bq$ are in $U^*$. Since $X$ is minimal, this implies that $U$ is finite. 

**Example 8.3.18** Let $X$ be the specular shift of Example 8.1.4. The letters $b, d$ are even and the letters $a, c$ are odd. The even code is

$$U = \{abc, ac, b, ca, cda, d\}.$$ 

Denote by $T_0, T_1$ the two trees such that $E(\varepsilon) = T_0 \cup T_1$. We consider the directed graph $G$ with vertices 0, 1 and edges all the triples $(i, a, j)$ for $0 \leq i, j \leq 1$ and $a \in A$ such that $(1 \otimes b, a \otimes 1) \in T_i$ and $(1 \otimes a, c \otimes 1) \in T_j$ for some $b, c \in A$. The graph $G$ is called the parity graph of $S$. Observe that for every letter $a \in A$ there is exactly one edge labeled $a$ because $a$ appears exactly once as a left (resp. right) vertex in $E(\varepsilon)$.

Note that, when $X$ is a specular shift obtained by a doubling map using a transducer $A$, the parity graph of $X$ is the output automaton of $A$.

**Example 8.3.19** Let $X$ be the specular shift of Example 8.3.14. The parity graph of $X$ is represented in Figure 8.3.4. It is the output automaton of the doubling transducer of Figure 8.3.3.

**Proposition 8.3.20** Let $X$ be a specular shift and let $G$ be its parity graph. Let $S_{i,j}$ be the set of words in $S = \mathcal{L}(X)$ which are the label of a path from $i$ to $j$ in the graph $G$.

1. The family $(S_{i,j} \setminus \{\varepsilon\})_{0 \leq i, j \leq 1}$ is a partition of $S \setminus \{\varepsilon\}$.
2. For $u \in S_{i,j} \setminus \{\varepsilon\}$ and $v \in S_{k,l} \setminus \{\varepsilon\}$, if $uv \in S$, then $j = k$.
3. $S_{0,0} \cup S_{1,1}$ is the set of even words.
4. $S_{i,j}^{-1} = S_{1-j,1-i}$. 
Proof. We first note that for \( a, b \in A \) such that \( ab \in S \), there is a path in \( G \) labeled \( ab \). Since \( (a, b) \in E(\varepsilon) \), there is a \( k \) such that \((1 \otimes a, b \otimes 1) \in T_k \). Then we have \( a \in S_{i,k} \) and \( b \in S_{k,j} \) for some \( i, j \in \{0, 1\} \). This shows that \( ab \) is the label of a path from \( i \) to \( j \) in \( G \).

Let us prove by induction on the length of a nonempty word \( w \in S \) that there exists a unique pair \( i, j \) such that \( w \in S_{i,j} \). The property is true for a letter, by definition of the extension graph \( E(\varepsilon) \) and for words of length 2 by the above argument. Let next \( w = ax \) be in \( S \) with \( a \in A \) and \( x \) nonempty. By induction hypothesis, there is a unique pair \( (k, j) \) such that \( x \in S_{k,j} \). Let \( b \) be the first letter of \( x \). Then the edge of \( G \) with label \( b \) starts in \( k \). Since \( ab \) is the label of a path, we have \( a \in S_{i,k} \) for some \( i \) and thus \( ax \in S_{i,j} \). The other assertions follow easily (Assertion (4), follows from Proposition 8.3.10).

Note that Assertion (4) implies that no nonempty even word is its own inverse. Indeed, \( S_{0,0}^{-1} = S_{1,1} \) and \( S_{1,1}^{-1} = S_{0,0} \).

**Proposition 8.3.21** Let \( X \) be a specular shift and let \( S = T(X) \). If \( x, y \in S \) are nonempty words such that \( yx^{-1} \in S \), then \( y \) is odd.

**Proof.** Let \( i, j \) be such that \( x \in S_{i,j} \). Then \( x^{-1} \in S_{1-j-1} \) by Assertion (4) of Proposition 8.3.20 and thus \( y \in S_{j,1-j} \) by Assertion (2). Thus \( y \) is odd by Assertion (3).

Recall that for a shift space \( X \), a finite bifix code \( U \subset T(X) \) which is an \( X \)-maximal prefix and suffix code and a coding morphism \( f \) for \( U \), the shift space \( Y \) such that \( L(Y) = f^{-1}(L(X)) \) is called a decoding of \( X \) by \( U \). We denote \( Y = f^{-1}(X) \).

**Theorem 8.3.22** The decoding of a minimal specular shift by the even code is a union of two minimal dendric shifts. More precisely, let \( X \) be a minimal specular shift and let \( f \) be a coding morphism for the even code. The shifts \( Y_0, Y_1 \) such that \( L(Y_0) = f^{-1}(S_{0,0}) \) and \( L(Y_1) = f^{-1}(S_{1,1}) \) are isomorphic minimal dendric shifts.

**Proof.** We show that the shift \( Y_0 \) such that \( L(Y_0) = f^{-1}(S_{0,0}) \) is a minimal dendric shift. The proof for \( f^{-1}(S_{1,1}) \) is the same. Set \( T_0 = L(Y_0) \).

Set \( S = T(X) \). Since \( X \) is minimal, for every \( u \in S \), there exists \( n \geq 1 \) such that \( u \) is a factor of any word \( w \) in \( S \) of length \( n \). But if \( u, w \in S_{0,0} \) are such that \( w = tu \), then \( t, r \in S_{0,0} \). Thus \( Y_0 \) is minimal.

We now show that \( Y_0 \) is dendric. Let \( U \) be the even code. Set \( U_0 = U \cap S_{0,0} \), \( U_1 = U \cap S_{1,1} \) and \( \mathcal{E}_0(w) = \mathcal{E}_{U_0}(w) \). It is enough to show that the graph \( \mathcal{E}_0(w) \) is a tree for any \( w \in S_{0,0} \).

Assume first that \( w \) is nonempty. Note first that \( \mathcal{E}_0(w) = \mathcal{E}_{U,U}(w) \). Indeed, if \( x, y \in U \) are such that \( xwy \in S \), one has \( x, y \in U_0 \) and \( xwy \in S_{0,0} \). But the graph \( \mathcal{E}_{U,U}(w) \) is a tree by Proposition 8.1.12.

Suppose now that \( w = \varepsilon \). First, since \( \mathcal{E}(\varepsilon) \) is a union of two trees, it is acyclic, and thus the graph \( \mathcal{E}_0(\varepsilon) \) is acyclic by Proposition 8.1.10. Next, since
every nonempty word in $S$ is neutral, by Lemma [8.1.13] we have $m_{U,U}(\varepsilon) = m(\varepsilon) = -1$. This implies that $E_{U,U}(\varepsilon)$ is a union of two trees. Since $E_{U,U}(\varepsilon)$ is the disjoint union of $E_{0}(\varepsilon)$ and $E_{U_1,U_1}(\varepsilon)$, this implies that each one is a tree.

Clearly, $Y_0$ and $Y_1$ are isomorphic. Indeed, let $f : B^* \to A^*$ be a coding morphism for $U$. Set $B_0 = f^{-1}(U_0)$ and $B_1 = f^{-1}(U_1)$. Then $\alpha : B_0 \to B_1$ defined by $\alpha(b) = f^{-1}(b^{-1})$ defines an isomorphism from $Y_0$ onto $Y_1$. 

Note that the decoding of a dendric shift $X$ by an $X$-maximal prefix and suffix code is again dendric (Exercise [8.17]).

**Example 8.3.23** Let $X$ be the shift space of Example [8.1.4] We have seen that it generated by the morphism

$$\sigma : a \mapsto ab, \ b \mapsto cda, \ c \mapsto cd, \ d \mapsto abc.$$ 

The even code $U$ is given in Example [8.3.18] Let $\Sigma = \{a, b, c, d, e, f\}$ and let $g$ be the coding morphism for $X$ given by

$$a \mapsto abc, \ b \mapsto ac, \ c \mapsto b, \ d \mapsto ca, \ e \mapsto cda, \ f \mapsto d.$$ 

The decoding of $X$ by $U$ is a union of two dendric shifts which generated by the two morphisms

$$a \mapsto afbf, \ b \mapsto af, \ f \mapsto a$$ 

and

$$c \mapsto e, \ d \mapsto ec, \ e \mapsto ecdc$$

These two morphisms are actually the restrictions to $\{a, b, f\}$ and $\{c, d, e\}$ of the morphism $g^{-1}\sigma g$.

### 8.3.5 Complete return words

Let $X$ be a shift space and let $U \subset \mathcal{L}(X)$ be a bifix code. An internal factor of a word $u$ is a word $v$ such that $u = pvs \in U$ for nonempty words $p, s$. A complete return word to $U$ is a word of $\mathcal{L}(X)$ with a proper prefix in $U$, a proper suffix in $U$ but no internal factor in $U$. We denote by $CR_X(U)$ the set of complete return words to $U$.

The set $CR_X(U)$ is a bifix code. If $X$ is minimal, $CR_X(U)$ is finite for any finite set $U$. For $x \in \mathcal{L}(X)$, we denote $CR_X(x)$ instead of $CR_X(\{x\})$.

**Example 8.3.24** Let $X$ be the specular shift of Example [8.3.14] One has

$$CR_X(a) = \{abca, abcd, acda\}$$
$$CR_X(b) = \{bcab, bcdacdb, bcdacdacdb\}$$
$$CR_X(c) = \{cabc, cdabc, cda\}$$
$$CR_X(d) = \{dabcabcab, dababcd, dacd\}.$$
The kernel of a bifix code $U$ is the set of words of $U$ which are factor of another word in $U$. The following result is a generalization of Theorem 8.1.15. The proof is very similar (Exercise 8.18).

**Theorem 8.3.25** Let $X$ be a minimal specular shift containing the alphabet $A$. For any finite nonempty bifix code $U \subset S$ with empty kernel, we have

$$\text{Card}(CR_X(U)) = \text{Card}(U) + \text{Card}(A) - 2.$$ \hfill (8.3.3)

The following example illustrates Theorem 8.3.25.

**Example 8.3.26** Let $X$ be the specular shift on the alphabet $A = \{a, b, c, d\}$ of Example 8.1.4. We have

$$CR_X(\{a, b\}) = \{ab, acda, bca, bcda\}.$$

It has four elements in agreement with Theorem 8.3.25.

### 8.3.6 Right return words

We now come to right return words in specular shifts. Note that when $S$ is a laminary set $R_S(x)^{-1} = R'_S(x^{-1})$.

**Proposition 8.3.27** Let $X$ be a specular shift and let $u \in L(X)$ be a nonempty word. All the words of $R_X(u)$ are even.

**Proof.** If $w \in R_X(u)$, we have $uw = vu$ for some $v \in L(X)$. If $u$ is odd, assume that $u \in S_{0,1}$. Then $w \in S_{1,1}$. Thus $w$ is even. If $u$ is even, assume that $u \in S_{0,0}$. Then $w \in S_{0,0}$ and $w$ is even again. $\blacksquare$

We now establish the following result, which replaces, for specular shifts, Theorem 8.1.15 for dendric shifts.

**Theorem 8.3.28** Let $X$ be a minimal specular shift. For any $u \in L(X)$, the set $R_X(u)$ has $\text{Card}(A) - 1$ elements.

**Proof.** This follows directly from Theorem 8.3.25 with $U = \{u\}$ since $\text{Card}(R_X(u)) = \text{Card}(CR_X(u))$. $\blacksquare$

**Example 8.3.29** Let $X$ be the specular shift of Example 8.3.14. We have

$$R_X(a) = \{bca, bcda, cda\},$$
$$R_X(b) = \{cab, cdacdb, cdacdacdb\},$$
$$R_X(c) = \{abc, dabc, dac\},$$
$$R_X(d) = \{abcabcd, abcabcabcd, acd\}.$$
8.3.7 Mixed return words

Let $S$ be a laminar set. For $w \in S$ such that $w \neq w^{-1}$, we consider complete return words to the set $X = \{w, w^{-1}\}$.

**Example 8.3.30** Let $X$ be the substitution shift generated by the substitution $f : a \rightarrow cb^{-1}, b \rightarrow c, c \rightarrow ab^{-1}$. We shall verify later that $X$ is specular (Example 9.4.4). We have

\[
CR_S(\{a, a^{-1}\}) = \{ab^{-1}cba^{-1}, ab^{-1}bca^{-1}a, a^{-1}cb^{-1}c^{-1}a, ab^{-1}c^{-1}ba^{-1}, a^{-1}c^{-1}ab^{-1}c^{-1}c^{-1}ba^{-1}\}
\]

\[
CR_S(\{b, b^{-1}\}) = \{ba^{-1}cb, ba^{-1}c^{-1}b, bc^{-1}ab^{-1}, b^{-1}cb, b^{-1}c^{-1}ab^{-1}, b^{-1}c^{-1}b\},
\]

\[
CR_S(\{c, c^{-1}\}) = \{cba^{-1}c, cbc^{-1}, cb^{-1}c^{-1}, c^{-1}ab^{-1}c, c^{-1}ab^{-1}c^{-1}, c^{-1}ba^{-1}c\}.
\]

The following result shows that, at the cost of taking return words to a set of two words, we recover a situation similar to that of dendric shifts.

**Theorem 8.3.31** Let $X$ be a minimal specular shift on the alphabet $A$ such that $A \subset \mathcal{L}(X)$. For any $w \in \mathcal{L}(X)$ such that $w \neq w^{-1}$, the set of complete return words to $\{w, w^{-1}\}$ has $\text{Card}(A)$ elements.

*Proof.* The statement results directly of Theorem 8.3.25. □

**Example 8.3.32** Let $X$ be the specular shift of Example 8.3.14. In view of the values of $CR_S(b)$ and $CR_S(d)$ given in Example 8.3.24, we have

\[
CR_S(\{b, d\}) = \{bcab, bcd, dab, dacd\}.
\]

Two words $u, v$ are said to overlap if a nonempty suffix of one of them is a prefix of the other. In particular a nonempty word overlaps with itself.

We now consider the return words to $\{w, w^{-1}\}$ with $w$ such that $w$ and $w^{-1}$ do not overlap. This is true for every $w$ in a laminar set $S$ where the involution $\theta$ has no fixed point (in particular when $X$ is the natural coding of a linear involution, as we shall see). In this case, the group $G_\theta$ is free and for any $w \in S$, the words $w$ and $w^{-1}$ do not overlap.

With a complete return word $u$ to $\{w, w^{-1}\}$, we associate a word $N(u)$ obtained as follows. If $u$ has $w$ as prefix, we erase it and if $u$ has a suffix $w^{-1}$, we also erase it. Note that these two operations can be made in any order since $w$ and $w^{-1}$ cannot overlap.

The mixed return words to $w$ are the words $N(u)$ associated with complete return words $u$ to $\{w, w^{-1}\}$. We denote by $MR_X(w)$ the set of mixed return words to $w$ in $X$.

Note that $MR_X(w)$ is symmetric and that $wMR_X(w)w^{-1} = MR_S(w^{-1})$. Note also that if $S$ is orientable, then

\[
MR_X(w) = R_X(w) \cup R_X(w)^{-1} = R_X(w) \cup R'_X(w^{-1}).
\]
Example 8.3.33 Let $X$ be the substitution shift generated by the morphism $f : a \rightarrow cb^{-1}, b \rightarrow c, c \rightarrow ab^{-1}$ extended to an automorphism of the free group on $\{a, b, c\}$. We shall see later that $X$ is actually specular (Example 9.4.4). We have

$$M_{RS}(a) = \{b^{-1}cb, b^{-1}cb^{-1}a, a^{-1}cb^{-1}a, b^{-1}c^{-1}b, a^{-1}cb^{-1}a, a^{-1}cb^{-1}c^{-1}b\}$$

$$M_{RS}(b) = \{a^{-1}cb, a^{-1}c, a^{-1}bcb, b^{-1}c^{-1}a, b^{-1}c^{-1}b\},$$

$$M_{RS}(c) = \{ba^{-1}c, b^{-1}, c^{-1}ab^{-1}c, c^{-1}ab^{-1}, c^{-1}ba^{-1}\}.$$

Observe that any uniformly recurrent biinfinite word $x$ such that $F(x) = S$ can be uniquely written as a concatenation of mixed return words (see Figure 8.3.5). Note that successive occurrences of $w$ may overlap but that successive occurrences of $w$ and $w^{-1}$ cannot.

Figure 8.3.5: A uniformly recurrent infinite word factorized as an infinite product $\cdots rstu \cdots$ of mixed return words to $w$.

We have the following cardinality result.

**Theorem 8.3.34** Let $X$ be a minimal specular shift on the alphabet $A$ such that $A \subset L(X)$. For any $w \in S$ such that $w, w^{-1}$ do not overlap, the set $M_{RS}(w)$ has $\text{Card}(A)$ elements.

**Proof.** This is a direct consequence of Theorem 8.3.31 since $\text{Card}(M_{RS}(w)) = \text{Card}(C_{RS}(\{w, w^{-1}\})$ when $w$ and $w^{-1}$ do not overlap. \[\blacksquare\]

Note that the bijection between $C_{RS}(\{w, w^{-1}\})$ and $M_{RS}(w)$ is illustrated in Figure 8.3.5.

**Example 8.3.35** Let $X$ be the specular shift of Example 8.3.14. The value of $C_{RS}(\{b, d\})$ is given in Example 8.3.32. Since $b, d$ do not overlap, the set

$$M_{RS}(b) = \{cab, c, dac, dab\}$$

has four elements in agreement with Theorem 8.3.34.

### 8.3.8 The Return Theorem for specular shifts

By Theorem 8.3.14, the set of right return words to a given word in a minimal dendric shift on the alphabet $A$ such that $A \subset L(X)$ is a basis of the free group on $A$. We will see a counterpart of this result for specular shifts.

Let $S$ be a specular set. The even subgroup is the group formed by the even words. It is a subgroup of index 2 of $G_{\theta}$ with symmetric rank $2(\text{Card}(A) - 1)$
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by \( \text{(8.3.2)} \) generated by the even code. Since no even word is its own inverse (by Proposition \( \text{(8.3.20)} \)), it is a free group. Thus its rank is \( \text{Card}(A) - 1 \).

The following result replaces, for specular shifts, the Return Theorem of dendric shifts (Theorem \( \text{(8.1.14)} \)).

**Theorem 8.3.36** Let \( X \) be a minimal specular shift. For any \( w \in \mathcal{L}(X) \), the set of right return words to \( w \) is a basis of the even subgroup.

**Proof.** Set \( S = \mathcal{L}(X) \). We first consider the case where \( w \) is even. Let \( f : B^* \to A^* \) be a coding morphism for the even code \( U \subset S \). Consider the partition \( (S_{i,j}) \), as in Proposition \( \text{(8.3.20)} \), and set \( U_0 = U \cap S_{0,0} \), \( U_1 = U \cap S_{1,1} \). By Theorem \( \text{(8.3.22)} \) the shift \( f^{-1}(X) \) is the union of two minimal dendric shifts, \( Y_0 \) and \( Y_1 \) on the alphabets \( B_0 = f^{-1}(U_0) \) and \( B_1 = f^{-1}(U_1) \) respectively. We may assume that \( w \in S_{0,0} \). Then \( R_X(w) \) is the image by \( f \) of the set \( R = R_{Y_0}(f^{-1}(w)) \). By Theorem \( \text{(8.1.14)} \) the set \( R \) is a basis of the free group on \( B_0 \). Thus \( R_X(w) \) is a basis of the image of \( F_{B_0} \) by \( f \), which is the even subgroup.

Suppose now that \( w \) is odd. Since the even code is an \( X \)-maximal bifix code, there exists an odd word \( u \) such that \( uw \in S \). Then \( R_X(uw) \subset R_X(w)^* \). By what precedes, the set \( R_X(uw) \) generates the even subgroup and thus the group generated by \( R_X(w) \) contains the even subgroup. Since all words in \( R_X(w) \) are even, the group generated by \( R_X(w) \) is contained in the even subgroup, whence the equality. We conclude by Theorem \( \text{(8.3.28)} \).

**Example 8.3.37** Let \( X \) be the specular shift of Example \( \text{(8.3.14)} \). The sets of right return words to \( a, b, c, d \) are given in Example \( \text{(8.3.29)} \). Each one is a basis of the even subgroup.

Concerning mixed return words, we have the following statement.

**Theorem 8.3.38** Let \( X \) be a minimal specular shift. For any \( w \in \mathcal{L}(X) \) such that \( w, w^{-1} \) do not overlap, the set \( MR_X(w) \) is a monoidal basis of the group \( G_\theta \).

**Proof.** Since \( w \) and \( w^{-1} \) do not overlap, we have \( R_X(w) \subset MR_X(w)^* \). Thus, by Theorem \( \text{(8.3.30)} \) the group \( \langle MR_X(w) \rangle \) contains the even subgroup. But \( MR_X(w) \) always contains odd words. Indeed, assume that \( w \in S_{i,j} \). Then \( w^{-1} \in S_{1-j,1-i} \) and thus any \( u \in MR_X(w) \) such that \( uww^{-1} \in S \) is odd. Since the even group is a maximal subgroup of \( G_\theta \), this implies that \( MR_X(w) \) generates the group \( G_\theta \). Finally since \( MR_S(w) \) has \( \text{Card}(A) \) elements by Theorem \( \text{(8.3.34)} \) we obtain the conclusion by Proposition \( \text{(8.3.6)} \).

**Example 8.3.39** Let \( X \) be the specular shift of Example \( \text{(8.3.14)} \). We have seen in Example \( \text{(8.3.35)} \) that

\[
MR_X(b) = \{c, cab, dab, dac\}.
\]

This set is a monoidal basis of \( G_\theta \) in agreement with Theorem \( \text{(8.3.38)} \).
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8.3.9 Dimension groups of specular shifts.

We have the following description of dimension groups of minimal specular shifts. It shows that they are dimension groups of dendric shifts, except possibly for the order unit.

**Theorem 8.3.40** Let \( (X, S) \) be a minimal specular shift on a \( k \) letter alphabet \( A \). The dimension group of \( (X, S) \) is, as ordered group, isomorphic to the dimension group of a minimal dendric shift on \( k - 1 \) letters.

**Proof.** Let \( w \in \mathcal{L}(X) \). By Theorem 8.3.39, the set \( \mathcal{R}_X(w) \) is a basis of the even group. Let \( Y \) be the shift space induced by \( (X, S) \) on \([w]\). Assume that \( w \) is even and, for instance that \( w \in S_{0,0} \). Let \( f \) be a coding morphism for the even code \( U \) with \( U_0 = U \cap S_{0,0} \). Let \( Y_0 \) be the shift such that \( \mathcal{L}(Y_0) = f^{-1}(S_{0,0}) \). Then \( \mathcal{R}_X(w) \subseteq U_0^* \) and thus \( Y \subseteq Y_0 \). Since \( Y_0 \) is a minimal dendric shift by Proposition 8.3.22, we have \( Y = Y_0 \) and we obtain the conclusion that \( Y \) is a minimal dendric shift. Since \( \text{Card}(\mathcal{R}_X(w)) = k - 1 \), this completes the proof by Proposition 8.4.12.

**Example 8.3.41** Let \( X \) be the specular shift generated by the morphism \( \varphi : a \to ab, b \to cda, c \to cd, d \to abc \) (see Example 8.3.19). The set of return words to \( a \) is \( \mathcal{R}_X(a) = \{bec, bcda, cda\} \). It is a basis of the even group, itself generated by the even code \( U = \{abc, ac, ba, cda, cd, d\} \). Let \( f : \{ab, ac, bc, ba, ca, cd, da\} \to A_2 = \{u, v, w, x, y, z\} \) and let \( \varphi_2 : u \to uw, v \to wv, w \to yzv, x \to yz, y \to yz, z \to uw \) be the 2-block presentation of \( \varphi \). Let \( B = \{r, s, t\} \) and let \( \phi : B^* \to A_2^* \) be a coding morphism for \( f(a\mathcal{R}_X(a)) = \{uw, uwyz, vyz\} \). The morphism \( \tau : r \to st, s \to str, t \to sr \) is such that \( \varphi_2 \circ \phi = \phi \circ \tau \). The matrix \( M(\tau) \) is

\[
M(\tau) = \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

The matrix \( M(\tau) \) has eigenvalues \(-1, 1 - \sqrt{2}, 1 + \sqrt{2}\) which is its dominant eigenvalue. A row eigenvector corresponding to \( \lambda = 1 + \sqrt{2} \) is \( w = [1 \quad \sqrt{2} \quad 1] \).

The dimension group is thus \( G = \mathbb{Z}^3 \) with \( G^+ = \{(a_r, a_s, a_t) \in \mathbb{Z}^3 \mid a_r + a_s \sqrt{2} + a_t > 0\} \cup \{0\} \) and unit \( u = [3 \quad 4 \quad 3]^t \). The quotient is the image of \( G \) by the map \( v \to v \cdot v \). It is isomorphic to \( \frac{1}{2}\mathbb{Z}[\sqrt{2}] \) (the unit is sent by this map to \( 6 + 4\sqrt{2} = 2(1 + \sqrt{2})^2 \)).

It is interesting to make the following observation. We have seen before (Example 8.3.15) that the shift \( X \) is obtained by a doubling map from the Sturmian shift \( Y \) generated by the morphism \( \sigma : a \to ab, b \to aba \). Since the map sending \( a, c \) to \( a \) and \( b, d \) to \( b \) is a morphism from \( X \) onto \( Y \), we know from Proposition 4.6.11 that there is a natural embedding of \( K^0(Y, S) \) in \( K^0(X, T) \). Let us look in more detail how this is related to the doubling map.
The morphism $\sigma$ is eventually proper and thus the dimension group $K^0(Y, S)$ is the group of the matrix
\[
M(\sigma) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.
\]
The group is found to be $\mathbb{Z}[\sqrt{2}]$. Thus, up to the unit, $K^0(Y, S)$ is the same as the quotient of $K^0(X, T)$ by the infinitesimal group. This can be verified directly as follows. We have in $X$, first writing down all possible extensions of $c$ and next using the form of $R_X(a)$

\[
\chi[c] = \chi[ab-ca] + \chi[a-cda] + \chi[ab-cda] \\
\sim \chi[abca] + \chi[acda] + \chi[abcd]\]

where $\sim$ denotes the cohomology equivalence. Since

\[
M(\varphi) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}
\]

the values of the invariant probability measure $\mu$ of $(X, S)$ on $a, b, c, d$ are proportional to the left eigenvector of $M(\varphi)$ for $\lambda = 1 + \sqrt{2}$, which is

\[
\begin{bmatrix} \sqrt{2} & 1 & \sqrt{2} & 1 \end{bmatrix}
\]

Thus, using the basis the characteristic functions of $[a], [b], [d]$, we find the group $K^0(X, S)$ as $G = \{(\alpha, \beta, \delta) \in \mathbb{Z}^3 \mid \alpha\sqrt{2} + \beta + \delta > 0\} \cup \{0\}$. The natural embedding of $H(Y, S, \mathbb{Z})$ in $H(X, S, \mathbb{Z})$ is induced by the map $(\alpha, \beta) \mapsto (2\alpha, \beta, \beta)$. The image is as expected embedded in $\frac{1}{2}\mathbb{Z}[\sqrt{2}]$.

### 8.4 Exercises

#### Section 8.1

8.1 Let $X$ be the substitution shift generated by the substitution $a \rightarrow ab, b \rightarrow cda, c \rightarrow cd, d \rightarrow abc$. Show that the graph of every nonempty word in $L(X)$ is a tree.

8.2 Show that the Chacon ternary shift is not eventually dendric.

8.3 Let $B = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$. Let $\tau : A^* \rightarrow B^*$ be the morphism $a \rightarrow 12, b \rightarrow 2, c \rightarrow 3$ and $d \rightarrow 13$. Let $X$ be the shift on $A$ generated by the morphism $a \rightarrow ab, b \rightarrow cda, c \rightarrow cd, d \rightarrow abc$ of Example 8.1.4. Show that the shift $Y = \tau(X)$ is neutral.
8.4 Let $X$ be a shift space. A bifix code $U \subset \mathcal{L}(X)$ is $X$-maximal if it is not strictly contained in another bifix code $V \subset \mathcal{L}(X)$. Show that, if $X$ recurrent, every finite $X$-maximal bifix code is also $X$-maximal as a prefix code (and, symmetrically as a suffix code). Hint: show that if $U$ is neither $X$-maximal as a prefix code and as a suffix code, it is not $X$-maximal as a bifix code.

8.5 Let $X$ be a minimal dendric shift on the alphabet $A$. Denote by $F(A)$ the free group on $A$. Let $U \subset \mathcal{L}(X)$ be a finite $X$-maximal bifix code. Let $H$ be the subgroup of $F(A)$ generated by $U$. Show that $H \cap \mathcal{L}(X) = U^* \cap \mathcal{L}(X)$. Hint: consider the coset graph of $U$ and the set $V$ of labels of simple paths from $\varepsilon$ to $\varepsilon$.

8.6 Let $X$ be a recurrent shift space and let $U \subset \mathcal{L}(X)$ be a finite $X$-maximal bifix code. A parse of a word $w$ is a triple $(s, x, p)$ such that $w = sxp$ with $s$ a proper suffix of a word in $U$, $x \in U^*$ and $p$ a proper prefix of a word of $U$. Let $d_U(w)$ be the number of parses of $w$. Show that for every $u \in \mathcal{L}(X)$ and $a \in R_X(u)$

$$d_U(ua) = \begin{cases} d_U(u) & \text{if } ua \text{ has a suffix in } U \\ d_U(u) + 1 & \text{otherwise} \end{cases} \quad (8.4.1)$$

Show that one has for $u, v, w \in \mathcal{L}(X)$, the inequality

$$d_U(v) \leq d_U(ww) \quad (8.4.2)$$

with equality if $v$ is not a factor of a word in $U$.

8.7 Let $X$ be a recurrent shift space and let $U$ be a finite $X$-maximal bifix code. The $X$-degree of $U$, denoted $d_U(X)$ is the maximum of the numbers $d_U(w)$ for $w \in \mathcal{L}(X)$. Show that every word in $\mathcal{L}(X)$ which is not a factor of a word in $U$ has $d_U(X)$ parses.

8.8 Show that the $X$-degree of $\mathcal{L}_n(X)$ is equal to $n$.

8.9 Let $X$ be a recurrent shift space. Let $U \subset \mathcal{L}(X)$ be a finite $X$-maximal bifix code. Set $d = d_U(X)$. Show that the set $S$ of nonempty proper suffixes of $U$ is a disjoint union of $d - 1$ $X$-maximal prefix codes. Hint: consider for $2 \leq i \leq d$ the set $S_i$ of proper suffixes $s$ of $U$ such that $d_U(s) = i$.

8.10 Let $X$ be a minimal dendric shift on the alphabet $A$ such that $A \subset \mathcal{L}(X)$. Show that for every finite $X$-maximal bifix code $U$, one has

$$\text{Card}(U) = (\text{Card}(A) - 1)d_U(X) + 1. \quad (8.4.3)$$

Hint: use Exercise 8.9.
Let $X$ be a minimal dendric shift on the alphabet $A$ such that $A \subseteq L(X)$. Show that a finite bifix code $U \subseteq L(X)$ is $X$-maximal with $X$-degree $d$ if and only if it is a basis of a subgroup of index $d$ of the free group $F(A)$. Hint: consider a word $w \in L(X)$ which is not a factor of $U$. Let $Q$ be the set of suffixes of $w$ which are proper prefixes of $U$. Show that $K = \{ v \in F(A) \mid Qv \subseteq HQ \}$ is equal to the free group $F(A)$.

Show the the set $V$ of words $v$ such that $Qv \subseteq \langle U \rangle$ is a subgroup of the free group containing $R_X(w)$. This implies that $U$ generates a subgroup of index $d = d_U(X)$. Conclude using Exercise 8.10 and Schreier’s Formula asserting that a basis $U$ of a subgroup of index $d$ of the free group on $A$ has $\text{Card}(U) = d(\text{Card}(U) - 1) + 1$ elements.

Show that the substitution shift generated by $\sigma : a \rightarrow ac, b \rightarrow bac, c \rightarrow cbac$ of Example 8.1.41 is dendric.

Let $X$ be a shift space. A bispecial word $w \in L(X)$ is called regular if there are unique $a \in L(w)$ and $b \in R(w)$ such that $aw$ is right special and $wb$ is left special. In other words, the graph $\mathcal{E}(w)$ is a tree with paths of length at most 3. The shift $X$ is said to satisfy the regular bispecial condition if there is an $n \geq 1$ such that every bispecial word in $L_n(X)$ is regular. Show that $X$ is eventually dendric if and only if satisfies the regular bispecial condition.

A primitive $S$-adic shift $X$ is called a Brun shift if it generated by a directive sequence $\tau = (\tau_n)$ such that every $\tau_n$ is an elementary automorphism $\beta_{ab} = \tilde{\alpha}_{ba}$ (which places $a$ before $b$) and for all $n$, we have that $\tau_n \circ \tau_{n+1}$ is either equal to $\beta_{ab}^2$ or to $\beta_{ab}\alpha_{ca}$ for some $a,b,c \in A$ with $a \neq b$ and $a \neq c$.

1. Prove that the morphism $\sigma : a \rightarrow cbccba, b \rightarrow cbccb, c \rightarrow cbccbacbc$ generates a Brun shift which is not dendric.

2. Show that a Brun shift is a proper unimodular $S$-adic shift.

Let $X$ be a Sturmian shift of slope $\alpha = [a_0, a_1, \ldots]$. Show that the following conditions are equivalent.

(i) $X$ is linearly recurrent.

(ii) The coefficients $a_i$ are bounded.

(iii) $L(X)$ is $K$-power free for some $K \geq 1$.

Let $X$ be a recurrent specular shift. Show that the even code is an $X$-maximal bifix code of $X$-degree 2.
8.17 Prove that the decoding of dendric shift $X$ by an $X$-maximal prefix and suffix code is dendric.

8.18 Prove Theorem 8.3.25. Hint: Proceed as for the proof of Theorem 8.1.15.

8.5 Solutions

Section 8.1

8.1 Let $x$ be the fixed point $\sigma^{\omega}(a)$. Let $\pi$ be the morphism from $A^*$ onto $\{a, b\}^*$ defined by $\pi(a) = \pi(c) = a$ and $\pi(b) = \pi(d) = b$. The image of $x$ by $\pi$ is the Sturmian word $y$ which is the fixpoint of the morphism $\tau : a \mapsto ab, b \mapsto aba$. The word $x$ can be obtained back from $y$ by changing one every other letter $a$ into a $c$ and any letter $b$ after a $c$ into a $d$ (see Figure 8.5.1). Thus every word in $L(y)$ gives rise to 2 words in $L(x)$. In this way every bispecial word $w$ of

\[ a \quad | \quad b \]

Figure 8.5.1: The inverse of the map $\pi$.

$L(y)$ gives two bispecial words $w', w''$ of $L(x)$ and their extension graphs are isomorphic to $E(w)$. This proves the claim.

8.2 This follows from the fact that, for every $n$, the word $\alpha^n(012)$ is bispecial not neutral by Exercise 7.15.

8.3 Let $g : \{a, c\}A^* \cap A^* \{a, c\} \to B^*$ be the map defined by

\[
g(w) = \begin{cases} 
3\tau(w) & \text{if } w \text{ begins and ends with } a \\
3\tau(w)1 & \text{if } w \text{ begins with } a \text{ and ends with } c \\
2\tau(w) & \text{if } w \text{ begins with } c \text{ and ends with } a \\
2\tau(w)1 & \text{if } w \text{ begins with } c \text{ and ends with } c 
\end{cases}
\]

It can be verified that the set of bispecial words of $L(Y)$ is the union of $\{\varepsilon, 2, 31\}$ and of the images by $g$ of nonempty bispecial words of $L(X)$ (described in the solution of Exercise 8.1). One may verify that these words are neutral. Since the words $\varepsilon, 2, 31$ are also neutral, the shift space $X$ is neutral.

8.4 Consider a word $u \in L(X)$ which is not a factor of a word in $U$. Since $X$ is recurrent, there is a word $v$ such that $uvu \in L(X)$. Define a relation $\rho$ on the
set $P$ of proper prefixes of $u$ by $(p, q) \in \rho$ if one of the following conditions is satisfied (see Figure 8.5.2).

1. $q \in pU$.
2. $u = ps = qt$ and $svq = xyz$ with $x, z \in U$, $y \in U^*$, $s$ a proper prefix of $x$, $q$ a proper suffix of $z$.

![Figure 8.5.2: The relation $\rho$.](image)

Since $U$ is bifix, the relation $\rho$ is a partial bijection from $P$ to itself. Assume first that $U$ is $X$-maximal as a suffix code. Then the partial map $\rho$ is onto. This implies that it is a bijection and thus $u$ has a prefix in $U$. Since this is true for every long enough $u \in \mathcal{L}(X)$, it implies that $U$ is $X$-maximal as a prefix code.

Assume now that $U$ is neither $X$-maximal as a suffix code nor as a prefix code. Let $y, z \in \mathcal{L}(X)$ be such that $U \cup y$ is a prefix code and $U \cup z$ is a suffix code. Since $X$ is recurrent, there is a $v$ such that $yvz \in \mathcal{L}(X)$. Then $U \cup yvz$ is a bifix code, a contradiction.

8.5 One has clearly $U^* \cap \mathcal{L}(X) \subseteq \langle U \rangle \cap \mathcal{L}(X)$. Conversely, consider the coset graph $C$ of $U$ and let $V$ be the set of labels of simple paths from $\varepsilon$ to itself in $C$. By Proposition 8.4.32, we have $U \subseteq V$. Since $C$ is Stallings reduced, $V$ is a bifix code and since $U$ is $X$-maximal, this implies $U = V \cap \mathcal{L}(X)$. Thus

$$\langle U \rangle \cap \mathcal{L}(X) \subseteq \langle V \rangle \cap \mathcal{L}(X) = V^* \cap \mathcal{L}(X) = U^* \cap \mathcal{L}(X)$$

whence the conclusion.

8.6 If $ua$ has a suffix in $X$, the number of parses of $ua$ and $u$ are the same. Otherwise, since $U$ is a maximal suffix code, $ua$ is a suffix of a word in $U$ and thus $ua$ has one more parse than $u$, namely $(ua, \varepsilon, \varepsilon)$. This proves 8.4.1.

Next $d_U(v) \leq d_U(uvw)$ since $U$ is $X$-maximal as a prefix code and as a suffix code. Indeed, every parse of $v$ extends to a parse of $uvw$. Next, if $v$ is not a factor of a word in $U$, let $(s, x, p)$ be a parse of $uvw$. Since $v$ is not a factor of a word of $U$, it cannot be a factor of any of $s, x$ or $p$. Thus there is a parse $(q, y, r)$ of $v$ and a factorisation $x = zyt$ with $z, y, t \in U^*$ such that $sz = uq$ and $rw = zp$. This shows that every parse of $uvw$ is an extension of a parse of $v$ and thus that $d_U(uvw) = d_U(v)$. This proves 8.4.2.

8.7 Let $w \in \mathcal{L}(X)$ be such that $d_U(w) = d_U(X)$. Let $u \in \mathcal{L}(X)$ not a factor of a word in $U$. Since $X$ is recurrent, there is a $v \in \mathcal{L}(X)$ such that $uvw \in \mathcal{L}(X)$. By Equation 8.4.2, we have $d_U(u) = d_U(uvw) \geq d_U(v)$. Thus $d_U(u) = d_U(X)$. 


8.5. SOLUTIONS

8.8
Every word of length at least \( n \) has clearly \( n \) parses.

8.9
For each \( i \) with \( 2 \leq i \leq d_U(X) \), let \( S_i \) be the set of proper suffixes \( s \) of \( U \) such that \( d_U(s) = i \). Then \( S_i \) is a prefix code. Indeed, if \( s,t \in S_i \) and if \( s \) is a proper prefix of \( t \), then \( d_U(s) = d_U(t) \) implies that \( t \) has a suffix in \( U \) by Equation (8.4.1) of Exercise 8.6, a contradiction. Next, \( S_i \) is an \( X \)-maximal prefix code. Indeed, let \( w \in L(X) \) be long enough so that \( d_U(w) = i \). Then \( w \) has nonempty prefixes \( s_2,\ldots,s_d \) such that \( s_i \in S_i \) for \( 2 \leq i \leq d \). We conclude that the set \( S \) of nonempty proper prefixes of \( U \) is a disjoint union of \( d-1 \) \( X \)-maximal prefix codes.

8.10
Set \( d = d_U(X) \). Let \( P \) be the set of proper prefixes of \( U \). By the well-known formula relating the number of leaves of a tree to the number of children of its interior nodes, we have

\[
\text{Card}(U) - 1 = \sum_{p \in P} (r_X(p) - 1).
\]

By (the dual of) Exercise 8.9, the set \( P \setminus \{ \varepsilon \} \) is a disjoint union of \( d-1 \) \( X \)-maximal suffix codes \( V_1,\ldots,V_{d-1} \). Set \( \rho(u) = r_X(u) - 1 \) and, for \( V \subset L(X) \), denote \( \rho(V) = \sum_{v \in V} \rho(v) \). By Lemma 8.1.10 we have \( \rho(u) = \sum_{a \in L(u)} \rho(au) \). This implies that for any \( X \)-maximal suffix code \( V \), one has

\[
\rho(V) = \rho(\varepsilon) = \text{Card}(A) - 1
\]

where the last equality results of the hypothesis \( A \subset L(X) \). Thus, we have

\[
\text{Card}(U) - 1 = \rho(P) = \rho(\varepsilon) + \sum_{i=1}^{d-1} \rho(V_i)
\]

\[
= (\text{Card}(A) - 1)d.
\]

8.11
Suppose first that \( U \) is a finite \( X \)-maximal bifix code of \( X \)-degree \( d = d_U(X) \) and let \( H = \langle U \rangle \) be the subgroup generated by \( U \). Let \( w \in L(X) \) be a word which not a factor of a word in \( U \). Let \( Q \) be the set of suffixes of \( w \) which are proper prefixes of \( U \). Then, by Exercise 8.7 \( w \) has \( d \) parses and thus \( \text{Card}(Q) = d \).

Moreover, we claim that it follows from Exercise 8.5 that the cosets \( Hq \) for \( q \in Q \) are distinct. Indeed, let \( p,q \in Q \) be such that \( Hp = Hq \). Since \( p,q \) are suffixes of \( w \), one is a suffix of the other. Assume that \( q = tp \). Then \( Hp = Htp \) implies \( Ht = H \) and thus \( t \in H \). By Exercise 8.3 this implies \( t \in U^* \). Since \( t \) is a proper prefix of \( U \), we conclude that \( t = \varepsilon \) and thus \( p = q \), which establishes the claim.

Consider the set

\[
K = \{ v \in F(A) \mid Qv \subset HQ \}.
\]
It is a subgroup of \( F(A) \). Indeed, by what precedes, the map \( p \mapsto q \) if \( pv \in Hq \) is a permutation of \( Q \) for every \( v \in K \).

Next, we have \( R_X(w) \subset K \). In fact, consider \( v \in R_X(w) \). For every \( p \in Q \), since \( U \) is \( X \)-complete, there is some \( x \in U^* \) and some proper prefix \( q \) of \( U \) such that \( pv = qx \). But since \( v \) is in \( R_X(w) \), \( pv \) ends with \( w \) and thus \( q \in Q \).

Now, by Theorem \( \ref{thm:8.1.14} \), \( R_X(w) \) generates \( F(A) \) and thus \( K = F(A) \). We conclude that \( F(A) \subset HQ \) and thus that \( Q \) is a set of representatives of the cosets of \( H \). Thus \( H \) has index \( d \). Since \( U \) generates a subgroup of index \( d \) and since \( \text{Card}(U) - 1 = d(\text{Card}(A) - 1) \), we conclude by Schreier’s Formula that \( U \) is a basis of \( H \).

Conversely, if the bifix code \( U \subset L(X) \) is a basis of a subgroup \( H \) of index \( d \), let \( C \) be the coset graph of \( U \). By Proposition \( \ref{prop:8.132} \), \( C \) is the Stallings graph of a subgroup of index \( d \). Moreover \( U \) is contained in the set \( V \) of labels of simple paths from \( \varepsilon \) to \( \varepsilon \). Then \( W = V \cap L(X) \) is an \( X \)-maximal bifix code of \( X \)-degree at most \( d \). By Exercise \( \ref{ex:8.10} \), we have

\[
\text{Card}(W) \leq d(\text{Card}(A) - 1)
\]

and thus, by Schreier’s Formula \( \text{Card}(W) \leq \text{Card}(U) \). This forces \( U = W \) and concludes the proof.

\section{8.12} The right-special words are the suffixes of the words \( \sigma^n(u) \) for \( n \geq 1 \) and the left-special words are the prefixes of the words \( \sigma^n(u) \) or \( \sigma^n(c) \) for \( n \geq 1 \), as one may verify. Let us show by induction on the length of \( w \) that for any bispecial word \( w \in L(X) \), the graph \( \mathcal{E}(w) \) is a tree. It is true for \( w = c \) and \( w = ac \). Assume that \( |w| \geq 2 \). Either \( w \) begins with \( a \) or with \( c \). Assume the first case. Then \( w \) begins and ends with \( ac \). We must have \( w = ac^\sigma(u) \) where \( u \) is a bispecial word beginning and ending with \( c \). In the second case, \( w \) begins with \( cbac \) and ends with \( ac \). We must have \( w = cbac^\sigma(u) \) where \( u \) is a bispecial word beginning with \( a \). In both cases, by induction hypothesis, \( \mathcal{E}(u) \) is a tree and thus \( \mathcal{E}(w) \) is a tree.

\section{8.13} Denote by \( LS(X) \) (resp. \( LS_{\geq n}(X) \)) the set of left-special words in \( L(X) \) (resp. \( L_{\geq n}(X) \)).

Assume first that \( X \) is eventually dendric with threshold \( m \). Then any word \( w \) in \( LS_{\geq m}(X) \) has at least one right extension in \( LS(X) \). Indeed, since \( R_1(w) \) has at least two elements and since the graph \( \mathcal{E}_1(w) \) is connected, there is at least one element \( v \) of \( R_1(w) \) which is connected by an edge to more than one element of \( L_1(w) \) and thus that \( rv \in LS(X) \).

Next, the symmetric of Equation \( \ref{eq:8.12} \) shows that for any \( w \in LS_{\geq m}(X) \) which has more than one right extension in \( LS(X) \), one has \( \ell(wb) < \ell(w) \) for each such extension. Thus the number of words in \( LS_{\geq m}(X) \) which are prefix of one another, and which have more than one right extension, is bounded by \( \text{Card}(A) \). This proves that there exists an \( n \geq m \) such that for any \( w \in LS_{\geq n}(X) \) there is exactly one \( b \in A \) for which \( wb \in LS(X) \) is left-special. Moreover, one has then \( \ell(wb) = \ell(w) \) by the symmetric of Equation \( \ref{eq:8.12} \).
This proves the uniqueness of $r$. The proof for left extensions of right-special words is symmetric.

Conversely, assume that the regular bispecial condition is satisfied for some integer $n$. For any word $w$ in $L_{\geq n}(X)$, the graph $E_1(w)$ is acyclic since all vertices in $R_1(w)$ except at most one have degree 1. Let $w \in L_{\geq n}(X)$. If $w$ is not bispecial, there is exactly one $b$ such that $w \in L(X)$ and then $wb \in LS(X)$. Thus, in both cases, there is exactly one $b$ such that $w \in L(X)$ and then $wb \in LS(X)$.

8.14 1. We have $\sigma = \beta_{cb} \circ \beta_{bc} \circ \beta_{cb} \circ \beta_{ba} \circ \beta_{ac}$ since for example

$$a \xrightarrow{\beta_{ac}} a \xrightarrow{\beta_{ba}} ba \xrightarrow{\beta_{cb}} cba \xrightarrow{\beta_{bc}} cbca \xrightarrow{\beta_{cb}} cbccba.$$ 

The extension graph of the word $w = cbccb$ is shown in Figure 8.5.3. Since this

![Figure 8.5.3: The extension graph of $w$.](image)

graph has a cycle, the shift $X(\sigma)$ is not dendric.

2. Let $\alpha = (\beta_{a_n b_n})$ be the directive sequence of morphisms defining the Brun shift $X$. Since $\alpha$ is primitive, there exists an increasing sequence $(n_k)$ of integers such that the set $\{a_{n_i} \mid n_k \leq n_i < n_{k+1}\}$ is equal to $A$. Then all morphisms $\beta_{n_k} \circ \cdots \circ \beta_{n_{k+1}}$ are left proper. Indeed, let $\sigma = (\beta_{a_1 a_2})^{i_1} \circ (\beta_{a_2 a_3})^{i_2} \circ \cdots \circ (\beta_{a_n a_{n+1}})^{i_n}$ with $i_j = 1$ or 2. Then all words $\sigma(a_1), \ldots, \sigma(a_{n+1})$ begin with $a_1$. Using finally Lemma 7.4.8 we obtain the conclusion.

Section 8.2

8.15 The equivalence of (i) and (ii) is Corollary 8.2.6. Next, the implication (i) $\Rightarrow$ (iii) results from Proposition 2.2.7. Finally we have seen in Exercise 2.39 that (iii) $\Rightarrow$ (ii).

Section 8.3

8.16 This follows from the fact that any word of the even code $U$ of length at least 2 is not an internal factor of $U$ and has two parses.

8.17 Let $X$ be a dendric shift on the alphabet $A$. Let $U \subset L(X)$ be a bifix code which is both an $X$-maximal prefix and suffix code. Let $f: B^* \to A^*$ be a
coding morphism for \(U\). Let \(Y\) be the shift space such that \(\mathcal{L}(Y) = f^{-1}(\mathcal{L}(X))\). For every \(w \in \mathcal{L}(Y)\), the generalized extension graph \(E_{U,Y}(f(w))\) is a tree by Proposition 8.1.12. Since this graph is isomorphic to \(E_Y(w)\), this proves that \(Y\) is dendric.

Set \(S = \mathcal{L}(X)\). Let \(P\) be the set of proper prefixes of \(CR_X(U)\). For \(q \in P\), we define \(\alpha(q) = \text{Card}\{a \in A \mid qa \in P \cup CR_X(U)\} - 1\). For \(P' \subset P\), we set \(\alpha(P') = \sum_{p \in P'} \alpha(p)\). Since \(CR_X(U)\) is a finite nonempty prefix code, we have, by a well-known property of trees, \(\text{Card}(CR_X(U)) = 1 + \alpha(P)\). Let \(P'\) be the set of words in \(P\) which are proper prefixes of \(U\) and let \(Y = P \setminus P'\). Since \(P'\) is the set of proper prefixes of \(U\), we have \(\alpha(P') = \text{Card}(U) - 1\).

For \(u \in \mathcal{L}(X)\), set

\[
\rho(u) = \begin{cases} r_X(u) - 1 & \text{if } u \neq \varepsilon \\ \text{Card}(A) - 2 & \text{otherwise} \end{cases}
\]

In this way, we have \(\rho(u) = \sum_{a \in L(u)}(au)\) for every \(u \in \mathcal{L}(X)\). Thus, if \(Y\) is an \(X\)-maximal suffix code, we have \(\sum_{u \in Y} \rho(y) = \rho(\varepsilon) = \text{Card}(A) - 2\).

Since \(P \cup CR_X(U) \subset S\), one has \(\alpha(q) \leq \rho(q)\) for any nonempty \(q \in P\).

Moreover, since \(X\) is recurrent, and since \(U\) has empty kernel, any word of \(S\) with a prefix in \(U\) is comparable for the prefix order with a word of \(CR_X(U)\).

This implies that for any \(q \in Y\) and any \(b \in R(q)\), one has \(qb \in P \cup CR(U)\).

Consequently, we have \(\alpha(q) = \rho(q)\) for any \(q \in Y\). Thus we have shown that \(\text{Card}(CR_X(U)) = 1 + \alpha(P') + \rho(Y) = \text{Card}(U) + \rho(Y)\).

Let us show that \(Y\) is an \(X\)-maximal suffix code. This will imply our conclusion. Suppose that \(q, uq \in Y\) with \(u\) nonempty. Since \(q\) is in \(Y\), it has a proper prefix in \(U\). But this implies that \(uq\) has an internal factor in \(U\), a contradiction. Thus \(Y\) is a suffix code. Consider \(w \in S\). Then, for any \(x \in U\), there is some \(u \in S\) such that \(xuw \in S\). Let \(y\) be the shortest suffix of \(xuw\) which has a proper prefix in \(U\). Then \(y \in Y\). This shows that \(Y\) is an \(X\)-maximal suffix code.

8.6 Notes

8.6.1 Dendric shifts

The languages of dendric shifts were introduced in Berthé et al. (2015a) under the name of tree sets. The language of the shift of Example 8.1.3 is a tree set of characteristic 2 ((Berthé et al., 2017a, Example 4.2)) and it is actually a specular set. Tree sets of characteristic \(c \geq 1\) were introduced in Berthé et al. (2017a) (see also Dolce and Perrin (2017a)). The shift spaces of Example 8.1.4 and of Exercise 8.3 (a neutral shift which is not dendric) are due to Julien Cassaigne (2015).

It can be proved that the class of eventually dendric shifts is closed under conjugacy (see Dolce and Perrin (2019)).
Theorem 8.1.14 is from Berthé et al. (2015a). Theorem 8.1.15 was proved earlier in Balková et al. (2008) and was proved even earlier for episturmian shifts in Justin and Vuillon (2000).

Theorem 8.1.23 is from Berthé et al. (2015b, Theorem 6.5.19). It generalizes the fact that the derived word of a Sturmian word is Sturmian (see Justin and Vuillon (2000)).

Theorem 8.1.27 is from Berthé et al. (2015b). It is obtained as a corollary of a result (called the Finite Index Basis Theorem) proved in Berthé et al. (2015c) (see Exercise 8.11).

The fact that the group of positive automorphisms of a free group on three letters is not finitely generated is from Tan et al. (2004). The word ‘tame’ used for tame automorphisms (as opposed to wild) is used here on analogy with its use in ring theory (see Cohn (1985)). The tame automorphisms as introduced here should, strictly speaking, be called positive tame automorphisms since the group of all automorphisms, positive or not, is tame in the sense that it is generated by the elementary automorphisms.

Theorem 8.1.40 is from Berthé et al. (2015b).

In the case of a ternary alphabet, a characterization of tree sets by their S-adic representation can be proved, showing that there exists a Büchi automaton on the alphabet $S_e$ recognizing the set of $S_e$-adic representations of uniformly recurrent tree sets (Leroy, 2014a,b).

The Brun shifts of Exercise 8.14 arise in the generalization of continued fractions expansion to triples of integers instead of pairs (Brun, 1958). The study of these algorithms as dynamical system was initiated in Berthé et al. (2019a). The relation of these shifts with dendric shifts was studied in Labbé and Leroy (2016).

The regular bispecial condition (Exercise 8.13) was introduced by Damron and Fickenscher (2019). They proved that for an irreducible shift satisfying this condition, the number of ergodic measures is at most $(K + 1)/2$ where $K$ is the limiting value of $p_{n+1}(X) - p_n(X)$. This generalizes a result proved independently by Katok (1973) and Veech (1978) concerning interval exchange transformations.

### 8.6.2 Sturmian shifts

Theorem 8.2.1 is already in Morse and Hedlund (1940). Assertion 1 follows from Theorem 7.1 in Morse and Hedlund (1940) and Assertion 2 is Theorem 8.1.

Corollary 8.2.6 is from Durand et al. (1999). It is of course related to the fact that a Sturmian word of slope $\alpha$ is $k$-power free if and only if the coefficients of the expansion of $\alpha$ as a continued fraction are bounded (Exercise 2.39).

Theorem 8.2.7 is from Dartnell et al. (2000). It is proved as a consequence of the fact that a Sturmian shift is a substitution shift if and only if the sequence $(\zeta_n)_n$ is ultimately periodic (see also Kurka (2003) and Araújo and Bruyère (2005)). For similar theorems characterizing purely substitutive Sturmian words, see the references in Lothaire (2002). In particular, it is proved in Allauzen (1998) that, for $0 < \alpha < 1$ the characteristic sequence $c_n$ of slope $\alpha$ is purely substitutive if and only if $\alpha$ is quadratic and its conjugate is $> 1$. 

It is not surprising that the condition on $\alpha$ for $c_{\alpha}$ to be purely substitutive is more restrictive than for the shift of slope $\alpha$ to be substitutive. Indeed, if $\alpha$ is quadratic, the Sturmian shift of slope $\alpha$ contains a purely substitutive sequence, but it need not be the characteristic sequence $c_{\alpha}$.

### 8.6.3 Specular shifts

The notion of specular shift was introduced in [Berthé et al. (2017a)](http://example.com). The idea of considering laminary sets is from [Coulbois et al. (2008)](http://example.com) (see also [Lopez and Narbel (2013)](http://example.com)).

For a proof of Kurosh subgroup Theorem, see [Magnus et al. (2004)](http://example.com). Specular groups are characterized by the property of their Cayley graphs to be regular (see [de la Harpe (2000)](http://example.com)). See also [de la Harpe (2000)](http://example.com) concerning the notion of virtually free group. Actually, specular groups can be studied as groups acting on trees as developed in the Bass-Serre theory ([Serre, 2003)](http://example.com).

A group having a free subgroup of finite index is called *virtually free*. On the other hand, a finitely generated group is said to be *context-free* if, for some presentation, the set of words equivalent to 1 is a context-free language. By Muller and Schupp’s theorem ([Muller and Schupp, 1983](http://example.com)), a finitely generated group is virtually free if and only if it is context-free. Thus a specular group is context-free. One may verify this directly as follows. A context-free grammar generating the words equivalent to 1 for the natural presentation of a specular group $G = G_\theta$ is the grammar with one nonterminal symbol $\sigma$ and the rules

$$
\sigma \rightarrow a\sigma a^{-1}\sigma \quad (a \in A), \quad \sigma \rightarrow 1. \quad (8.6.1)
$$

The proof that the grammar given by Equation (8.6.1) generates the set of words equivalent to 1 is similar to that used in [Berstel (1979)](http://example.com) for the so-called Dyck-like languages.

Theorem 8.3.22 is the counterpart for minimal specular shifts of the main result of [Berthé et al. (2015b)](http://example.com) Theorem 6.1 asserting that the family of uniformly recurrent tree sets of characteristic 1 is closed under maximal bifix decoding.

Theorem 8.3.25 is proved in ([Dolce and Perrin, 2017a](http://example.com) Theorem 3).

The definition of mixed return words comes from the fact that, when $S$ is the natural coding of a linear involution, we are interested in the transformation induced on $I_w \cup \sigma_2(I_w)$ (see [Berthé et al. (2017b)](http://example.com)). The natural coding of a point in $I_w$ begins with $w$ while the natural coding of a point $z$ in $\sigma_2(I_w)$ ‘ends’ with $w^{-1}$ in the sense that the natural coding of $T^{-|w|}(z)$ begins with $w^{-1}$.

A geometric proof and interpretation of Theorem 8.3.38 is given in [Berthé et al. (2017b)](http://example.com). It is shown that the set of mixed return words are a symmetric basis of a fundamental group corresponding to a surface built above the linear involution.
Chapter 9

Interval exchange transformations

In the chapter, we study a class of dynamical systems obtained by iterating simple geometric transformations on an interval. These transformations, called interval exchange, form a classical family of dynamical systems. They can be seen as a generalization of the rotations of the circle, already met several times. They define by a natural coding shift spaces which turn out to be dendric shifts. Thus we obtain, after Arnoux-Rauzy shifts a large class of dendric shifts on which we will describe more precisely return words and $S$-adic representations.

In the first part, we introduce interval exchange transformations. We define the natural representation of an interval exchange transformation, which is shown to be dendric shifts (Proposition 9.1.9). We introduce the notion of regular interval exchange transformation and prove that it implies the minimality of the transformation (Theorem 9.1.2). We develop the notion of Rauzy induction and characterize the subintervals reached by iterating the transformation (Theorem 9.1.22). We generalize Rauzy induction to a two-sided version and characterize the intervals reached by this more general transformation (Theorem 9.2.3). We link these transformations with automorphisms of the free group (Theorem 9.2.15). We also relate these results with the theorem of Boshernizan and Carrol giving a finiteness condition on the systems induced by an interval exchange when the lengths of the intervals belong to a quadratic field (Theorem 9.3.2).

In a second part, we present linear involutions, which form a larger family by adding the possibility of symmetries in addition to translations. We introduce the notion of connexion and relate it to the minimality of the transformation (Proposition 9.4.6). We show that the natural coding of a linear involution without connexion is specular (Theorem 9.4.9).
9.1 Interval exchange transformations

A semi-interval is a nonempty subset of the real line of the form \([\ell, r) = \{z \in \mathbb{R} \mid \ell \leq z < r\}\). Thus it is a left-closed and right-open interval. For two semi-intervals \(\Delta, \Gamma\), we denote \(\Delta < \Gamma\) if \(x < y\) for every \(x \in \Delta\) and \(y \in \Gamma\). A partition \(\{I_1, \ldots, I_k\}\) of the semi-interval \(I\) indexed by \(1, \ldots, k\) is ordered if \(\Delta_1 < \ldots < \Delta_k\).

Let \(\{\Delta_1, \ldots, \Delta_k\}\) be an ordered partition of the semi-interval \(I = [\ell, r)\) into \(k \geq 2\) disjoint semi-intervals. A \(k\)-interval exchange transformation on \(I\) is an onto map \(T: [\ell, r) \to [\ell, r)\) where \(T: \Delta_i \to [\ell, r)\) is a translation.

For every \(i\) with \(1 \leq i \leq k\), there is a real number \(\alpha_i\) such that \(Tx = x + \alpha_i\) for every \(x \in \Delta_i\). The numbers \(\alpha_i\) are called the translation values of \(T\).

We denote by \(\pi\) the permutation of \(\{1, \ldots, k\}\) such that \(T\Delta_{\pi(1)}, \ldots, T\Delta_{\pi(k)}\) is an ordered partition (see Figure 9.1.1).

If \(\lambda_i\) denotes the length of the interval \(\Delta_i\) and \(\lambda = (\lambda_1, \ldots, \lambda_k)\), the transformation is defined by \(\lambda\) and the permutation \(\pi\). We denote \(T = T_{\lambda, \pi}\).

One may imagine an interval exchange transformation as the following ‘physical’ operation. Break the semi-interval \(I\) into \(k\) pieces and rearrange them in a different order.

![Figure 9.1.1: A \(k\)-interval exchange transformation.](image)

Such a transformation is not continuous and thus does not fit well in the framework of topological dynamical systems. It is actually a measure preserving transformation and thus an interval exchange transformation is a measure theoretic dynamical system.

We shall see below how the interval can be modified to obtain a topological dynamical system.

For \(k = 2\), an interval exchange transformation on \(I = [0, 1)\) is a rotation. We will often consider the semi-interval interval \([0, 1)\) to simplify the notation. Indeed, for \(0 < \alpha < 1\), set \(\Delta_1 = [0, 1-\alpha)\) and \(\Delta_2 = [1-\alpha, 1)\). The corresponding interval exchange transformation \(T\) is \(T(x) = x + \alpha \mod 1\). In this particular case, the transformation is continuous provided one identifies the two endpoints of the interval (see Example 2.1.5).

An interval exchange transformation is invertible since a translation is one-to-one. Its inverse is again an interval exchange transformation, on the intervals \(\Delta_i' = T\Delta_i\).
A power $T^n$ of an interval exchange transformation is again an interval exchange transformation. Since we have seen that $T^{-1}$ is also an interval exchange transformation, we may assume $n > 0$. Consider the nonempty sets of the form

$$\Delta_{i_0, \ldots, i_{n-1}} = \Delta_{i_0} \cap T^{-1} \Delta_{i_1} \cap \ldots \cap T^{-n+1} \Delta_{i_{n-1}}.$$  

These sets are semi-intervals because for every $i, j$, the set

$$\Delta_i \cap T^{-1} \Delta_j = T^{-1}(\Delta_j \cap T \Delta_i) = T^{-1}(\Delta_j \cap \Delta'_i)$$

is a semi-interval or empty and next

$$\Delta_{i_0} \cap T^{-1} \Delta_{i_1} \cap \ldots \cap T^{-n+1} \Delta_{i_{n}} =$$

$$\Delta_{i_0} \cap T^{-1}(\Delta_{i_1} \cap T^{-1}(\Delta_{i_2} \cap \ldots \cap T^{-1}(\Delta_{i_{n-2}} \cap T^{-1} \Delta_{i_{n-1}}) \ldots)).$$

Thus $T^n$ is an exchange of the nonempty intervals $\Delta_{i_0, \ldots, i_{n-1}}$.

**Example 9.1.1** A 3-interval exchange transformation is represented in Figure 9.1.2. The associated permutation is the cycle $\pi = (123)$.

![Figure 9.1.2: A 3-interval exchange transformation.](image)

### 9.1.1 Minimal interval exchange transformations

The *separation points* of an interval exchange are the left end points $d_1 = 0, d_2, \ldots, d_k$ of the semi-intervals $\Delta_i$ for $1 \leq i \leq k$. We denote by Sep($T$) the set of separation points of $T$.

An interval exchange transformation $T$ on $I = [0, 1)$ is called *regular* if the orbits of the nonzero separation points are infinite and disjoint. Note that the orbit of 0 cannot be disjoint of the others, since one has $T(d_i) = 0$ for some $i$ with $1 \leq i \leq k$, but that the orbit of 0 is infinite when $T$ is regular.

Equivalently, an interval exchange transformation is regular if there is no triple $(i, j, n)$ for $2 \leq i, j \leq k$, and $n > 0$ such that $T^n d_i = d_j$. Such a triple is called a *connexion*. Indeed, a connexion $(i, j, n)$ with $i = j$ corresponds to a finite orbit and with $i \neq j$ to intersecting orbits.

As an example, a rotation of angle $\alpha$ is regular if and only if $\alpha$ is irrational. Indeed, $\alpha$ is rational if and only if the orbit of $\alpha$ is finite (in which case all orbits are finite).

We say that an interval exchange is *minimal* if the orbit of every point is dense.

**Theorem 9.1.2 (Keane)** A regular interval exchange transformation is minimal.
The converse is not true. Indeed, consider the the rotation of irrational angle $\alpha$ on $I = [0, 1)$ as a 3-interval exchange with the partition $[0, 1 - 2\alpha), [1 - 2\alpha, 1 - \alpha), [1 - \alpha, 0)$ (see Figure 9.1.3). The transformation is minimal, as any rotation of irrational angle but $T(1 - 2\alpha) = 1 - \alpha$ and thus there is a connexion.

We first prove the following result. Let $T$ be an interval exchange on $I = [0, 1)$. We shall consider the transformation $T_1$ induced by $T$ on a semi-interval $X_1 = [a, b) \subset [0, 1)$. It is defined by $T_1(x) = T^n(x)$ where $n(x)$ is the least integer $n \geq 1$ such that $T^n(x) \in X_1$. The return time $n(x)$ exists, even if $T$ is not minimal, by the Poincaré Recurrence Theorem, because $T$ preserves the Lebesgue measure. Such an induction on a semi-interval is called a Rauzy induction and we will have more to say on this later.

**Theorem 9.1.3 (Rauzy)** Let $T$ be a $k$-interval exchange transformation on $I = [0, 1)$ and let $X_1 = [a, b) \subset [0, 1)$ be a semi-interval. The transformation $T_1$ induced by $T$ on $X_1$ is a $k_1$-interval exchange transformation with $k_1 \leq k + 2$.

**Proof.** Consider the set $Y = \{a, b, d_2, \ldots, d_k\}$ where $d_2, \ldots, d_k$ are the nonzero separation points. For $y \in Y$, let $s(y)$ be the least integer $s \geq 0$ if it exists such that $T^{-s(y)}(y) \in (a, b)$. The points $T^{-s(y)}$ divide the semi-interval $[a, b)$ in $k_1 \leq k + 2$ semi-intervals $\Delta'_1, \ldots, \Delta'_{k_1}$. For each semi-interval $\Delta'_i$ consider the number $n_i$ which is the least $n \geq 1$ such that $T^n\Delta'_i \cap [a, b) \neq \emptyset$.

Then, for $1 \leq p \leq n_i$, the transformations $T^p$ are continuous on $\Delta'_i$ and we have $T^n\Delta'_i \subset [a, b)$. Indeed, otherwise, for some $p$ with $1 \leq p \leq n_i - 1$, the semi-interval $T^p\Delta'_i$ contains one of the points $y \in Y$ and then $s(y) = p$ so that the point $T^{-p}y$ would lie within $\Delta'_i$, a contradiction with the definition of the semi-intervals $\Delta'_i$. This shows that $n_i$ is the return time of all $x \in \Delta'_i$ and thus that $T_1$ is the interval exchange of $[a, b)$ corresponding to the intervals $\Delta'_i$.

**Proof of Theorem 9.1.4** Let us first show that $T$ has no periodic point. Assume that $T^n$ has a fixed point, that is a point $x \in I$ such that $T^n x = x$. Since $T^n$ is an interval exchange transformation on the intervals $\Delta^{(n)}_{b_0, \ldots, b_{n-1}}$ one of this intervals is formed of fixed points. But the left end of this interval is in the orbit of some of the separation points $d_i$. This contradicts the hypothesis that the orbit of all nonzero separation points is infinite unless $i = 1$ and $n = 1$. But in this case, the interval $\Delta_1$ is fixed by $T$, which implies that $T d_j = d_2$ for some $j \geq 2$, a contradiction again with the hypothesis.

Suppose now that the orbit $O(x)$ of some $x \in [0, 1)$ is not dense. We can find a semi-interval $[a, b)$ disjoint of $O(x)$. Let $T_1$ be the transformation induced by $T$ on $[a, b)$. By Theorem 9.1.3 $T_1$ is an interval exchange on semi-intervals $\Delta'_j$. Let $n_j$ denote the return time to $\Delta'_j$. Set

$$F = \bigcup_{j} \bigcup_{n=0}^{n_j-1} T^n \Delta'_j.$$ 

The set $F$ can be written as a union of a finite number of nonintersecting semi-intervals. Hence its connected components are also semi-intervals, say $F_s$ and
their number is finite. Let $G$ be the union of the left end points of all the semi-intervals $F_s$. By definition, the set $F$ is invariant by $T$ and therefore, for every $x \in G$, we have either $Tx \in G$ or the point $x$ is a point of discontinuity of $T$ or $x = 0$, that is $x = d_i$ for some $i$ with $1 \leq i \leq k$. Since $G$ is finite and since $T$ has no periodic points, there is for every $g \in G$ some $n \geq 0$ such that $T^n x = d_i$ with $1 \leq i \leq k$. Similarly, there is for every $x \in G$ an integer $m > 0$ such that $T^{-m} x = d_j$ with $2 \leq i \leq k$. Then, we have $T^n x = T^{n + m} d_j = d_i$.

Since the orbits of $d_2, \ldots, d_k$ are disjoint, the only possibility is $i = 1$. Then $d_j = T^{-1} d_1$ and thus $m = 1, n = 0$ and $x = 0$. Thus $G = \{0\}$, which implies $F = [0, 1)$, a contradiction with the fact that, by construction $F \cap O(x) = \emptyset$.

The following necessary condition for minimality of an interval exchange transformation is useful. A permutation $\pi$ of an ordered set $A$ is called decomposable if there exists an element $b \in A$ such that the set $B$ of elements strictly less than $b$ is nonempty and such that $\pi(B) = B$. Otherwise it is called indecomposable. If an interval exchange transformation $T = T_{\lambda, \pi}$ is minimal, the permutation $\pi$ is indecomposable. Indeed, if $B$ is a set as above, the set of orbits of the points in the set $S = \cup_{a \in B} I_a$ is closed and strictly included in $[\ell, r[$. The following example shows that the indecomposability of $\pi$ is not sufficient for $T$ to be minimal.

**Example 9.1.4** Let $A = \{a, b, c\}$ and $\lambda$ be such that $\lambda_a = \lambda_c$. Let $\pi$ be the transposition $(ac)$. Then $\pi$ is indecomposable but $T_{\lambda, \pi}$ is not minimal since it is the identity on $I_b$.

The iteration of a $k$-interval exchange transformation is, in general, an interval exchange transformation operating on a larger number of semi-intervals.

**Proposition 9.1.5** Let $T$ be a regular $k$-interval exchange transformation. Then, for any $n \geq 1$, $T^n$ is a regular $n(k - 1) + 1$-interval exchange transformation.

**Proof.** Since $T$ is regular, the set $\bigcup_{i=0}^{n-1} T^{-i}(d)$ where $d$ runs over the set of $s - 1$ nonzero separation points of $T$ has $n(k - 1)$ elements. These points partition the interval $[\ell, r]$ in $n(k - 1) + 1$ semi-intervals on which $T$ is a translation.

We close this subsection with a lemma that will be useful in Section ??.

**Lemma 9.1.6** Let $T$ be a minimal interval exchange transformation. For every $N > 0$ there exists an $\varepsilon > 0$ such that for every $z \in D(T)$ and for every $n > 0$, one has $|T^n(z) - z| < \varepsilon \implies n \geq N$. 

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be the translation values of $T$. For every $N > 0$ it is sufficient to choose 
\[ \varepsilon = \min \left\{ \left| \sum_{i,j=1}^{M} \alpha_{ij} \right| \mid 1 \leq i, j \leq s \text{ and } M \leq N \right\}. \]

9.1.2 Natural coding

To every interval exchange transformation $T$ on $I = [\ell, r)$, we may associate a shift space called its natural representation defined as follows.

Let $\phi : \{1, 2, \ldots, k\} \to A$ be a bijection onto an alphabet $A$ and let $\gamma : I \to A^\mathbb{Z}$ be the map defined by $y = \gamma(x)$ with $y = (y_n)$ and $y_n = \phi(i)$ if $T^n(x) \in \Delta_i$.

The two-sided sequence $y = \gamma(x)$ is called the natural coding of $x$. The natural representation of $T$, denoted $X(T)$, is the closure in $A^\mathbb{Z}$ of $\gamma([\ell, r))$.

An interval exchange shift is the natural representation $X(T)$ of an interval exchange $T$. It is said to be regular if the interval exchange is regular. We denote $\mathcal{L}(T) = \mathcal{L}(X(T))$.

The map $\gamma$ satisfies $\gamma \circ T = S \circ \gamma$ where $S$ denotes as usual the shift. Thus the natural representation of a minimal interval exchange transformation is a minimal shift space. Note that if $T$ is minimal, the map $\gamma$ is injective. Indeed, if $x \neq x'$, there is for every $i$ an $n$ such that $T^n(x) \in \Delta_i$ but $T^n(x') \notin \Delta_i$.

Example 9.1.7 Consider the interval exchange transformation $T$ represented in Figure 9.1.3 with $\alpha = (3 - \sqrt{5})/2$. It is the rotation $x \mapsto x + 2\alpha$ represented

![Figure 9.1.3: The rotation of angle $2\alpha$.](image)

as a regular interval exchange on on three semi-intervals $\Delta_a = [0, 1 - 2\alpha)$, $\Delta_b = [1 - 2\alpha, 1 - \alpha)$, and $\Delta_c = [1 - \alpha, 1)$.

Since $T$ is minimal, the set $\mathcal{L}(T)$ is uniformly recurrent. The words of length at most 6 of the set $\mathcal{L}(T)$ are represented in Figure 9.1.4.

The shift $X = X(T)$ is actually a primitive substitution shift. Indeed, since $T$ is the rotation of angle $2\alpha$, the shift $X(T)$ is the coding by nonoverlapping blocks of length 2 of the Fibonacci shift, which is the natural coding of the rotation of angle $\alpha$. But the Fibonacci morphism $\varphi : a \to ab, b \to a$ is such that $\varphi^3 : a \to abaab, b \to aba$ sends words of even length to words of even length. Using the coding $a \to aa, b \to ab, c \to ba$, the action of $\varphi^3$ on $\{a, b, c\}$ is $\psi : a \to baccb, b \to bacc, c \to bacb$. Thus $X = X(\psi)$. 


9.1. INTERVAL EXCHANGE TRANSFORMATIONS

9.1.3 Cantor version of interval exchange

The natural coding can be made continuous at the cost of modifying the space $I = [0, 1]$ as described below.

Suppose that $T$ is minimal, that is, all its orbits are dense in $[0, 1)$. Let $\text{Sep}(T) = \{d_1, \ldots, d_k\}$ be the set of separation points and set $\mathcal{O}(T) = \{T^jd \mid j \in \mathbb{Z}, \ d \in \text{Sep}(T)\}$. We define

$$X = (\{0, 1\} \setminus \mathcal{O}(T)) \cup \{z^-, z^+ \mid z \in \mathcal{O}(T)\}$$

where $0^- = 1$. Defining $x < z^- < z^+ < y$ for all $z \in \mathcal{O}(T)$ and $x, y \in [0, 1) \setminus \mathcal{O}(T)$ such that $x < z < y$ (with the exception of $0^- \geq x$ for all $x \in X$), this extends the natural order on $[0, 1)$ to $X$. Endowed with the topology of intervals, $X$ is a Cantor space because $\mathcal{O}(T)$ is dense in $[0, 1)$.

Let $F : X \to X$ defined by $F(x) = T(x)$ if $x \in [0, 1) \setminus \mathcal{O}(T)$ and $F(z^\epsilon) = T(z)^\epsilon$ if $z \in \mathcal{O}(T)$ where $\epsilon \in \{+, -\}$. The pair $(X, F)$ is a minimal Cantor dynamical system, we will refer to as the Cantor version of the interval exchange $T$.

Let $\phi : X \to [0, 1)$ be defined by $\phi(z^+) = \phi(z^-) = z$ and $\phi(x) = x$ when $x \not\in \mathcal{O}(T)$. This is an onto continuous map. It is one-to-one everywhere except on a countable set of points. Moreover, $\phi \circ F = T \circ \phi$.

Let $T$ be a $k$-interval exchange transformation corresponding to semi-intervals $\Delta_i$. Let $A = \{1, 2, \ldots, k\}$ and let $X$ be the natural representation of $T$. For $w \in A^*$, denote by $I(w)$ the set defined by $I(\varepsilon) = I$ and by

$$I(au) = \Delta_a \cap T^{-1}(I(u)). \quad (9.1.1)$$

It can be verified that every nonempty $I(w)$ is a semi-interval (Exercise 9.2). Note that $I(ua) = I(u) \cap T^{-|u|}\Delta_a$. Note also that

$$b \in R(w) \text{ if and only if } I(w) \cap T^{-|u|}I(b) \neq \emptyset \quad (9.1.2)$$

Figure 9.1.4: The words of length $\leq 6$ of the set $L(T)$. 
Note that the hypothesis that \( T \) right boundary of \( I \) of alphabet \( a \) \( T \).

Thus, placing the vertices of \( L(w) \) on a vertical line in the order given by \( <_1 \) and those of \( R(w) \) on a parallel line in the order given by \( <_2 \), the tree \( \mathcal{E}(w) \) becomes planar. Note that the orders \( <_1 \) and \( <_2 \) do not depend on \( w \).

**Proposition 9.1.8** One has \( w \in \mathcal{L}(X) \) if and only if \( I(w) \neq \emptyset \).

**Proof.** The statement follows from the fact that \( w \in \mathcal{L}(X) \) if and only if \([w] \neq \emptyset\) and that an easy induction shows that \([w] \neq \emptyset\) if and only if \( I(w) \neq \emptyset \).

Besides the semi-intervals \( I(w) \), we will also need symmetrically the sets \( J(w) \) defined, for \( w \in A^* \), by \( J(\varepsilon) = I \) and by

\[
J(ua) = TJ(u) \cap T\Delta_a
\]

for \( a \in A \) and \( u \in A^* \). As for the \( I(w) \), the nonempty sets \( J(w) \) are semi-intervals (Exercise 9.3).

### 9.1.4 Planar dendric shifts

A shift space is called *planar dendric* if it is dendric and if there are two orders \( \leq_1 \) and \( \leq_2 \) on the alphabet \( A \) such that for every \( w \in \mathcal{L}(X) \), the tree \( \mathcal{E}(w) \) is compatible with these orders, that is, for every \((a, b), (c, d) \in \mathcal{E}(w)\), one has \( a \leq_1 c \) if and only if \( b \leq_2 d \).

Thus, placing the vertices of \( L(w) \) on a vertical line in the order given by \( <_1 \) and those of \( R(w) \) on a parallel line in the order given by \( <_2 \), the tree \( \mathcal{E}(w) \) becomes planar. Note that the orders \( <_1 \) and \( <_2 \) do not depend on \( w \).

**Proposition 9.1.9** A regular interval exchange transformation shift is a minimal and planar dendric shift.

**Proof.** Assume that \( T \) is a regular interval exchange transformation relative to \((\Delta_a)_{a \in A}\). Let \( X \) be the natural representation of \( T \). We consider, on the alphabet \( A \), the two orders defined by

1. \( a <_{\text{top}} b \) if \( \Delta_a \) is to the left of \( \Delta_b \),
2. \( a <_{\text{bottom}} b \) if \( T\Delta_a \) is to the left of \( T\Delta_b \).

Note that, if \( wb, wc \) are in \( \mathcal{L}(X) \) then \( b <_{\text{top}} c \) if and only if \( I(wb) \) is to the left of \( I(wc) \). Indeed, \( I(wb), I(wc) \subset I(w) \) and \( T|w| \) is a translation on \( I(w) \).

For \( a, a' \in L(w) \) with \( a <_{\text{bottom}} a' \), there is a unique reduced path in \( \mathcal{E}(w) \) from \( a \) to \( a' \) which is the sequence \( a_1, b_1, \ldots, a_n \) with \( a_1 = a \) and \( a_n = a' \) with \( a_1 <_{\text{bottom}} a_2 <_{\text{bottom}} \cdots <_{\text{bottom}} a_n, b_1 <_{\text{top}} b_2 <_{\text{top}} \cdots <_{\text{top}} b_{n-1} \) and \( T\Delta_{a_i} \cap I_{wb_i} \neq \emptyset, T\Delta_{a_i+1} \cap I_{wb_i} \neq \emptyset \) for \( 1 \leq i \leq n - 1 \) (see Figure 9.1.5). Note that the hypothesis that \( T \) is regular is needed here since otherwise the right boundary of \( T\Delta_a \) could be the left boundary of \( I(wb) \). Thus \( \mathcal{E}(w) \) is a
tree. It is compatible with the orders $<_{\text{bottom}}, <_{\text{top}}$ since the above shows that $a <_{\text{bottom}} a'$ implies that the letters $b_1, b_{n-1}$ such that $(a, b_1), (a', b_{n-1}) \in E(w)$ satisfy $b_1 \leq_{\text{top}} b_{n-1}$.

Figure 9.1.5: A path from $a_1$ to $a_n$ in $E(w)$.

Example 9.1.10 The extension graph of $\varepsilon$ is represented in Figure 9.1.6. The orders $<_{\text{top}}$ and $<_{\text{bottom}}$ are

$$1 <_{\text{top}} 2 <_{\text{top}} 3 \quad \text{and} \quad 2 <_{\text{bottom}} 3 <_{\text{bottom}} 1$$

The following example shows that the Tribonacci shift is not a planar dendric shift.

Example 9.1.11 Let $X$ be the Tribonacci shift (see Example 2.5.2). The words $a, aba$ and $abacaba$ are bispecial. Thus the words $ba, caba$ are right-special and the words $ab, abac$ are left-special. The graphs $E(\varepsilon), E(a)$ and $E(aba)$ are shown in Figure 9.1.7. One sees easily that it is not possible to find two orders on $A$

Figure 9.1.7: The graphs $E(\varepsilon), E(a)$ and $E(aba)$ in the Tribonacci set.

making the representation of the three graphs simultaneously planar.
9.1.5 Rauzy induction

Since the natural representation of a minimal interval exchange $T$ is dendric, it has by Theorem 8.1.40 a primitive $S_e$-adic representation with a directive sequence $\tau = (\tau_n)$ formed of elementary morphisms $\tau_n \in S_e$. We will now give a method, called Rauzy induction which allows to build directly this representation from the interval exchange $T$.

Proposition 9.1.12 Let $T$ be the $k$-interval exchange transformation on $[0,1)$ on the intervals $\Delta_i$ of lengths $\lambda_i$ with the permutation $\pi$. Let $r = \min\{\lambda_k, \lambda_{\pi(k)}\}$. If $T$ is minimal, the transformation induced by $T$ on the interval $[0, r)$ is again a minimal $k$-interval exchange transformation $T'$.

Proof. Set $\ell = \pi(k)$. Assume first that $\lambda_\ell < \lambda_k$ (see Figure 9.1.8). Let $\Delta'_k = \Delta_k \cap T \Delta_\ell$ and let $\Delta_k' = \Delta_k' \cup \Delta_k''$ be a partition of $\Delta_k$. The transformation $T'$ induced by $T$ on $[0, r)$ is such that for every $x \in [0, r)$,

$$T'x = \begin{cases} T x & \text{if } T x \notin \Delta_\ell \\ T^2 x & \text{otherwise.} \end{cases}$$

Thus $T'$ is an interval exchange on the $k$ intervals $\Delta_1, \ldots, \Delta_{k-1}, \Delta_k'$.

In the case $\lambda_k < \lambda_{\pi(k)}$, we replace $T$ by $T^{-1}$ and we find the first case again. The case $\lambda_{\pi(k)} = \lambda_k$ cannot occur because $T$ is minimal.

The natural representation of $T'$ is the shift space $X' = \sigma^{-1}(X)$ where $X$ is the natural coding of $X$ and $\sigma$ is the elementary morphism

$$\alpha_{\ell,k}(i) = \begin{cases} \ell k & \text{if } i = \ell \\ i & \text{otherwise} \end{cases}$$

which places a $k$ after each $\ell$.

In the case $\lambda_k < \lambda_{\pi(k)}$, we obtain the morphism $\tilde{\alpha}_{k,\ell}$ which places an $\ell$ before each $k$. 

Figure 9.1.8: The first case of Rauzy induction.
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Example 9.1.13 Consider the 3-interval exchange of Example 9.1.7 which is a rotation of angle $2\alpha$. The effect of Rauzy induction is represented in Figure 9.1.9. The corresponding morphism is $\alpha_{a,c} : a \rightarrow ac, b \rightarrow b, c \rightarrow c$.

Iterating Rauzy induction, one may obtain a transformation defined on $\Delta_1 \cup \ldots \cup \Delta_{k-1}$. This allows to obtain a BV-representation described as follows.

Theorem 9.1.14 Let $(X, S)$ be the natural representation of a minimal $k$-interval exchange. Then, $(X, S)$ has a BV-representation $(X_E, \varphi_E)$ where $(V, E, \leq)$ is such that

1. $\text{Card}(V(1)) = k$ and $\text{Card}(V(i)) - \text{Card}(V(i+1)) \in \{0, 1\}$ for all $i \geq 1$,

2. for all $i \geq 1$ when $V(i-1) = V(i)$ the incidence matrix $M(i)$ of $E(i)$ has the following form

$$M(i) = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & 0 & s_1 \\
0 & 1 & \ldots & 0 & 0 & \ldots & 0 & s_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 & s_l \\
0 & 0 & \ldots & 1 & 0 & \ldots & 0 & s_{l+1} \\
0 & 0 & \ldots & 0 & 1 & \ldots & 0 & s_{l+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 1 & s_k
\end{bmatrix},$$

where $s_i \in \{0, m, m + 1\}$, $s_l = m$ and $s_{l+1} = m + 1$ for some $m \geq 0$. When $\text{Card}(V(i)) - \text{Card}(V(i+1)) = 1$, the line $l + 1$ does not exist. All the entries of $M(1)$ are equal to 1.

Example 9.1.15 Let us consider again the interval exchange which is the rotation of angle $2\alpha$ of Example 9.1.7. A first step of Rauzy induction gives the interval exchange of Example 9.1.13. A second step gives the interval exchange represented in Figure 9.1.10. The combination of the two steps corresponds to the morphism $\varphi : a \rightarrow ac, b \rightarrow b, c \rightarrow cac$. The composition matrix of this morphism is of the form given in Theorem 9.1.14 with the order $b < a < c$. 
Indeed, with this order
\[
M(\varphi) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}
\]
which is of form given in case 2 with \(s_1 = 0, s_2 = 1, s_3 = 2\) and \(\ell = 1, m = 1\).

### 9.1.6 Induced transformations and admissible semi-intervals

Let \(T\) be a minimal interval exchange transformation on \(I = [\ell, r)\). Let \(J \subseteq [\ell, r)\) be a semi-interval. Since \(T\) is minimal, for each \(z \in [\ell, r)\) there is an integer \(n > 0\) such that \(T^n(z) \in J\).

As seen before, the transformation induced by \(T\) on \(J\) is the transformation \(S: J \rightarrow J\) defined for \(z \in J\) by \(S(z) = T^n(z)\) with \(n = \min\{n > 0 \mid T^n(z) \in J\}\). We also say that \(S\) is the first return map (of \(T\)) on \(J\). The semi-interval \(J\) is called the domain of \(S\), denoted \(D(S)\).

**Example 9.1.16** Let \(T\) be the transformation of Example 9.1.7. Let \(J = [0, 2\alpha)\). The transformation induced by \(T\) on \(J\) is
\[
S(z) = \begin{cases}
T^2(z) & \text{if } 0 \leq z < 1 - 2\alpha \\
T(z) & \text{otherwise.}
\end{cases}
\]

Let \(T\) be an interval exchange transformation relative to \((\Delta_a)_{a \in A}\). Denote by \(d_a\) for \(a \in A\) the separation points. For \(\ell < t < r\), the semi-interval \([\ell, t)\) is right admissible for \(T\) if there is a \(k \in \mathbb{Z}\) such that \(t = T^k(d_a)\) for some \(a \in A\) and

(i) if \(k > 0\), then \(t < T^h(d_a)\) for all \(h\) such that \(0 < h < k\),

(ii) if \(k \leq 0\), then \(t < T^h(d_a)\) for all \(h\) such that \(k < h \leq 0\).

We also say that \(t\) itself is right admissible. Note that all semi-intervals \([\ell, d_a)\) with \(\ell < d_a\) are right admissible. Similarly, all semi-intervals \([\ell, Td_a)\) with \(\ell < Td_a\) are right admissible.

**Example 9.1.17** Let \(T\) be the interval exchange transformation of Example 9.1.7. The semi-interval \([0, t)\) for \(t = 1 - 2\alpha\) or \(t = 1 - \alpha\) is right admissible since \(1 - 2\alpha = d_b\) and \(1 - \alpha = d_c\). On the contrary, for \(t = 2 - 3\alpha\), it is not right admissible because \(t = T^{-1}(d_c)\) but \(d_c < t\) contradicting (ii).
The following result is the basis for the definition of Rauzy induction.

**Theorem 9.1.18 (Rauzy)** Let $T$ be a regular $k$-interval exchange transformation and let $J$ be a right admissible interval for $T$. The transformation induced by $T$ on $J$ is a regular $k$-interval exchange transformation.

Note that a $k$-interval exchange transformation $T$ on $[\ell, r)$, the transformation induced by $T$ on any semi-interval included in $[\ell, r)$ is always an interval exchange transformation on at most $k + 2$ intervals by Theorem 9.1.3.

**Example 9.1.19** Consider again the transformation of Example 9.1.7. The transformation induced by $T$ on the semi-interval $I = [0, 2\alpha)$ is the 3-interval exchange transformation represented in Figure 9.1.11.

![Figure 9.1.11: The transformation induced on $I$.](image)

The notion of left admissible interval is symmetrical to that of right admissible. For $\ell < t < r$, the semi-interval $[t, r)$ is left admissible for $T$ if there is a $k \in \mathbb{Z}$ such that $t = T^k(a)$ for some $a \in A$ and

(i) if $k > 0$, then $T^h(a) < t$ for all $h$ such that $0 < h < k$,

(ii) if $k \leq 0$, then $T^h(a) < t$ for all $h$ such that $k < h \leq 0$.

We also say that $t$ itself is left admissible. Note that, as for right induction, the semi-intervals $[d_a, r)$ and $[T^k d_a, r)$ are left admissible. The symmetrical statements of Theorem 9.1.18 also hold for left admissible intervals.

Let us now generalize the notion of admissibility to a two-sided version. For a semi-interval $J = [u, v) \subset [\ell, r)$, we define the following functions on $[\ell, r)$:

\[
\rho_{J,T}^+(z) = \min \{ n > 0 \mid T^n(z) \in (u, v) \}, \quad \rho_{J,T}^-(z) = \min \{ n \geq 0 \mid T^{-n}(z) \in (u, v) \}.
\]

We then define for every $z \in [\ell, r)$ three sets. First, let $E_{J,T}(z)$ be the following set of indices.

\[
E_{J,T}(z) = \{ k \mid -\rho_{J,T}^- (z) \leq k < \rho_{J,T}^+ (z) \}.
\]

Next, the set of neighbors of $z$ with respect to $J$ and $T$ is

\[
N_{J,T}(z) = \{ T^k(z) \mid k \in E_{J,T}(z) \}.
\]
Finally, the set of division points of $I$ with respect to $T$ is the finite set

$$\text{Div}(J,T) = \bigcup_{i=1}^{n} N_{J,T}(d_i).$$

We now formulate the following definition. For $\ell \leq u < v \leq r$, we say that the semi-interval $J = [u,v)$ is admissible for $T$ if $u, v \in \text{Div}(J,T) \cup \{r\}$.

Note that a semi-interval $[\ell, v)$ is right admissible if and only if it is admissible and that a semi-interval $[u, r)$ is left admissible if and only if it is admissible. Note also that $[\ell, r)$ is admissible.

Note also that for a regular interval exchange transformation relative to a partition $(\Delta_a)_{a \in A}$, each of the semi-intervals $\Delta_a$ (or $T\Delta_a$) is admissible although only the first one is right admissible (and the last one is left admissible). Actually, we will prove that for every word $w$, the semi-intervals $I(w)$ and $J(w)$ are admissible. In order to do that, we need the following lemma.

**Lemma 9.1.20** Let $T$ be a $k$-interval exchange transformation on the semi-interval $[\ell, r)$. For any $n \geq 1$, the set $P_n = \{T^h(d_i) \mid 1 \leq i \leq k, 1 \leq h \leq n\}$ is the set of $(k-1)n+1$ left boundaries of the semi-intervals $J(y)$ for all words $y$ with $|y| = n$.

*Proof.* Let $Q_n$ be the set of left boundaries of the intervals $J(y)$ for $|y| = n$. Since $\text{Card}((L(T) \cap A^n) = (k-1)n+1$ by Proposition 9.1.15, we have $\text{Card}(Q_n) = (k-1)n+1$. Since $T$ is regular the set $R_n = \{T^h(d_i) \mid 2 \leq i \leq k, 1 \leq h \leq n\}$ is made of $(k-1)n$ distinct points. Moreover, since

$$d_1 = T(d_{n(1)}), \quad T(d_1) = T^2(d_{n(1)}), \ldots, T^{n-1}(d_1) = T^n(d_{n(1)}),$$

we have $P_n = R_n \cup \{T^n(d_1)\}$. This implies $\text{Card}(P_n) \leq (k-1)n+1$. On the other hand, if $y = b_0 \cdots b_{n-1}$, then $J(y) = \cap_{i=0}^{n-1} T^{-i}J(b_i)$. Thus the left boundary of each $J(y)$ is the left boundary of some $T^hI(a)$ for some $h$ with $1 \leq h \leq n$ and some $a \in A$. Consequently $Q_n \subset P_n$. This proves that $\text{Card}(P_n) = (k-1)n+1$ and that consequently $P_n = Q_n$. \hfill $\blacksquare$

A dual statement holds for the semi-intervals $I(y)$.

**Proposition 9.1.21** Let $T$ be a $k$-interval exchange transformation on the semi-interval $[\ell, r)$. For any $w \in F(T)$, the semi-interval $J(w)$ is admissible.

*Proof.* Set $|w| = n$ and $J(w) = [u,v)$. By Lemma 9.1.20, we have $u = T^g(d_i)$ for $1 \leq i \leq k$ and $1 \leq g \leq n$. Similarly, we have $v = r$ or $v = T^d(d_j)$ for $1 \leq j \leq k$ and $1 \leq d \leq n$.

For $1 \leq h < g$, the point $T^h(d_i)$ is the left boundary of some semi-interval $J_y$ with $|y| = n$ and thus $T^h(d_i) \notin J(w)$. This shows that $g \in E_{J(w),T}(d_i)$ and thus that $u \in \text{Div}(J(w), T)$.

If $v = r$, then $v \in \text{Div}(J(w), T)$. Otherwise, one shows in the same way as above that $v \in \text{Div}(J(w), T)$. Thus $J(w)$ is admissible. \hfill $\blacksquare$
Note that the same statement holds for the semi-intervals $I(w)$ instead of the semi-intervals $J(w)$ (using the dual statement of Lemma 9.1.20).

It can be useful to reformulate the definition of a division point and of an admissible pair using the terminology of graphs. Consider the graph with vertex set $[\ell, r)$ and edges the pairs $(z, T(z))$ for $z \in [\ell, r]$. Then, if $T$ is minimal and $I$ is a semi-interval, for any $z \in [\ell, r)$, there is a path $P_{I,T}(z)$ such that its origin $x$ and its end $y$ are in $I$, $z$ is on the path, $z \neq y$ and no vertex of the path except $x, y$ are in $I$ (actually $x = T^{-n}(z)$ with $n = \rho_{I,T}(z)$ and $y = T^m(z)$ with $m = \rho_{I,T}^+(z)$). Then the division points of $I$ are the vertices which are on a path $P_{I,T}(d_i)$ but not at its end (see Figure 9.1.12).

The following is a generalization of Theorem 9.1.18. Recall that Sep($T$) denotes the set of separation points of $T$, i.e. the points $d_1 = 0, d_2, \ldots, d_k$ (which are the left boundaries of the semi-intervals $\Delta_1, \ldots, \Delta_k$).

**Theorem 9.1.22** Let $T$ be a regular $k$-interval exchange transformation on $[\ell, r)$. For any admissible semi-interval $I = [u, v)$, the transformation $S$ induced by $T$ on $I$ is a regular $k$-interval exchange transformation with separation points $\text{Sep}(S) = \text{Div}(I,T) \cap I$.

**Proof.** Since $T$ is regular, it is minimal. Thus for each $i \in \{2, \ldots, k\}$ there are points $x_i, y_i \in (u, v)$ such that there is a path from $x_i$ to $y_i$ passing by $d_i$ but not containing any point of $I$ except at its origin and its end. Since $T$ is regular, the $x_i$ are all distinct and the $y_i$ are all distinct.

Since $I$ is admissible, there exist $\lambda, \rho \in \{1, \ldots, k\}$ such that $u \in N_{I,T}(d_\lambda)$ and $v \in N_{I,T}(d_\rho)$. Moreover, since $u$ is a neighbor of $d_\lambda$ with respect to $I$, $u$ is on the path from $x_\lambda$ to $y_\lambda$ (it can be either before or after $d_\lambda$). Similarly, $v$ is on the path from $x_\rho$ to $y_\rho$ (see Figure 9.1.13 where $u$ is before $d_\lambda$ and $v$ is after $d_\rho$).

Set $x_1 = y_1 = u$. Let $(I_j)_{1 \leq j \leq k}$ be the partition of $I$ in semi-intervals such that $x_j$ is the left boundary of $I_j$ for $1 \leq j \leq k$. Let $J_j$ be the partition of $I$ such that $y_j$ is the left boundary of $J_j$ for $1 \leq j \leq k$. We will prove that

$$S(I_j) = \begin{cases} J_j & \text{if } j \neq 1, \lambda \\ J_1 & \text{if } j = \lambda \\ J_\lambda & \text{if } j = 1 \end{cases}$$
and that the restriction of $S$ to $I_j$ is a translation.

Assume first that $j \neq 1, \lambda$. Then $S(x_j) = y_j$. Let $n$ be such that $y_j = T^n(x_j)$ and denote $I'_j = I_j \setminus x_j$. We will prove by induction on $h$ that for $0 \leq h \leq k-1$, the set $T^h(I'_j)$ does not contain $u, v$ or any $x_i$. It is true for $h = 0$. Assume that it holds up to $h < n - 1$.

For any $h'$ with $0 \leq h' \leq h$, the set $T^{h'}(I'_j)$ does not contain any $\gamma_i$. Indeed, otherwise there would exist $h''$ with $0 \leq h'' \leq h'$ such that $x_i \in T^{h''}(I'_j)$, a contradiction. Thus $T$ is a translation on $T^{h}(I'_j)$. This implies that $T^h$ is a translation on $I_j$. Note also that $T^{h}(I'_j) \cap I = \emptyset$. Assume the contrary. We first observe that we cannot have $T^{h}(x_j) \in I$. Indeed, $h < n$ implies that $T^{h}(x_j) \notin (u, v)$. And we cannot have $T^{h}(x_j) = u$ since $j \neq \lambda$. Thus $T^{h}(I'_j) \cap I \neq \emptyset$ implies that $u \in T^{h}(I'_j)$, a contradiction.

Suppose that $u = T^{h+1}(z)$ for some $z \in I'_j$. Since $u$ is on the path from $x_\lambda$ to $y_\lambda$, it implies that for some $h'$ with $0 \leq h' \leq h$ we have $x_\lambda = T^{h'}(z)$, a contradiction with the induction hypothesis. A similar proof (using the fact that $v$ is on the path from $x_\rho$ to $y_\rho$) shows that $T^{h+1}(I'_j)$ does not contain $v$.

Finally suppose that some $x_i$ is in $T^{h+1}(I'_j)$. Since the restriction of $T^h$ to $I_j$ is a translation, $T^h(I'_j)$ is a semi-interval. Since $T^{h+1}(x_j) \notin I$ the fact that $T^{h+1}(I'_j) \cap I$ is not empty implies that $u \in T^h(I'_j)$, a contradiction.

This shows that $T^n$ is continuous at each point of $I'_j$ and that $S = T^n(x)$ for all $x \in I_j$. This implies that the restriction of $S$ to $I_j$ is a translation into $I_j$.

If $j = 1$, then $S(x_1) = S(u) = y_\lambda$. The same argument as above proves that the restriction of $S$ to $I_1$ is a translation form $I_1$ into $J_\lambda$. Finally if $j = \lambda$, then $S(x_\lambda) = x_1 = u$ and, similarly, we obtain that the restriction of $S$ to $I_\lambda$ is a translation into $I_1$.

Since $S$ is the transformation induced by the transformation $T$ which is one to one, it is also one to one. This implies that the restriction of $S$ to each of the semi-intervals $I_j$ is a bijection onto the corresponding interval $J_j, J_1$ or $J_\lambda$ according to the value of $j$.

This shows that $S$ is an $s$-interval exchange transformation. Since the orbits of the points $x_2, \ldots, x_k$ relative to $S$ are included in the orbits of $d_2, \ldots, d_k$, they are infinite and disjoint. Thus $S$ is regular.

Let us finally show that $\text{Sep}(S) = \text{Div}(I, T) \cap I$. We have $\text{Sep}(S) = \{x_1, x_2, \ldots, x_k\}$ and $x_1 \in N_{I, T}(d_1)$. Thus $\text{Sep}(S) \subset \text{Div}(I, T) \cap I$. Conversely,
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Let \( x \in \text{Div}(I,T) \cap I \). Then \( x \in N_{I,T}(d_i) \cap I \) for some \( 1 \leq i \leq k \). If \( i \neq 1, \lambda \), then \( x = x_i \). If \( i = 1 \), then either \( x = u \) (if \( u = l \)) or \( x = x_{\pi(1)} \) since \( d_1 = T(d_{\pi(1)}) \). Finally, if \( i = \lambda \) then \( x = u \) or \( x = x_\lambda \). Thus \( x \in \text{Sep}(S) \) in all cases.

9.1.7 A closure property

We will prove that the family of regular interval exchange shifts is closed under derivation. The same property holds for minimal dendric shifts by Theorem 8.1.23. Thus regular interval exchange shifts form a subfamily closed under derivation of minimal dendric shifts.

**Theorem 9.1.23** Any derived shift of a regular \( k \)-interval exchange shift \( X \) with respect to some \( w \in \mathcal{L}(X) \) is a regular \( k \)-interval exchange shift.

We first prove the following lemma.

**Lemma 9.1.24** Let \( T \) be a regular interval exchange transformation and let \( (X,S) \) be its natural representation. For \( w \in \mathcal{L}(X) \), let \( T' \) be the transformation induced by \( T \) on \( J_w \). One has \( x \in \mathcal{R}_X(w) \) if and only if

\[
\Sigma_T(z) = x\Sigma_T(T'(z))
\]

for some \( z \in J_w \).

**Proof.** Assume first that \( x \in \mathcal{R}_X(w) \). Then for any \( z \in J_w \cap I_z \), we have \( T'(z) = T^{\vert z \vert}(z) \) and

\[
\Sigma_T(z) = x\Sigma_T(T^{\vert z \vert}(z)) = x\Sigma_T(T'(z)).
\]

Conversely, assume that \( \Sigma_T(z) = x\Sigma_T(T'(z)) \) for some \( z \in J_w \). Then \( T^{\vert z \vert}(z) \in J_w \) and thus \( wx \in A^*w \) which implies that \( x \in \Gamma_X(w) \). Moreover, \( x \) does not have a proper prefix in \( \Gamma_X(w) \) and thus \( x \in \mathcal{R}_X(w) \).

Since a regular interval exchange shift is recurrent, the previous lemma says that the natural coding of a point in \( J_w \) is a concatenation of first return words to \( w \). Moreover, note also that \( T^n(z) \in J_w \) if and only if the prefix of length \( n \) of \( \Sigma_T(z) \) is a return word to \( w \).

**Proof of Theorem 9.1.23** Let \( T \) be a regular \( s \)-interval exchange transformation and let \( X \) be its natural representation.

Let \( w \in \mathcal{L}(X) \). Since the semi-interval \( J_w \) is admissible according to Proposition 9.1.21, the transformation \( T' \) induced by \( T \) on \( J_w \) is, by Theorem 9.1.22, a \( k \)-interval exchange transformation. The corresponding partition of \( J_w \) is the family \( (J_{w,x})_{x \in \mathcal{R}_X(w)} \).

Using Lemma 9.1.24 and the observation following it, it is clear that \( \Sigma_T(z) = \varphi(\Sigma_S(z)), \) where \( z \) is a point of \( J_w \) and \( \varphi : A^* \to \mathcal{R}_X(w)^* \) is a coding morphism for \( \mathcal{R}_X(w) \).

Set \( x = \Sigma_T(T^{-\vert w \vert}(z)) \) and \( y = \Sigma_T(z) \). Then \( x = wy \) and thus \( \Sigma_S(z) = \mathcal{D}_{w}(x) \). This shows that the derived shift of \( X \) with respect to \( w \) is the natural representation of \( T' \).
Example 9.1.25 Consider the interval exchange of Example 9.1.7. The set of return words to $c$ is $R_X(c) = \{bac, bbac, c\}$. The natural coding of transformation induced on the interval $\Delta_c$ is the interval exchange represented in Figure 9.1.14. The intervals of the upper level are labeled by $R'_X(c)$ and those of the lower level by $R_X(c)$ (we shall describe this transformation in more detail in Examples 9.2.8 and 9.2.17).

![Figure 9.1.14: The transformation induced on $\Delta_c$](image)

9.2 Branching Rauzy induction

In this section we introduce a branching version of Rauzy induction and generalize Rauzy’s results to the two-sided case (Theorems 9.1.22 and 9.2.3). In particular we characterize in Theorem 9.2.3 the admissible semi-intervals for an interval exchange transformation.

Let $T = T_{\lambda, \pi}$ be a regular $k$-interval exchange transformation on $[\ell, r)$. Set $Z(T) = [\ell, \max\{d_k, Td_{\pi(k)}\})$.

Note that $Z(T)$ is the largest semi-interval which is right-admissible for $T$. We denote by $\psi(T)$ the transformation induced by $T$ on $Z(T)$.

**Theorem 9.2.1 (Rauzy)** Let $T$ be a regular interval exchange transformation. A semi-interval $I$ is right admissible for $T$ if and only if there is an integer $n \geq 0$ such that $I = Z(\psi^n(T))$. In this case, the transformation induced by $T$ on $I$ is $\psi^{n+1}(T)$.

The map $T \mapsto \psi(T)$ is called the right Rauzy induction. There are actually, as we have seen, two cases according to $d_k < Td_{\pi(k)}$ (Case 0) or $d_k > Td_{\pi(k)}$ (Case 1). We cannot have $d_k = Td_{\pi(k)}$ since $T$ is regular.

In Case 0, we have $Z(T) = [\ell, Td_{\pi(k)})$ and for any $z \in Z(T)$,

$$S(z) = \begin{cases} T^2(z) & \text{if } z \in I(a_{\pi(k)}) \\ T(z) & \text{otherwise.} \end{cases}$$

The transformation $S$ is the interval exchange transformation relative to $(K_a)_{a \in A}$ with $K(a) = I(a) \cap Z(T)$ for all $a \in A$. Note that $K(a) = I(a)$ for $a \neq a_k$. The
translation values $\beta_a$ are defined as follows, denoting $\alpha_i, \beta_i$ instead of $\alpha_{a_i}, \beta_{a_i},$

$$\beta_i = \begin{cases} 
\alpha_{\pi(k)} + \alpha_k & \text{if } i = \pi(k) \\
\alpha_i & \text{otherwise.}
\end{cases}$$

In summary, in Case 0, the semi-interval $J(a_{\pi(k)})$ is suppressed, the semi-interval $J(a_k)$ is split into $SK(a_k)$ and $SK(a_{\pi(k)})$. The left boundaries of the semi-intervals $K(a)$ are the left boundaries of the semi-intervals $I(a)$. The transformation is represented in Figure 9.2.1 in which the left boundary of the semi-interval $S(K(a_{\pi(k)})$ is $Sd_{\pi(k)}$.

![Figure 9.2.1: Case 0 in Rauzy induction.](image)

In Case 1, we have $Z(T) = [\ell, d_k]$ and for any $z \in Z(T)$,

$$S(z) = \begin{cases} 
T^2(z) & \text{if } z \in T^{-1}(I_{a_k}) \\
T(z) & \text{otherwise.}
\end{cases}$$

The transformation $S$ is the interval exchange transformation relative to $(K(a))_{a \in A}$ with

$$K(a) = \begin{cases} 
T^{-1}I(a) & \text{if } a = a_k \\
T^{-1}(TI(a) \cap Z(T)) & \text{otherwise.}
\end{cases}$$

Note that $K(a) = I(a)$ for $a \neq a_k$ and $a \neq a_{\pi(k)}$. Moreover $K(a) = S^{-1}(TI(a) \cap Z(T))$ in all cases. The translation values $\beta_i$ are defined by

$$\beta_i = \begin{cases} 
\alpha_{\pi(k)} + \alpha_k & \text{if } i = k \\
\alpha_i & \text{otherwise.}
\end{cases}$$

In summary, in Case 1, the semi-interval $I(a_{s})$ is suppressed, the semi-interval $I(a_{\pi(k)})$ is split into $K(a_{\pi(k)})$ and $K(a_k)$. The left boundaries of the semi-intervals $SK(a)$ are the left boundaries of the semi-intervals $J(a)$. The transformation is represented in Figure 9.2.2 where the left boundary of the semi-interval $K(a_k)$ is $d'_k$. 
Example 9.2.2 Consider again the transformation $T$ of Example 9.1.7. Since $Z(T) = [0, 2\alpha)$, the transformation $\psi(T)$ is represented in Figure 9.1.11. The transformation $\psi^2(T)$ is represented in Figure 9.2.3.

![Figure 9.2.3: The transformation $\psi^2(T)$](image)

The symmetrical notion of left Rauzy induction is defined similarly as follows. Let $T = T_{\lambda, \pi}$ be a regular $s$-interval exchange transformation on $[\ell, r)$. Set $Y(T) = \min\{d_2, Td_{\pi(2)}\}, r)$. We denote by $\varphi(T)$ the transformation induced by $T$ on $Y(T)$. The map $T \mapsto \varphi(T)$ is called the left Rauzy induction.

The symmetrical assertions of Theorem 9.2.1 also hold for left admissible intervals.

9.2.1 Branching induction

The following is an addition to Theorem 9.1.18.

**Theorem 9.2.3** Let $T$ be a regular $k$-interval exchange transformation on $[\ell, r)$. A semi-interval $I$ is admissible for $T$ if and only if there is a sequence $\chi \in \{\varphi, \psi\}^*$ such that $I$ is the domain of $\chi(T)$. In this case, the transformation induced by $T$ on $I$ is $\chi(T)$.
We first prove the following lemmas, in which we assume that $T$ is a regular $s$-interval exchange transformation on $[\ell, r)$. Recall that $Y(T), Z(T)$ are the domains of $\varphi(T), \psi(T)$ respectively.

**Lemma 9.2.4** If a semi-interval $I$ strictly included in $[\ell, r)$ is admissible for $T$, then either $I \subset Y(T)$ or $I \subset Z(T)$.

**Proof.** Set $I = [u, v)$. Since $I$ is strictly included in $[\ell, r)$, we have either $\ell < u$ or $v < r$. Set $Y(T) = [y, r)$ and $Z(T) = [\ell, z)$.

Assume that $v < r$. If $y \leq u$, then $I \subset Y(T)$. Otherwise, let us show that $v \leq z$. Assume the contrary. Since $I$ is admissible, we have $v = T^j(d_i)$ with $j \in E_{I,T}(d_i)$ for some $i$ with $1 \leq i \leq k$. But $j > 0$ is impossible since $u < T(d_i) < v$ implies $T(d_i) \in (u, v)$, in contradiction with the fact that $j < \rho^T_{I}(d_i)$. Similarly, $j \leq 0$ is impossible since $u < d_i < v$ implies $d_i \in [u, v)$. Thus $I \subset Z(T)$.

The proof in the case $\ell < u$ is symmetric. ■

**Lemma 9.2.5** Let $T$ be a regular $k$-interval exchange transformation on $[\ell, r)$. Let $J$ be an admissible semi-interval for $T$ and let $S$ be the transformation induced by $T$ on $J$. A semi-interval $I \subset J$ is admissible for $T$ if and only if it is admissible for $S$. Moreover $\text{Div}(J, T) \subset \text{Div}(I, T)$.

**Proof.** Set $J = [t, w)$ and $I = [u, v)$. Since $J$ is admissible for $T$, the transformation $S$ is a regular $k$-interval exchange transformation by Theorem 9.1.22.

Suppose first that $I$ is admissible for $T$. Then $u = T^s(d_i)$ with $g \in E_{I,T}(d_i)$ for some $1 \leq i \leq k$, and $v = T^d(d_j)$ with $d \in E_{I,T}(d_j)$ for some $1 \leq j \leq k$ or $v = r$.

Since $S$ is the transformation induced by $T$ on $J$, there is a separation point $x$ of the form $x = T^m(d_i)$ with $m = -\rho^T_{I,T}(d_i)$ and thus $m \in E_{I,T}(d_i)$.

Thus $u = T^{s-s^j}(x)$.

Assume first that $g - m > 0$. Since $u, x \in J$, there is an integer $n$ with $0 < n \leq g - m$ such that $u = S^n(x)$.

Let us show that $n \in E_{I,S}(x)$. Assume by contradiction that $\rho^T_{I,S}(x) \leq n$. Then there is some $j$ with $0 < j \leq n$ such that $S^j(x) \in (u, v)$. But we cannot have $j \leq n$ since $u \notin (u, v)$. Thus $j < n$.

Next, there is $h$ with $0 < h < g - m$ such that $T^h(x) = S^j(x)$. Indeed, setting $y = S^j(x)$, we have $u = T^{g-m-h}(y) = S^{n-j}(y)$ and thus $h < g - m$. If $0 < h \leq -m$, then $T^h(x) = T^{m+h}(d_i) \in I \subset J$ contradicting the hypothesis that $m \in E_{I,T}(d_i)$. If $-m < h < g - m$, then $T^h(x) = T^{m+h}(d_i) \in I$, contradicting the fact that $g \in E_{I,T}(d_i)$. This shows that $n \in E_{I,S}(x)$ and thus that $u \in \text{Div}(I, S)$.

Assume next that $g - m \leq 0$. There is an integer $n$ with $g - m \leq n \leq 0$ such that $u = S^n(x)$. Let us show that $n \in E_{I,S}(x)$. Assume by contradiction that $n < -\rho^T_{I,S}(x)$. Then there is some $j$ with $n < j < 0$ such that $S^j(x) = T^h(x)$. Then $T^h(x) = T^{h+m}(d_i) \in I$ with $g < h + m < m$, in contradiction with the hypothesis that $m \in E_{I,T}(d_i)$.
We have proved that \( u \in \text{Div}(I, S) \). If \( v = r \), the proof that \( I \) is admissible for \( S \) is complete. Otherwise, the proof that \( v \in \text{Div}(I, S) \) is similar to the proof for \( u \).

Conversely, if \( I \) is admissible for \( S \), there is some \( x \in \text{Sep}(S) \) and \( g \in E_{I, S}(x) \) such that \( u = S^g(x) \). But \( x = T^m(d_i) \) and since \( u, x \in J \) there is some \( n \) such that \( u = T^n(d_i) \).

Assume for instance that \( n > 0 \) and suppose that there exists \( k \) with \( 0 < k < n \) such that \( T^k(d_i) \in (u, v) \). Then, since \( I \subset J \), \( T^k(d_i) \) is of the form \( S^h(x) \) with \( 0 < h < g \) which contradicts the fact that \( g \in E_{I, S}(x) \). Thus \( n \in E_{I, T}(d_i) \) and \( u \in \text{Div}(I, T) \).

The proof is similar in the case \( n \leq 0 \).

If \( v = r \), we have proved that \( I \) is admissible for \( T \). Otherwise, the proof that \( v \in \text{Div}(I, T) \) is similar.

Finally, assume that \( I \) is admissible for \( T \) (and thus for \( S \)). For any \( d_i \in \text{Sep}(T) \), one has

\[
\rho^-_{I, T}(d_i) \geq \rho^-_{J, T}(d_i) \quad \text{and} \quad \rho^+_{I, T}(d_i) \geq \rho^+_{J, T}(d_i)
\]

showing that \( \text{Div}(J, T) \subset \text{Div}(I, T) \).

The last lemma is the key argument to prove Theorem 9.2.3.

**Lemma 9.2.6** For any admissible interval \( I \subset [\ell, r] \), the set \( \mathcal{F} \) of sequences \( \chi \in \{\varphi, \psi\}^* \) such that \( I \subset D(\chi(T)) \) is finite.

**Proof.** The set \( \mathcal{F} \) is suffix-closed. Indeed it contains the empty word because \([\ell, r] \) is admissible. Moreover, for any \( \xi, \chi \in \{\varphi, \psi\}^* \), one has \( D(\xi \chi(T)) \subset D(\chi(T)) \) and thus \( \xi \chi \in \mathcal{F} \) implies \( \chi \in \mathcal{F} \).

The set \( \mathcal{F} \) is finite. Indeed, by Lemma 9.2.5 applied to \( J = D(\chi(T)) \), for any \( \chi \in \mathcal{F} \), one has \( \text{Div}(D(\chi(T)), T) \subset \text{Div}(I, T) \). In particular, the boundaries of \( D(\chi(T)) \) belong to \( \text{Div}(I, T) \). Since \( \text{Div}(I, T) \) is a finite set, this implies that there is a finite number of possible semi-intervals \( D(\chi(T)) \). Thus there is no infinite word with all its suffixes in \( \mathcal{F} \). Since the sequences \( \chi \) are binary, this implies that \( \mathcal{F} \) is finite.

**Proof of Theorem 9.2.3.** We consider \( \chi \) as a word on the alphabet \( \varphi, \psi \). We first prove by induction on the length of \( \chi \) that the domain \( I \) of \( \chi(T) \) is admissible and that the transformation induced by \( T \) on \( I \) is \( \chi(T) \). It is true for \( |\chi| = 0 \) since \([\ell, r] \) is admissible and \( \chi(T) = T \). Next, assume that \( J = D(\chi(T)) \) is admissible and that the transformation induced by \( T \) on \( J \) is \( \chi(T) \). Then \( D(\varphi \chi(T)) \) is admissible for \( \chi(T) \) since \( D(\varphi \chi(T)) = Y(\chi(T)) \). Thus \( I = D(\varphi \chi(T)) \) is admissible for \( T \) by Lemma 9.2.5 and the transformation induced by \( T \) on \( I \) is \( \varphi \chi(T) \). The same proof holds for \( \psi \chi \).

Conversely, assume that \( I \) is admissible. By Lemma 9.2.6, the set \( \mathcal{F} \) of sequences \( \chi \in \{\varphi, \psi\}^* \) such that \( I \subset D(\chi(T)) \) is finite.

Thus there is some \( \chi \in \mathcal{F} \) such that \( \varphi \chi, \psi \chi \notin \mathcal{F} \). If \( I \) is strictly included in \( D(\chi(T)) \), then by Lemma 9.2.4 applied to \( \chi(T) \), we have \( I \subset Y(\chi(T)) = I(S) \).
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\[D(\varphi\chi(T)) \text{ or } I \subset Z(\chi(T)) = D(\psi\chi(T)), \text{ a contradiction. Thus } I = D(\chi(T)).\]

We close this subsection with a result concerning the dynamics of the branching induction.

**Theorem 9.2.7** For any sequence \((T_n)_{n \geq 0}\) of regular interval exchange transformations such that \(T_{n+1} = \varphi(T_n)\) or \(T_{n+1} = \psi(T_n)\) for all \(n \geq 0\), the length of the domain of \(T_n\) tends to 0 when \(n \to \infty\).

**Proof.** Assume the contrary and let \(I\) be an open interval included in the domain of \(T_n\) for all \(n \geq 0\). The set \(\text{Div}(I, T) \cap I\) is formed of \(s\) points. For any pair \(u, v\) of consecutive elements of this set, the semi-interval \([u, v)\) is admissible. By Lemma 9.2.6, there is an integer \(n\) such that the domain of \(T_n\) does not contain \([u, v)\), a contradiction. \(\blacksquare\)

### 9.2.2 Equivalent transformations

Let \(\ell_1, r_1\), \(\ell_2, r_2\) be two semi-intervals of the real line. Let \(T_1 = T_{\lambda_1, \pi_1}\) be an \(s\)-interval exchange transformation relative to a partition of \([\ell_1, r_1)\) and \(T_2 = T_{\lambda_2, \pi_2}\) another \(s\)-interval exchange transformations relative to \([\ell_2, r_2)\). We say that \(T_1\) and \(T_2\) are equivalent if \(\pi_1 = \pi_2\) and \(\lambda_1 = c\lambda_2\) for some \(c > 0\). Thus, two interval exchange transformations are equivalent if we can obtain the second from the first by a rescaling following by a translation. We denote by \([T_{\lambda,\pi}]\) the equivalence class of \(T_{\lambda,\pi}\).

**Example 9.2.8** Let \(S = T_{\mu,\pi}\) be the 3-interval exchange transformation on a partition of the semi-interval \((2\alpha, 1)\), with \(\alpha = (3 - \sqrt{5})/2\), represented in Figure 9.2.4. \(S\) is equivalent to the transformation \(T = T_{\lambda,\pi}\) of Example 9.1.7 with length vector \(\lambda = (1 - 2\alpha, \alpha, \alpha)\) and permutation the cycle \(\pi = (132)\). Indeed the length vector \(\mu = (8\alpha - 3, 2 - 5\alpha, 2 - 5\alpha)\) satisfies \(\mu = \frac{2 - 5\alpha}{\alpha} \lambda\).

![Figure 9.2.4: The transformation S.](image_url)

Note that if \(T\) is a minimal (resp. regular) interval exchange transformation and \([S] = [T]\), then \(S\) is also minimal (resp. regular).

For an interval exchange transformation \(T\) we consider the directed labelled graph \(G(T)\), called the induction graph of \(T\), defined as follows. The vertices are
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the equivalence classes of transformations obtained starting from \( T \) and applying all possible \( \chi \in \{ \psi, \varphi \}^* \). There is an edge labelled \( \psi \) (resp. \( \varphi \)) from a vertex \([S]\) to a vertex \([U]\) if and only if \( U = \psi(S) \) (resp \( \varphi(S) \)) for two transformations \( S \in [S] \) and \( U \in [U] \).

**Example 9.2.9** Let \( \alpha = \frac{3 - \sqrt{5}}{2} \) and \( R \) be a rotation of angle \( \alpha \). It is a 2-interval exchange transformation on \([0, 1)\) relative to the partition \([0, 1 - \alpha), [1 - \alpha, 1)\).

The induction graph \( G(R) \) of the transformation is represented in the left of Figure 9.2.6.

Note that for a 2-interval exchange transformation \( T \), one has \([\psi(T)] = [\varphi(T)]\), whereas in general the two transformations are not equivalent.

The induction graph of an interval exchange transformation can be infinite. A sufficient condition for the induction graph to be finite is given in Section 9.3.

Let us now introduce a variant of this equivalence relation (and of the related graph).

For a \( k \)-interval exchange transformation \( T = T_{\lambda, \pi} \), with length vector \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), we define the mirror transformation \( \tilde{T} = T_{\tilde{\lambda}, \tau \pi} \) of \( T \), where \( \tilde{\lambda} = (\lambda_k, \lambda_{k-1}, \ldots, \lambda_1) \) and \( \tau : i \mapsto (k - i + 1) \) is the permutation that reverses the names of the semi-intervals.

Given two interval exchange transformations \( T_1 \) and \( T_2 \) on the same alphabet relative to two partitions of two semi-intervals \([\ell_1, r_1]\) and \([\ell_2, r_2]\) respectively, we say that \( T_1 \) and \( T_2 \) are similar either if \([T_1] = [T_2]\) or \([T_1] = [\tilde{T_2}]\). Clearly, similarity is also an equivalent relation. We denote by \( \langle T \rangle \) the class of transformations similar to \( T \).

**Example 9.2.10** Let \( T \) be the interval exchange transformation of Example 9.1.7. The transformation \( U = \varphi^6(T) \) is represented in Figure 9.2.5 (see also Example 9.2.18). It is easy to verify that \( U \) is similar to the transformation \( S \) of Example 9.2.8. Indeed, we can obtain the second transformation (up to the separation points and the end points) by taking the mirror image of the domain.

Note that the order of the labels, i.e. the order of the letters of the alphabet, may be different from the order of the original transformation.

![Figure 9.2.5: The transformation \( U \).](image)
As the equivalence relation, similarity also preserves minimality and regularity.

Let $T$ be an interval exchange transformation. We denote by

$$S(T) = \bigcup_{n \in \mathbb{Z}} T^n(Sep(T))$$

the union of the orbits of the separation points. Let $S$ be an interval exchange transformation similar to $T$. Thus, there exists a bijection $f : D(T) \setminus S(T) \rightarrow D(S) \setminus S(S)$. This bijection is given by an affine transformation, namely a rescaling following by a translation if $T$ and $S$ are equivalent and a rescaling following by a translation and a reflection otherwise. By the previous remark, if $T$ is a minimal exchange interval transformation and $S$ is similar to $T$, then the two interval exchange sets $L(T)$ and $L(S)$ are equal up to permutation, that is there exists a permutation $\pi$ such that one for every

$$w = a_0a_1 \cdots a_{n-1} \in L(T)$$

there exists a unique word $v = b_0b_1 \cdots b_{n-1} \in L(S)$ such that $b_i = \pi(a_i)$ for all $i = 1, 2, \ldots, n-1$.

In a similar way as before, we can use the similarity in order to construct a graph. For an interval exchange transformation $T$ we define $\tilde{G}(T)$ the modified induction graph of $T$ as the directed (unlabelled) graph with vertices the similar classes of transformations obtained starting from $T$ and applying all possible $\chi \in \{\psi, \varphi\}^*$ and an edge from $\langle S \rangle$ to $\langle U \rangle$ if $U = \psi(S)$ or $U = \varphi(S)$ for two transformations $S \in \langle S \rangle$ and $U \in \langle U \rangle$.

Note that this variant appears naturally when considering the Rauzy induction of a 2-interval exchange transformation as a continued fraction expansion. There exists a natural bijection between the closed interval $[0, 1]$ of the real line and the set of 2-interval exchange transformation given by the map $x \mapsto T_{\lambda, \pi}$ where $\pi = (12)$ and $\lambda = (\lambda_1, \lambda_2)$ is the length vector such that $x = \frac{\lambda_1}{\lambda_2}$.

In this view, the Rauzy induction corresponds to the Euclidean algorithm, i.e. the map $E : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ given by

$$E(\lambda_1, \lambda_2) = \begin{cases} (\lambda_1 - \lambda_2, \lambda_2) & \text{if } \lambda_1 \geq \lambda_2 \\ (\lambda_1, \lambda_2 - \lambda_2) & \text{otherwise.} \end{cases}$$

Applying iteratively the Rauzy induction starting from $T$ corresponds then to the continued fraction expansion of $x$.

**Example 9.2.11** Let $\alpha$ and $R$ be as in Example 9.2.9. The modified induction graph $\tilde{G}(R)$ of the transformation is represented on the right of Figure 9.2.6. Note that the ratio of the two lengths of the semi-intervals exchanged by $T$ is

$$\frac{1 - \alpha}{\alpha} = \frac{1 + \sqrt{5}}{2} = \phi = 1 + \frac{1}{1 + \frac{1}{1 + \cdots}}.$$
The morphisms $\theta$.

If $Td$ alphabet.

We will prove by induction on the length of $T$

natural coding of

Proposition 9.2.12

Let

$\psi, \varphi$

Assume first that $\pi$

Proof.

Assume next that $w$

$\alpha$

of angle $346$

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Let $T = T_{\lambda, \pi}$ be a regular interval exchange on $[\ell, r)$ relative to $(I(a))_{a \in A}$. Set $A = \{a_1, \ldots, a_k\}$. Recall now from Subsection 9.1.2 that for any $z \in [\ell, r)$, the natural coding of $T$ relative to $z$ is the infinite word $\Sigma_T(z) = b_0b_1\cdots$ on the alphabet $A$ with $b_n \in A$ defined for $n \geq 0$ by $b_n = a$ if $T^n(z) \in I_a$.

Denote by $\theta_1, \theta_2$ the morphisms from $A^\ast$ into itself defined by

$$
\theta_1(a) = \begin{cases} a_{\pi(k)}a_k & \text{if } a = a_{\pi(k)} \\ a & \text{otherwise} \end{cases},
\theta_2(a) = \begin{cases} a_{\pi(k)}a_k & \text{if } a = a_k \\ a & \text{otherwise} \end{cases}.
$$

The morphisms $\theta_1, \theta_2$ extend to automorphisms of the free group on $A$.

Proposition 9.2.12 Let $T$ be a regular interval exchange transformation on the alphabet $A$ and let $S = \psi(T)$, $I = Z(T)$. There exists an automorphism $\theta$ of the free group on $A$ such that $\Sigma_T(z) = \theta(\Sigma_S(z))$ for any $z \in I$.

Proof. Assume first that $\gamma_k < \delta_{\pi(k)}$ (Case 0). We have $Z(T) = [\ell, \delta_{\pi(k)}]$ and for any $x \in Z(T)$,

$$
S(z) = \begin{cases} T^2(z) & \text{if } z \in K(a_{\pi(k)}) = I(a_{\pi(k)}) \\ T(z) & \text{otherwise}. \end{cases}
$$

We will prove by induction on the length of $w$ that for any $z \in I$, $\Sigma_S(z) \in wA^\ast$ if and only if $\Sigma_T(z) \in \theta_1(w)A^\ast$. The property is true if $w$ is the empty word. Assume next that $w = av$ with $a \in A$ and thus that $z \in I(a)$. If $a \neq a_{\pi(k)}$, then $\theta_1(a) = a$, $S(z) = T(z)$ and

$$
\Sigma_S(z) \in avA^\ast \Leftrightarrow \Sigma_S(S(z)) \in vA^\ast \Leftrightarrow \Sigma_T(T(z)) \in \theta_1(v)A^\ast \Leftrightarrow \Sigma_T(z) \in \theta_1(w)A^\ast.
$$

Otherwise, $\theta_1(a) = a_{\pi(k)}a_k$, $S(z) = T^2(z)$. Moreover, $\Sigma_T(z) = a_{\pi(k)}a_k\Sigma_T(T^2(z))$ and thus

$$
\Sigma_S(z) \in avA^\ast \Leftrightarrow \Sigma_S(S(z)) \in vA^\ast \Leftrightarrow \Sigma_T(T^2(z)) \in \theta_1(v)A^\ast \Leftrightarrow \Sigma_T(z) \in \theta_1(w)A^\ast.
$$

If $Td_{\pi(k)} < d_k$ (Case 1), we have $Z(T) = [\ell, d_k]$ and for any $z \in Z(T)$,

$$
S(z) = \begin{cases} T^2(z) & \text{if } z \in K(a_k) = T^{-1}I(a_k) \\ T(z) & \text{otherwise}. \end{cases}
$$

Figure 9.2.6: Induction graph and modified induction graph of the rotation $R$ of angle $\alpha = (3 - \sqrt{5})/2$. 9.2.3 Induction and automorphisms
As in Case 0, we will prove by induction on the length of \( w \) that for any \( z \in I \), \( \Sigma_S(z) \in wA^* \) if and only if \( \Sigma_T(z) \in \theta_2(w)A^* \).

The property is true if \( w \) is empty. Assume next that \( w = av \) with \( a \in A \). If \( a \neq a_k \), then \( \theta_2(a) = a, S(z) = T(z) \) and \( z \in K(a) \subset I(a) \). Thus
\[
\Sigma_S(z) \in avA^* \iff \Sigma_S(S(z)) \in vA^* \iff \Sigma_T(T(z)) \in \theta_2(v)A^* \iff \Sigma_T(z) \in \theta_2(w)A^* .
\]
Next, if \( a = a_k \), then \( \theta_2(a) = a_{\pi(k)}a_k \), \( S(z) = T^2(z) \) and \( z \in K_{a_k} = T^{-1}(I_{a_k}) \subset I(a_{\pi(k)}) \). Thus
\[
\Sigma_S(z) \in avA^* \iff \Sigma_S(S(z)) \in vA^* \iff \Sigma_T(T^2(z)) \in \theta_2(v)A^* \iff \Sigma_T(z) \in \theta_2(w)A^* .
\]
where the last equivalence results from the fact that \( \Sigma_T(z) \in a_{\pi(k)}a_kA^* \). This proves that \( \Sigma_T(z) = \theta_2(\Sigma_S(z)) \).

**Example 9.2.13** Let \( T \) be the transformation of Example 9.1.7. The automorphism \( \theta_1 \) is defined by
\[
\theta_1(a) = ac, \quad \theta_1(b) = b, \quad \theta_1(c) = c .
\]
The right Rauzy induction gives the transformation \( S = \psi(T) \) computed in Example 9.1.19. One has \( \Sigma_S(\alpha) = bac \cdots \) and \( \Sigma_T(\alpha) = baccbac \cdots = \theta_1(\Sigma_S(\alpha)) \).

We state the symmetrical version of Proposition 9.2.12 for left Rauzy induction. The proof is analogous.

**Proposition 9.2.14** Let \( T \) be a regular interval exchange transformation on the alphabet \( A \) and let \( S = \varphi(T) \), \( I = Y(T) \). There exists an automorphism \( \theta \) of the free group on \( A \) such that \( \Sigma_T(z) = \theta(\Sigma_S(z)) \) for any \( z \in I \).

Combining Propositions 9.2.12 and 9.2.14, we obtain the following statement.

**Theorem 9.2.15** Let \( T \) be a regular interval exchange transformation. For \( \chi \in \{ \varphi, \psi \}^* \), let \( S = \chi(T) \) and let \( I \) be the domain of \( S \). There exists an automorphism \( \theta \) of the free group on \( A \) such that \( \Sigma_T(z) = \theta(\Sigma_S(z)) \) for all \( z \in I \).

**Proof.** The proof follows easily by induction on the length of \( \chi \) using Propositions 9.2.12 and 9.2.14.

Note that if the transformations \( T \) and \( S = \chi(T) \), with \( \chi \in \{ \psi, \varphi \}^* \), are equivalent, then there exists a point \( z_0 \in D(S) \subseteq D(T) \) such that \( z_0 \) is a fixpoint of the isometry that transforms \( D(S) \) into \( D(T) \) (if \( \chi \) is different from the identity map, this point is unique). In that case one has \( \Sigma_S(z_0) = \Sigma_T(z_0) = \theta(\Sigma_S(z_0)) \) for an appropriate automorphism \( \theta \), i.e. \( \Sigma_T(z_0) \) is a fixpoint of an appropriate automorphism.

We now prove the following statement, which gives for regular interval exchanges, a direct proof of the Return Theorem (Theorem 8.1.14).
Corollary 9.2.16 Let \( T \) be a regular interval exchange transformation and let \( X = X(T) \). For \( w \in \mathcal{L}(T) \), the set \( R_X(w) \) is a basis of the free group on \( A \).

Proof. By Proposition 9.1.21 the semi-interval \( J(w) \) is admissible. By Theorem 9.2.3 there is a sequence \( \chi \in \{ \phi, \psi \}^* \) such that \( D(\chi(T)) = J(w) \). Moreover, the transformation \( S = \chi(T) \) is the transformation induced by \( T \) on \( J(w) \). By Theorem 9.2.15 there is an automorphism \( \theta \) of the free group on \( A \) such that \( \Sigma_T(z) = \theta(\Sigma_S(z)) \) for any \( z \in J(w) \).

By Lemma 9.1.24 we have \( x \in R_X(w) \) if and only if \( \Sigma_T(z) = \theta(\Sigma_S(S(z))) \) for some \( z \in J(w) \). This implies that \( R_X(w) = \theta(A) \). Indeed, for any \( z \in J(w) \), let \( a \) is the first letter of \( \Sigma_S(z) \). Then

\[
\Sigma_T(z) = \theta(\Sigma_S(z)) = \theta(a\Sigma_S(S(z))) = \theta(a)\theta(\Sigma_S(Sz)) = \theta(a)\Sigma_T(Sz).
\]

Thus \( x \in R_X(w) \) if and only if there is \( a \in A \) such that \( x = \theta(a) \). This proves that the set \( R_X(w) \) is a basis of the free group on \( A \).

We illustrate this result with the following examples.

Example 9.2.17 We consider again the transformation \( T \) of Example 9.1.7 and \( X = X(T) \). We have \( R_X(c) = \{ bac, bbac, c \} \) (see Example 9.1.7). We represent in Figure 9.2.7 the sequence \( \chi \) of Rauzy inductions such that \( J(c) \) is the domain of \( \chi(T) \).

![Figure 9.2.7: The sequence \( \chi \in \{ \phi, \psi \}^* \)](image)

The sequence is composed of a right induction followed by two left inductions. We have indicated on each edge the associated automorphism (indicating only the image of the letter which is modified). We have \( \chi = \phi^2\psi \) and the resulting composition \( \theta \) of automorphisms gives

\[
\theta(a) = bac, \quad \theta(b) = bbac, \quad \theta(c) = c.
\]

Thus \( R_X(c) = \theta(A) \).

Example 9.2.18 Let \( T \) and \( X \) be as in the preceding example. Let \( U \) be the transformation induced by \( T \) on \( J_a \). We have \( U = \phi^6(T) \) and a computation shows that for any \( z \in J_a \), \( \Sigma_T(z) = \theta(\Sigma_U(z)) \) where \( \theta \) is the automorphism of
the free group on $A = \{a, b, c\}$ which is the coding morphism for $R_X(a)$ defined by:

$\theta(a) = ccba, \quad \theta(b) = cbba, \quad \theta(c) = ccbba.$

One can verify that $L(U) = L(S)$, where $S$ is the transformation obtained from $T$ by permuting the labels of the intervals according to the permutation $\pi = (acb)$.

Note that $L(U) = L(S)$ although $S$ and $U$ are not identical, even up to rescaling the intervals. Actually, the rescaling of $U$ to a transformation on $[0,1)$ corresponds to the mirror image of $S$, obtained by taking the image of the intervals by a symmetry centered at $1/2$.

Note that in the above examples, all lengths of the intervals belong to the quadratic number field $\mathbb{Q}[\sqrt{5}]$.

In the next Section we will prove that if a regular interval exchange transformation $T$ is defined over a quadratic field, then the family of transformations obtained from $T$ by the Rauzy inductions contains finitely many distinct transformations up to rescaling.

### 9.3 Interval exchange over a quadratic field

An interval exchange transformation is said to be defined over a number field $K \subset \mathbb{R}$ if the lengths of all exchanged semi-intervals belong to $K$. Let $T$ be a minimal interval exchange transformation on semi-intervals defined over a quadratic number field. Let $(T_n)_{n \geq 0}$ be a sequence of interval exchange transformation such that $T_0 = T$ and $T_n + 1$ is the transformation induced by $T_n$ on one of its exchanged semi-intervals $I_n$.

**Theorem 9.3.1 (Boshernitzan, Carrol)** If $T$ is defined over a quadratic number field, up to rescaling all semi-intervals $I_n$ to the same length, the sequence $(T_n)$ contains finitely many distinct transformations.

It is possible to generalize this result and prove that, under the above hypothesis on the lengths of the semi-intervals and up to rescaling and translation, there are finitely many transformations obtained by the branching Rauzy induction defined in Section 9.2.

**Theorem 9.3.2** Let $T$ be a regular interval exchange transformation defined over a quadratic field. The family of all induced transformation of $T$ over an admissible semi-interval contains finitely many distinct transformations up to equivalence.

An immediate corollary of Theorem 9.3.2 is the following.

**Corollary 9.3.3** Let $T$ be a regular interval exchange transformation defined over a quadratic field. Then the induction graph $\mathcal{G}(T)$ and the modified induction graph $\tilde{\mathcal{G}}(T)$ are finite.
Example 9.3.4 Let $T$ be the regular interval exchange transformation of Example 9.1.7. The modified induction graph $\tilde{G}(T)$ is represented in Figure 9.3.1. The transformation $T$ belongs to the similarity class $\langle T_1 \rangle$ as well as transformations $S$ of Example 9.2.8 and $U$ of Example 9.2.10. The transformations $\psi(T)$ and $\psi^2(T)$ of Example 9.2.2 belongs respectively to classes $\langle T_2 \rangle$ and $\langle T_4 \rangle$, while the two last transformations of Figure 9.2.7, namely $\varphi \psi(T)$ and $\varphi^2 \psi(T)$, belongs respectively to $\langle T_5 \rangle$ and $\langle T_7 \rangle$. Finally, the left Rauzy induction sequence from $T$ to $U = \varphi^6(T)$ corresponds to the loop $\langle T_1 \rangle \to \langle T_3 \rangle \to \langle T_4 \rangle \to \langle T_6 \rangle \to \langle T_7 \rangle \to \langle T_8 \rangle \to \langle T_1 \rangle$ in $\tilde{G}(T)$ (indicated by thick edges in Figure 9.3.1).

![Figure 9.3.1: Modified induction graph of the transformation $T$.](image)

### 9.3.1 Primitive substitution shifts

In this section we prove an important property of interval exchange transformations defined over a quadratic field, namely that the related interval exchange shifts are primitive substitution shifts.

**Theorem 9.3.5** Let $T$ be a regular interval exchange transformation defined over a quadratic field. The interval exchange shift $X(T)$ is a primitive substitution shift.

**Example 9.3.6** Let $T = T_{\lambda, \pi}$ be the transformation of Example 9.1.7 (see also 9.1.7). The shift $X(T)$ is a primitive substitution shift, as we have seen. This can be obtained as a consequence of Theorem 9.3.5. Indeed the transformation $T$ is regular and the length vector $\lambda = (1 - 2\alpha, \alpha, \alpha)$ belongs to $\mathbb{Q}[\sqrt{5}]^3$.

In order to prove Theorem 9.3.5 we need some preliminary results.
Proposition 9.3.7 Let $T, \chi(T)$ be two equivalent regular interval exchange transformations with $\chi \in \{\varphi, \psi\}^\ast$. There exists a primitive morphism $\theta$ and a point $z \in D(T)$ such that the natural coding of $T$ relative to $z$ is a fixpoint of $\theta$.

Proof. Since $T$ is regular, it is minimal and thus the set $L(T)$ is uniformly recurrent. Thus, there exists a positive integer $N$ such that every letter of the alphabet appears in every word of length $N$ of $L(T)$. Moreover, by Theorem 9.2.7, applying iteratively the Rauzy induction, the length of the domains tends to zero.

Consider $T' = \chi^m(T)$, for a positive integer $m$, such that $D(T') < \varepsilon$, where $\varepsilon$ is the positive real number for which, by Lemma 9.1.6, the first return map for every point of the domain is “longer” than $N$, i.e. $T'(z) = T^m(z)(z)$, with $m(z) \geq N$, for every $z \in D(T)$.

By Theorem 9.2.15 and the remark following it, there exists an automorphism $\theta$ of the free group and a point $z \in D(T) \subseteq D(S)$ such that the natural coding of $T$ relative to $z$ is a fixpoint of $\theta$, that is $\Sigma_T(z) = \theta(\Sigma_T(z))$.

By the previous argument, the image of every letter by $\theta$ is longer than $N$, hence it contains every letter of the alphabet as a factor. Therefore, $\theta$ is a primitive morphism.

Using the previous results we can finally prove Theorem 9.3.5.

Proof of Theorem 9.3.5. By Theorem 9.3.2 there exists a regular interval transformation $S$ such that we can find in the induction graph $G(T)$ a path from $[T]$ to $[S]$ followed by a cycle on $[S]$. Thus, by Theorem 9.2.15 there exist a point $z \in D(S)$ and two automorphisms $\theta, \eta$ of the free group such that $\Sigma_T(z) = \theta(\Sigma_T(z))$, with $\Sigma_S(z)$ a fixpoint of $\eta$.

By Proposition 9.3.7 we can suppose, without loss of generality, that $\eta$ is primitive. Therefore, $X(T)$ is a primitive substitution shift.

9.4 Linear involutions

Let $A$ be an alphabet of cardinality $k$ with an involution $\theta$ and the corresponding specular group $G_\theta$. Note that we allow $\theta$ to have fixed points. Recall that, in the group $G_\theta$, we have $\theta(a) = a^{-1}$. Thus, when $\theta$ has no fixed points, the alphabet $A$ can be identified with $B \cup B^{-1}$ in such a way that $G_\theta$ is the free group on $B$.

We consider two copies $I \times \{0\}$ and $I \times \{1\}$ of an open interval $I$ of the real line and denote $\hat{I} = I \times \{0, 1\}$. We call the sets $I \times \{0\}$ and $I \times \{1\}$ the two components of $\hat{I}$. We consider each component as an open interval.

A generalized permutation on $A$ of type $(\ell, m)$, with $\ell + m = k$, is a bijection $\pi : \{1, 2, \ldots, k\} \rightarrow A$. We represent it by a two line array

$$\pi = \begin{pmatrix}
\pi(1) & \pi(2) & \ldots & \pi(\ell) \\
\pi(\ell + 1) & \ldots & \pi(\ell + m)
\end{pmatrix}$$
A length data associated with \((\ell, m, \pi)\) is a nonnegative vector \(\lambda \in \mathbb{R}_+^k = \mathbb{R}_+^{k}\) such that
\[
\lambda_{\pi(1)} + \ldots + \lambda_{\pi(\ell)} = \lambda_{\pi(\ell+1)} + \ldots + \lambda_{\pi(k)} \quad \text{and} \quad \lambda_a = \lambda_{a-1} \quad \text{for all} \quad a \in A.
\]

We consider a partition of \(I \times \{0\}\) (minus \(\ell - 1\) points) in \(\ell\) open intervals \(I_{\pi(1)}, \ldots, I_{\pi(\ell)}\) of lengths \(\lambda_{\pi(1)}, \ldots, \lambda_{\pi(\ell)}\) and a partition of \(I \times \{1\}\) (minus \(m - 1\) points) in \(m\) open intervals \(I_{\pi(\ell+1)}, \ldots, I_{\pi(\ell+m)}\) of lengths \(\lambda_{\pi(\ell+1)}, \ldots, \lambda_{\pi(\ell+m)}\). Let \(\Sigma\) be the set of \(k - 2\) division points separating the intervals \(I_a\) for \(a \in A\).

The linear involution on \(I\) relative to these data is the map \(T = \sigma_2 \circ \sigma_1\) defined on the set \(\hat{I} \setminus \Sigma\), formed of \(\hat{I}\) minus the \(k - 2\) division points, and which is the composition of two involutions defined as follows.

(i) The first involution \(\sigma_1\) is defined on \(\hat{I} \setminus \Sigma\). It is such that for each \(a \in A\), its restriction to \(I_a\) is either a translation or a symmetry from \(I_a\) onto \(I_{a-1}\).

(ii) The second involution \(\sigma_2\) exchanges the two components of \(\hat{I}\). It is defined, for \((x, \delta) \in \hat{I}\), by \(\sigma_2(x, \delta) = (x, 1 - \delta)\). The image of \(z\) by \(\sigma_2\) is called the mirror image of \(z\).

We also say that \(T\) is a linear involution on \(I\) and relative to the alphabet \(A\) or that it is a \(k\)-linear involution to express the fact that the alphabet \(A\) has \(k\) elements.

**Example 9.4.1** Let \(A = \{a, b, c, d, a^{-1}, b^{-1}, c^{-1}, d^{-1}\}\) and
\[
\pi = \begin{pmatrix} a & b & a^{-1} & c \\ c^{-1} & d^{-1} & b^{-1} & d \end{pmatrix}
\]

Let \(T\) be the 8-linear involution corresponding to the length data represented in Figure 9.4.1 (we represent \(I \times \{0\}\) above \(I \times \{1\}\)) with the assumption that the restriction of \(\sigma_1\) to \(I_a\) and \(I_d\) is a symmetry while its restriction to \(I_b, I_c\) is a translation.

We indicate on the figure the effect of the transformation \(T\) on a point \(z\) located in the left part of the interval \(I_a\). The point \(\sigma_1(z)\) is located in the right part of \(I_{a-1}\) and the point \(T(z) = \sigma_2 \sigma_1(z)\) is just below on the left of \(I_{b-1}\). Next, the point \(\sigma_1 T(z)\) is located on the left part of \(I_b\) and the point \(T^2(z)\) just below.
Thus the notion of linear involution is an extension of the notion of interval exchange transformation in the following sense. Assume that

(i) \( \ell = m \),

(ii) for each letter \( a \in A \), the interval \( I_a \) belongs to \( I \times \{0\} \) if and only if \( I_{a-1} \) belongs to \( I \times \{1\} \),

(iii) the restriction of \( \sigma_1 \) to each subinterval is a translation.

Then, the restriction of \( T \) to \( I \times \{0\} \) is an interval exchange (and so is its restriction to \( I \times \{1\} \) which is the inverse of the first one). Thus, in this case, \( T \) is a pair of mutually inverse interval exchange transformations.

Note that we consider here interval exchange transformations defined by a partition of an open interval minus \( \ell - 1 \) points in \( \ell \) open intervals. The usual notion of interval exchange transformation uses a partition of a semi-interval in a finite number of semi-intervals. One recovers the usual notion of interval exchange transformation on a semi-interval by attaching to each open interval its left endpoint.

A linear involution \( T \) is a bijection from \( \hat{I} \setminus \Sigma \) onto \( \hat{I} \setminus \sigma_2(\Sigma) \). Since \( \sigma_1, \sigma_2 \) are involutions and \( T = \sigma_2 \circ \sigma_1 \), the inverse of \( T \) is \( T^{-1} = \sigma_1 \circ \sigma_2 \).

The set \( \Sigma \) of division points is also the set of singular points of \( T \) and their mirror images are the singular points of \( T^{-1} \) (which are the points where \( T \) (resp. \( T^{-1} \)) is not defined). Note that these singular points \( z \) may be 'false' singularities, in the sense that \( T \) can have a continuous extension to an open neighborhood of \( z \).

Two particular cases of linear involutions deserve attention.

A linear involution \( T \) on the alphabet \( A \) relative to a generalized permutation \( \pi \) of type \((\ell, m)\) is said to be nonorientable if there are indices \( i, j \leq \ell \) such that \( \pi(i) = \pi(j)^{-1} \) (and thus indices \( i, j \geq \ell + 1 \) such that \( \pi(i) = \pi(j)^{-1} \)). In other words, there is some \( a \in A \) for which \( I_a \) and \( I_{a-1} \) belong to the same component of \( \hat{I} \). Otherwise \( T \) is said to be orientable.

A linear involution \( T = \sigma_2 \circ \sigma_1 \) on \( I \) relative to the alphabet \( A \) is said to be coherent if, for each \( a \in A \), the restriction of \( \sigma_1 \) to \( I_a \) is a translation if and only if \( I_a \) and \( I_{a-1} \) belong to distinct components of \( \hat{I} \).

**Example 9.4.2** The linear involution of Example 9.4.1 is coherent.

Linear involutions which are orientable and coherent correspond to interval exchange transformations, whereas orientable but noncoherent linear involutions are called *interval exchanges with flip*.

A *connexion* of a linear involution \( T \) is a triple \((x, y, n)\) where \( x \) is a singularity of \( T^{-1} \), \( y \) is a singularity of \( T \), \( n \geq 0 \) and \( T^n x = y \).

**Example 9.4.3** Let us consider the linear involution \( T \) which is the same as in Example 9.4.1 but such that the restriction of \( \sigma_1 \) to \( I_c \) is a symmetry. Thus \( T \) is not coherent. We assume that \( I = [0, 1] \), that \( \lambda_a = \lambda_d \). Let \( x = (1 - \lambda_d, 0) \) and \( y = (\lambda_a, 0) \).
Then $x$ is a singularity of $T^{-1}$ ($\sigma_2(x)$ is the left endpoint of $I_a$), $y$ is a singularity of $T$ (it is the right endpoint of $I_a$) and $T(x) = y$. Thus $(x, 1, y)$ is a connexion.

**Example 9.4.4** Let $T$ be the linear involution on $I = [0, 1]$ represented in Figure 9.4.2. We assume that the restriction of $\sigma_1$ to $I_a$ is a translation whereas the restriction to $I_b$ and $I_c$ is a symmetry. We choose $(3 - \sqrt{5})/2$ for the length of the interval $I_c$ (or $I_b$). With this choice, $T$ has no connexions.

![Figure 9.4.2: A linear involution without connexions.](image)

Let $T$ be a linear involution without connexions. Let

$$O = \bigcup_{n \geq 0} T^{-n}(\Sigma) \quad \text{and} \quad \hat{O} = O \cup \sigma_2(O)$$

be respectively the negative orbit of the singular points and its closure under mirror image. Then $T$ is a bijection from $\hat{I} \setminus \hat{O}$ onto itself. Indeed, assume that $T(z) \in \hat{O}$. If $T(z) \in O$ then $z \in O$. Next if $T(z) \in \sigma_2(O)$, then $T(z) \in \sigma_2(T^{-n}(\Sigma)) = T^n(\sigma_2(\Sigma))$ for some $n \geq 0$. We cannot have $n = 0$ since $\sigma_2(\Sigma)$ is not in the image of $T$. Thus $z \in T^{n-1}(\sigma_2(\Sigma)) = \sigma_2(T^{-n+1}(\Sigma)) \subset \sigma_2(O)$. Therefore in both cases $z \in \hat{O}$. The converse implication is proved in the same way.

### 9.4.1 Minimal linear involutions

A linear involution $T$ on $I$ without connexions is *minimal* if for any point $z \in \hat{I} \setminus \hat{O}$ the nonnegative orbit of $z$ is dense in $\hat{I}$.

Note that when a linear involution is orientable, that is, when it is a pair of interval exchange transformations (with or without flips), the interval exchange transformations can be minimal although the linear involution is not since each component of $\hat{I}$ is stable by the action of $T$.

**Example 9.4.5** Let us consider the noncoherent linear involution $T$ which is the same as in Example 9.4.1 but such that the restriction of $\sigma_1$ to $I_c$ is a symmetry, as in Example 9.4.3. We assume that $I = [0, 1]$, that $\lambda_a = \lambda_d$, that $1/4 < \lambda_c < 1/2$ and that $\lambda_a + \lambda_b < 1/2$. Let $z = 1/2 + \lambda_c$ (see Figure 9.4.3). We have then $T^3(z) = z$, showing that $T$ is not minimal. Indeed, since $z \in I_c$, we have $T(z) = 1 - z = 1/2 - \lambda_c$. Since $T(z) \in I_a$ we have $T^2(z) = (\lambda_a + \lambda_b) + (\lambda_a - 1 + z) = z - \lambda_c = 1/2$. Finally, since $T^2(z) \in I_{d-1}$, we obtain $1 - T^3(z) = T^2(z) - \lambda_c = 1 - z$ and thus $T^3(z) = z.$
9.4. LINEAR INVOLUTIONS

We quote without proof the following result, analogous to Keane Theorem (Theorem 9.1.2) for interval exchange transformations.

**Proposition 9.4.6** Let $T$ be a linear involution without connexions on $I$. If $T$ is nonorientable, it is minimal. Otherwise, its restriction to each component of $I$ is minimal.

9.4.2 Natural coding

Let $T$ be a linear involution on $I$, let $\hat{I} = I \times \{0, 1\}$ and let $\hat{O}$ be the set defined by Equation (9.4.1).

Given $z \in I \setminus \hat{O}$, the infinite natural coding of $T$ relative to $z$ is the infinite word $\Sigma_T(z) = (a_n)_{n \in \mathbb{Z}}$ on the alphabet $A$ defined by

$$a_n = a \quad \text{if} \quad T^n(z) \in I_a.$$

We first observe that the factors of $\Sigma_T(z)$ are reduced. Indeed, assume that $a_n = a$ and $a_{n+1} = a^{-1}$ with $a \in A$. Set $x = T^n(z)$ and $y = T(x) = T^{n+1}(z)$. Then $x \in I_a$ and $y \in I_{a^{-1}}$. But $y = \sigma_2(u)$ with $u = \sigma_1(x)$. Since $x \in I_a$, we have $u \in I_{a^{-1}}$. This implies that $y = \sigma_2(u)$ and $u$ belong to the same component of $I$, a contradiction.

We denote by $X(T)$ the set of infinite natural codings of $T$. We say that $X(T)$ is the natural coding of $T$. We also denote $\mathcal{L}(T) = \mathcal{L}(X(T))$.

**Example 9.4.7** Let $T$ be the linear involution of Example 9.4.4. The words of length at most 3 of $S = \mathcal{L}(T)$ are represented in Figure 9.4.4.

The set $S$ can actually be defined directly as the set of factors of the substitution

$$f : a \mapsto cb^{-1}, \quad b \mapsto c, \quad c \mapsto ab^{-1},$$

which extends to an automorphism of the free group on $\{a, b, c\}$. Indeed, the application twice of Rauzy induction on $T$ gives a linear involution which is the same as $T$ (with the two copies of $[0, 1]$ interchanged). This gives the explanation of why the substitution shift of Example 8.3.30 is specular.

Define, as for an interval exchange transformation, for a word $w = a_0 a_1 \cdots a_n$ with $a_i \in A \cup A^{-1}$,

$$I_w = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \ldots \cap T^{-n}(I_{a_n}).$$
It is clear that

\[ w \in \mathcal{L}(T) \iff I_w \neq \emptyset. \tag{9.4.2} \]

**Proposition 9.4.8** Let \( T \) be a linear involution. The set \( \mathcal{L}(T) \) is a laminar set.

**Proof.** Set \( T = \sigma_2 \circ \sigma_1 \). We claim that for any nonempty word \( u \in \mathcal{L}(T) \), one has \( I_{a^{-1}} = \sigma_1 T^{\lfloor |u| \rfloor - 1}(I_u) \).

To prove the claim, we use an induction on the length of \( u \). The property holds for \( |u| = 1 \) by definition of \( \sigma_1 \). Next, consider \( u \in \mathcal{L}(T) \) and \( a \in A \cup A^{-1} \) such that \( ua \in \mathcal{L}(T) \).

Since \( T^{-1} = \sigma_1 \circ \sigma_2 \), we have, using the induction hypothesis,

\[
\sigma_1 T^{|u|}(I_{ua}) = \sigma_1 T^{|u|}(I_u \cap T^{-|u|}(I_a)) = \sigma_1 T^{|u|}(I_u) \cap \sigma_1(I_a) = \sigma_1 \sigma_2 \sigma_1 T^{|u| - 1}(I_u) \cap \sigma_1(I_a) = \sigma_1 \sigma_2 (I_{a^{-1}}) \cap I_{a^{-1}} = I_{a^{-1}u^{-1}}
\]

where the last equality results from \( I_{a^{-1}u^{-1}} = T^{-1}I_{u^{-1}} \cap I_{a^{-1}} \).

We easily deduce from the claim that the set \( \mathcal{L}(T) \) is closed under taking inverses. Furthermore, it is a factorial subset of the group \( G_\theta \). It is thus a laminar set.

We prove the following result.

**Theorem 9.4.9** The natural coding of a linear involution without connexions is a specular shift.

We first prove the following lemma, which replaces for linear involutions Conditions \([9.1.2]\) and \([9.1.3]\).
9.4. LINEAR INVOLUTIONS

Lemma 9.4.10 Let \( T \) be a linear involution. For every nonempty word \( w \) and letter \( a \in A \), one has

(i) \( a \in L(w) \Leftrightarrow \sigma_2(I_{a^{-1}}) \cap I_w \neq \emptyset \),

(ii) \( a \in R(w) \Leftrightarrow \sigma_2(I_a) \cap I_{w^{-1}} \neq \emptyset \).

Proof. By (9.4.2), we have \( a \in L(w) \) if and only if \( I_{aw} \neq \emptyset \), which is also equivalent to \( T(I_{aw}) \neq \emptyset \). By definition of \( I_{aw} \), we have \( T(I_{aw}) = T(I_a) \cap I_w \).

Since \( T = \sigma_2 \circ \sigma_1 \) and since \( \sigma_1(I_a) = I_{a^{-1}} \), we have \( a \in L(w) \) if and only if \( \sigma_2(I_{a^{-1}}) \cap I_w \neq \emptyset \). Next, since \( \mathcal{L}(T) \) is closed under taking inverses by Proposition 9.4.8, we have \( aw \in \mathcal{L}(T) \) if and only if \( w^{-1}a^{-1} \in \mathcal{L}(T) \). Thus \( a \in R(w) \) if and only if \( a^{-1} \in L(w^{-1}) \), whence the second equivalence.

Given a linear involution \( T \) on \( I \), we introduce two orders on \( \mathcal{L}(T) \) as follows.

For any \( u, v \in \mathcal{L}(T) \), one has

(i) \( u <_R v \) if and only if \( I_u < I_v \),

(ii) \( u <_L v \) if and only if \( I_{u^{-1}} < I_{v^{-1}} \).

Lemma 9.4.11 Let \( T \) be a linear involution on \( I \) without connexion. Let \( w \in \mathcal{L}(T) \) and \( a, a' \in L(w) \) (resp. \( b, b' \in R(w) \)). Then \( 1 \otimes a, 1 \otimes a' \) (resp. \( b \otimes 1, b' \otimes 1 \)) are in the same connected component of \( \mathcal{E}(w) \) if and only if \( I_{a^{-1}}, I_{a'^{-1}} \) (resp. \( I_b, I_{b'} \)) are in the same component of \( \hat{I} \).

Proof. If \( (1 \otimes a, b \otimes 1) \in \mathcal{E}(w) \), then \( \sigma_2(I_{a^{-1}}) \cap I_{wb} \neq \emptyset \). Thus \( I_{a^{-1}} \) and \( I_{wb} \) belong to distinct components of \( \hat{I} \). Consequently, if \( a, a' \in L(w) \) (resp. \( R(w) \)) belong to the same connected component of \( \mathcal{E}(w) \), then \( I_{a^{-1}}, I_{a'^{-1}} \) (resp. \( I_{wa}, I_{wa'} \)) belong to the same component of \( \hat{I} \). Conversely, let \( a, a' \in L(w) \) be such that \( a < L a' \). There is a reduced path (i.e., it does not use twice consecutively the same edge) in \( \mathcal{E}(w) \) from \( a \) to \( a' \) which is the sequence \( a_1, b_1, ..., b_{n-1}, a_n \) with \( a_1 = a \) and \( a_n = a' \) with \( a_1 < L a_2 < L a_n, wb_1 < R wb_2 < R wb_{n-1} \) and \( \sigma_2(I_{a_i^{-1}}) \cap I_{wb_i} \neq \emptyset \), \( \sigma_2(I_{a_{i+1}^{-1}}) \cap I_{wb_i} \neq \emptyset \) for \( 1 \leq i \leq n-1 \) (see Figure 9.4.5 for an illustration).

![Figure 9.4.5: A path from \( a_1 \) to \( a_n \) in \( \mathcal{E}(w) \).]

Note that the hypothesis that \( T \) is without connexion is needed since otherwise the right boundary of \( \sigma_2(I_{a_{i-1}}) \) could be the left boundary of \( I_{wb_i} \). The assertion concerning \( b, b' \in R(w) \) is a consequence of the first one since \( b, b' \in R(w) \) if and only if \( b^{-1}, b'^{-1} \in L(w^{-1}) \).
Proof of Theorem 9.4.9. Let $T$ be a linear involution without connexions. By Proposition 9.4.8 the set $\mathcal{L}(T)$ is symmetric. Since it is by definition extendable and formed of reduced words, it is a laminary set.

There remains to show that $X(T)$ is dendric of characteristic 2. Let us first prove that for any $w \in \mathcal{L}(T)$, the graph $E(w)$ is acyclic. Assume that $(1 \otimes a_1, b_1 \otimes 1, \ldots, 1 \otimes a_n, b_n \otimes 1)$ is a path in $E(w)$ with $(a_1, \ldots, a_n) \in L(w)$ and $b_1, \ldots, b_n \in R(w)$. We may assume that the path is reduced, that $n \geq 2$ and also that $a_1 <_L a_2$. It follows that $a_1 <_L \ldots <_L a_n$ and $wb_1 <_R \ldots <_R wb_n$ (see Figure 9.4.5). Thus it is not possible to have an edge $(a_1, b_n)$, which shows that $E(w)$ is acyclic.

Let $a, a' \in A$. If $I_{a^{-1}}$ and $I_{a'^{-1}}$ are in the same component of $\hat{I}$, then $1 \otimes a$ and $1 \otimes a'$ are in the same connected component of $E(\varepsilon)$. Thus $E(\varepsilon)$ is a union of two trees.

Next, if $w \in \mathcal{L}(T)$ is nonempty and $1 \otimes a, 1 \otimes a' \in L(w)$, then $I_{a^{-1}}$ and $I_{a'^{-1}}$ are in the same component of $\hat{I}$ (by Lemma 9.4.10), and thus $1 \otimes a, 1 \otimes a'$ are in the same connected component of $E(w)$. Thus $E(w)$ is a tree.

We now present an example of a linear involution on an alphabet $A$ where the involution $\theta$ has fixed points.

Example 9.4.12 Let $A = \{a, b, c, d\}$ be as in Example 8.3.1 (in particular, $d = b^{-1}, a = a^{-1}, c = c^{-1}$). Let $T$ be the linear involution represented in Figure 9.4.6.

$$\begin{align*}
a & \quad d \\
b & \quad c
\end{align*}$$

Figure 9.4.6: A linear involution on $A = \{a, b, c, d\}$.

with $\sigma_1$ being a translation on $I_b$ and a symmetry on $I_a, I_c$. Choosing $(3 - \sqrt{5})/2$ for the length of $I_b$, the involution is without connexions. Thus $\mathcal{L}(T)$ is a specular set.

Note that the natural coding of the linear involution $T$ is equal to the set of factors of the shift of Example 8.3.1. Indeed, consider the interval exchange $V$ on the interval $Y = [0, 1]$ represented in Figure 9.4.7 on the right, which is obtained by using two copies of the interval exchange $U$ defining the Fibonacci set (represented in Figure 9.4.7 on the left). Let $X = [0, 1] \times \{0, 1\}$ and let $\alpha : Y \to X$ be the map defined by

$$\alpha(z) = \begin{cases} (z, 0) & \text{if } z \in [0, 1] \\ (2 - z, 1) & \text{otherwise.} \end{cases}$$

Then $\alpha \circ V = T \circ \alpha$ and thus $\mathcal{L}(V) = \mathcal{L}(T)$.
9.5. **EXERCISES**

**Section 9.1**

9.1 Let \( T \) be the \( s \)-interval exchange transformation on the intervals \( \Delta_i \) defined by a permutation \( \pi \). Let \( \lambda_i \) be the length of \( \Delta_i \). Let \( \lambda \) be the column vector with coordinates \( \lambda_i \). Set \( Tx = x + \alpha_i \) for \( x \in \Delta_i \). Show that there is an antisymmetric matrix \( M \) such that the column vector \( \alpha \) with coordinates \( \alpha_i \) is defined by

\[
\alpha_i = M \lambda.
\]  

(9.5.1)

9.2 Set \( I(\varepsilon) = I \) for and \( I(au) = \Delta_a \cap T^{-1}I(u) \) for \( u \in A^* \). Show that every nonempty \( I(w) \) is a semi-interval.

9.3 Let \( J(w) \) be defined by \( J(\varepsilon) = I \) and by

\[
J(ua) = TJ(u) \cap T\Delta_a
\]  

(9.5.2)

for \( a \in A \) and \( u \in A^* \). Show that the nonempty sets \( J(w) \) are semi-intervals.

**Section 9.6**

9.1 We have

\[
\alpha_i = \sum_{k<\pi^{-1}(i)} \lambda_{\pi(k)} - \sum_{k<i} \lambda_k.
\]  

(9.6.1)

Indeed, the first term of the right hand side is the left boundary of \( T\Delta_i \) and the second one is the left boundary of \( \Delta_i \). Setting

\[
M_{ij} = \begin{cases} 
1 & \text{if } \pi^{-1}(i) > \pi^{-1}(j) \text{ and } i < j \\
-1 & \text{if } \pi^{-1}(i) < \pi^{-1}(j) \text{ and } i > j \\
0 & \text{otherwise}
\end{cases}
\]

we obtain the desired alternating matrix.

9.2 We use an induction on the length of \( w \). The property is true if \( w \) is the empty word. Next, assume that \( I(w) \) is a semi-interval and let \( a \) be a letter.
Then $T(I(aw)) = T(\Delta_a) \cap I(w)$ is a semi-interval since $T(\Delta_a)$ is a semi-interval and also $I(w)$ by induction hypothesis. Since $I(aw) \subset \Delta_a$, the set $T(I(aw))$ is a translation of $I(aw)$, which is therefore also a semi-interval.

9.3 The proof is symmetrical to the proof for $I(w)$, using this time the fact that $T^{-1}J(wa) = J(w) \cap T\Delta(a)$ is a semi-interval and thus that $J(wa)$ is a semi-interval since $J(wa) \subset T\Delta(a)$.

9.7 Notes

9.7.1 Interval exchange transformations

Interval exchange transformations were introduced by Keane (1975) who proved Theorem 9.1.2. The condition defining regular interval exchange transformations is also called the infinite disjoint orbit condition or idoc.

The idea of Rauzy induction and Theorem 9.1.3 are from Rauzy (1979). For more details on interval exchange transformations we refer to Cornfeld et al. (1982), that we follow closely for the proof of Theorems 9.1.3 and 9.1.2. The Cantor version of interval exchange transformations was introduced by Keane (1975).

The notion of planar dendric shifts and Proposition 9.1.9 are from Berthé et al. (2015). The converse of Proposition 9.1.9 characterizing the languages of regular interval exchange transformations, is proved in Ferenczi and Zamboni (2008).

Theorem 9.1.14 is due to Gjerde and Johansen (2002). They also showed that there are BV-dynamical systems satisfying the hypothesis of the theorem that are not isomorphic to a Cantor version of an interval exchange transformation.

The BV-representation of non-minimal Cantor systems can also be considered. In Medynets (2006) the author shows that for Cantor dynamical systems without periodic points (but not necessarily minimal) a BV-representation can also be given. It is applied in Bezgulya et al. (2009) to subshifts generated by non-primitive substitutions. The authors show that they have stationary BV-representations as in the minimal case.

Theorem 9.1.18 is Theorem 14 in Rauzy (1979) while Theorem 9.2.3 is Theorem 23. Lemma 9.2.5 is the two-sided version of Lemma 22 in Rauzy (1979).

We have noted that for any $s$-interval exchange transformation on $[\ell, r)$ and any semi-interval $I$ of $[\ell, r]$, the transformation $S$ induced by $T$ on $I$ is an interval exchange transformation on at most $s + 2$-intervals (Lemma 9.1.3). Actually, it follows from the proof of Lemma 2, page 128 in Cornfeld et al. (1982) that, if $T$ is regular and $S$ is an $s$-interval exchange transformation with separation points $\text{Sep}(S) = \text{Div}(I, T) \cap I$, then $I$ is admissible. Thus the converse of Theorem 9.1.22 is also true.

Branching Rauzy induction is introduced in Dolce and Perrin (2017) where Theorem 9.2.3 appears. Actually, left Rauzy induction is already considered in...
Veitch (1990) and Theorem 9.2.3 appears independently in Fickenscher (2017).
On the relation between Rauzy induction and continued fractions, see Miernowski and Nogueira (2013) for more details.
Theorem 9.3.1 is from Boshernitzan and Carroll (1997). In the same paper, an extension to right Rauzy induction is suggested (but not completely developed). Theorem 9.3.2 is from Dolce and Perrin (2017).

9.7.2 Linear involutions

Linear involutions were introduced in Danthony and Nogueira (1990). It is also an extension of the notion of interval exchange with flip Nogueira (1989); Nogueira et al. (2013). Our definition is somewhat more general than the one used in Danthony and Nogueira (1990) and also that of Berthé et al. (2017). Orientable linear involutions correspond to orientable laminations, whereas coherent linear involutions correspond to orientable surfaces. Thus coherent nonorientable involutions correspond to nonorientable laminations on orientable surfaces.

Contrary to what happens with interval exchanges, it is shown in Danthony and Nogueira (1990) that noncoherent linear involutions are almost surely not minimal.
Proposition 9.4.6 (already proved in Boissy and Lanneau 2009, Proposition 4.2) for the class of coherent involutions is (Berthé et al. 2017b, Proposition 3.7). The proof uses Keane’s theorem proving that an interval exchange transformation without connections is minimal (Keane 1975).
The interval exchange \( U \) built in Example 9.4.12 is actually the orientation covering of the linear involution \( T \) (see Berthé et al. 2017).
Chapter 10

Bratteli diagrams and $C^*$-algebras

In this chapter, we give a short introduction to the connexion between Bratteli diagrams and $C^*$-algebras. These notions are closely related to our subject. Indeed, one may associate to every ordered Bratteli diagram both a $C^*$-algebra and a Cantor system. Minimal Cantor systems correspond to the so-called properly ordered Bratteli diagrams and the dimension groups of the Cantor system and of the algebra are the same.

We define approximately finite dimensional algebras (AF algebras) as follows. A $C^*$-algebra $\mathfrak{A}$ is an AF algebra if it is the closure of an increasing sequence of finite dimensional subalgebras $(\mathfrak{A}_k)_{k \geq 0}$. Let $t_k$ be the dimension of $\mathfrak{A}_k$. To such an algebra $\mathfrak{A}$ we associate a dimension group as a direct limit of abelian groups $\mathbb{Z}^{t_k}$.

The main object of this chapter is to prove the theorem of Elliott (Theorem 10.3.17). It asserts that two unital approximately finite algebras are $*$-isomorphic if and only if their dimension groups are isomorphic. In this way the difficult problem of isomorphism of $C^*$-algebras is, in the case of AF algebras, reduced to the easier problem of isomorphism of dimension groups.

In the first section, we show how Bratteli diagrams can describe embeddings of sequences of finite dimensional algebras. In the next section (Section 10.2) we give a brief introduction to $C^*$-algebras. In Section 10.2.3 we introduce enveloping $C^*$-algebras and use them to define direct limits of $C^*$-algebras. Elliott's Theorem (Theorem 10.3.17) is presented in Section 10.3.

10.1 Bratteli diagrams

Let us first give a modified version of Bratteli diagrams adapted to the purpose of this chapter and which essentially differs by the addition of integer labels to the vertices. A Bratteli diagram, as defined in Chapter 6, is an infinite labeled multigraph. Its set of nodes of level $k$ consists of pairs $(k,j)$ with $p \geq 0$ and
CHAPTER 10. BRATTELI DIAGRAMS AND C*-ALGEBRAS

1 ≤ j ≤ t_k with t_0 = 1 (there is only one vertex at level 0). There are a_{ij}^{(k)} ≥ 0 arrows from (k, j) to (k + 1, i). We denote by M_k the t_{k+1} × t_k-matrix with coefficients a_{ij}^{(k)} . We label the vertex (k + 1, i) of the diagram by the integer

\[ n(k+1, i) = \sum_{j=1}^{t_k} a_{ij}^{(k)} n(k, j) \]  \hspace{1cm} (10.1.1)

with n(0, 1) = 1. Conversely, any sequence \((M_k)_{k \geq 1}\) of nonnegative integer matrices defines a Bratteli diagram as above. By convention, we assume that \(M_0 = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^t\) and that n(0, 1) = 1. Thus n(1, i) = 1 for 1 ≤ i ≤ t_1.

**Example 10.1.1** The graph represented in Figure 10.1.1 with all matrices \(M_k\) for \(k \geq 1\) equal to the matrix \(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\) is a Bratteli diagram. The levels \(k = 0, 1, 2, \ldots\) are growing horizontally from left to right and on each level the vertices are numbered vertically from bottom to top. The integer labelling each node \((k, j)\) is \(n(k, j)\).

![Figure 10.1.1: A Bratteli diagram.](image)

**Example 10.1.2** The graph represented in Figure 10.1.2 is a Bratteli diagram which corresponds to all matrices \(M_n\) equal to \(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\).

![Figure 10.1.2: A second Bratteli diagram.](image)

We will now explain how a Bratteli diagram describes a sequence

\[ \mathfrak{A}_1 \xrightarrow{\pi_1} \mathfrak{A}_2 \xrightarrow{\pi_2} \mathfrak{A}_3 \xrightarrow{\pi_3} \ldots \]

of algebras \(\mathfrak{A}_k\) and morphisms \(\pi_k : \mathfrak{A}_k \rightarrow \mathfrak{A}_{k+1}\).

Let \(M\) be an \(\ell \times k\)-matrix with coefficients \(a_{ij} \geq 0\), let \((m_1, \ldots, m_\ell)\) be integers and let \((n_1, \ldots, n_s)\) be such that

\[ n_i = \sum_{j=1}^{t_k} a_{ij} m_j \quad (1 \leq i \leq \ell). \]
Let \( \mathfrak{A}_1 = \mathcal{M}_{m_1} \oplus \cdots \oplus \mathcal{M}_{m_t} \) and \( \mathfrak{A}_2 = \mathcal{M}_{n_1} \oplus \cdots \oplus \mathcal{M}_{n_s} \) where \( \mathcal{M}_n \) denotes the algebra of \( n \times n \) matrices with complex coefficients. We associate to an \( s \times t \) matrix \( M \) with integer nonnegative coefficients \( a_{ij} \) the morphism \( \varphi : \mathfrak{A}_1 \to \mathfrak{A}_2 \) defined as

\[
\varphi = \varphi_1 \oplus \cdots \oplus \varphi_t
\]

where \( \text{id}_n \) is the identity matrix of \( \mathcal{M}_n \) and \( \text{id}_n^{(e)} : \mathcal{M}_n \to \mathcal{M}_{en} \) is the morphism \( x \mapsto (x, \ldots, x) \) (\( e \) times). The matrix \( M \) is called the matrix of partial multiplicities associated with \( \varphi \) and \( \varphi \) is the morphism determined by \( M \).

We associate to a Bratteli diagram the sequence

\[
\mathfrak{A}_1 \xrightarrow{\pi_1} \mathfrak{A}_2 \xrightarrow{\pi_2} \mathfrak{A}_3 \xrightarrow{\pi_3} \cdots
\]

of algebras \( \mathfrak{A}_k \) with algebra morphisms \( \pi_k : \mathfrak{A}_k \to \mathfrak{A}_{k+1} \) where

\[
\mathfrak{A}_k = \mathcal{M}_{n(k,1)} \oplus \cdots \oplus \mathcal{M}_{n(k,t_k)}
\]

and where \( \pi_k : \mathfrak{A}_k \to \mathfrak{A}_{k+1} \) is associated with the matrix \( M_k \) as above.

**Example 10.1.3** The sequence of algebras \( \mathfrak{A}_k \) associated to the Bratteli diagram of Example 10.1.1 is \( \mathfrak{A}_k = \mathcal{M}_{n_k} \oplus \mathcal{M}_{n_k-1} \) where \( (n_k) \) is the Fibonacci sequence defined by \( n_1 = n_2 = 1 \) and \( n_{k+1} = n_k + n_{k-1} \) for \( k \geq 2 \). The morphism \( \pi_k \) is defined by \( \pi_k(x,y) = (x \oplus y, x) \).

**Example 10.1.4** The sequence of algebras \( \mathfrak{A}_k \) associated to the Bratteli diagram of Figure 10.1.2 is \( \mathfrak{A}_k = \mathcal{M}_k \oplus \mathbb{C} \). The corresponding morphism \( \pi_k : \mathfrak{A}_k \to \mathfrak{A}_{k+1} \) is defined by \( \pi_k(x,\lambda) = (x \oplus \lambda, \lambda) \).

As a more general notion of Bratteli diagram, one may consider partial diagrams in which the labels \( n(p,i) \) satisfy the inequalities

\[
n(k+1,j) \geq \sum_{i=1}^{t_p} a_{ji}^{(k)} n(k,i).
\]

instead of an equality.

**Example 10.1.5** The diagram represented in Figure 10.1.3 with all matrices \( M_k \) equal to \([1]\) is a partial Bratteli diagram.

\[
1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots
\]

Figure 10.1.3: A partial Bratteli diagram.

A partial Bratteli diagram represents as above a sequence of \( * \)-morphisms between finite dimensional algebras, but the morphisms are this time not unital.

**Example 10.1.6** The Bratteli diagram of Figure 10.1.3 corresponds to the sequence of algebras \( (\mathcal{M}_k)_{k \geq 1} \) with the morphisms \( \pi_k : \mathcal{M}_k \to \mathcal{M}_{k+1} \) defined by \( \pi_k(x) = x \oplus 0 \).
10.2 $C^*$-algebras

A $*$-algebra $\mathfrak{A}$ is a complex algebra with an idempotent map (called the adjoint) $A \mapsto A^*$ related to the algebra structure by

$$(A + B)^* = A^* + B^*, \quad (AB)^* = B^*A^*, \quad \text{and} \quad (\lambda A)^* = \bar{\lambda}A^*$$

(10.2.1)

for every $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. It is usual to denote with capitals the elements of a $C^*$-algebra. The notation $A^*$ should not lead the reader into a confusion with an alphabet, nor the notation $A^*$ with the free monoid on $A$.

A $C^*$-algebra $\mathfrak{A}$ is a Banach $*$-algebra such that for all $A \in \mathfrak{A}$,

$$\|A^*A\| = \|A\|^2.$$  

(10.2.2)

A $*$-morphism from a $C^*$-algebra $\mathfrak{A}$ to a $C^*$-algebra $\mathfrak{B}$ is an algebra morphism $\pi$ such that $\pi(A^*) = \pi(A)^*$ for all $A \in \mathfrak{A}$. It can be shown that this implies $\|\pi(A)\| \leq \|A\|$ for every $A \in \mathfrak{A}$ (Exercise 10.2).

A $C^*$-algebra is unital if it has an identity element $I$. It follows from (10.2.1) that $I^* = I$ and from (10.2.2) that $\|I\| = 1$.

Each algebra $M_k$ is a $C^*$-algebra, using the usual conjugate transpose $A^*$ of $A$ as adjoint of $A$ and using the matrix norm induced by the Hermitian norm. Indeed, Equation (10.2.2) is a consequence of the Cauchy-Schwartz inequality $|\langle x, y \rangle| \leq \|x\|\|y\|$ which implies

$$\|A^*A\| = \sup_{\|x\| = \|y\| = 1} \langle A^*Ax, y \rangle = \sup_{\|x\| = \|y\| = 1} \langle Ax, Ay \rangle = \|A\|^2.$$  

More generally, the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is a $C^*$-algebra. The adjoint of $A$ is defined by $\langle A^*x, y \rangle = \langle x, Ay \rangle$ for all $x, y \in \mathcal{H}$.

As a second fundamental example, the space $C(X, \mathbb{C})$ of continuous complex valued functions on a compact space $X$ is a $C^*$-algebra with complex conjugation as adjoint operation. We have indeed

$$\|\overline{f}g\| = \sup_{x \in X} |f(x)\overline{g(x)}| = \sup_{x \in X} |f(x)|^2 = \|f\|^2$$

An element $A$ of a $C^*$-algebra is normal if $AA^* = A^*A$. Recall that $\text{spr}(A)$ denotes the spectral radius of $A$ (see Appendix A). The following property is clear when $\mathfrak{A} = M_k$ (as well as the necessity of the hypothesis that $A$ is normal).

**Proposition 10.2.1** In a $C^*$-algebra, one has $\|A\| = \text{spr}(A)$ for every normal element of $A$.

**Proof.** Let first $A$ be self adjoint. Then by repeated use of Equation (10.2.2), we have

$$\text{spr}(A) = \lim \|A^2\|^2 = \|A\|$$  

(10.2.3)
When $A$ is normal, using the preceding case, we have
\[
spr(A)^2 \leq \|A\|^2 = \|A^*A\| = \lim \|A^*A\|^n^{1/n} \\
\leq \lim \|A^*A\|^n\|A^n\|^{1/n} = spr(A)^2.
\]

One can deduce from this property that the norm is unique in a $C^*$-algebra (Exercise 10.1).

A subalgebra $\mathfrak{A}'$ of a $C^*$-algebra $\mathfrak{A}$ is self-adjoint if $A^*$ is in $\mathfrak{A}'$ for every $A \in \mathfrak{A}'$. It is easy to verify that a closed self-adjoint subalgebra of a $C^*$-algebra is again a $C^*$-algebra.

The direct sum $\mathfrak{A} \oplus \mathfrak{B}$ of two $C^*$-algebras $\mathfrak{A}, \mathfrak{B}$ is itself a $C^*$-algebra, using on the set $\mathfrak{A} \times \mathfrak{B}$ the componentwise sum, product and $*$, and defining $\|(A, B)\| = \max(\|A\|, \|B\|)$.

### 10.2.1 Ideals in $C^*$-algebras

An ideal in a $C^*$-algebra $\mathfrak{A}$ is a norm-closed two-sided ideal of the algebra $\mathfrak{A}$.

It can be shown that every ideal $\mathcal{J}$ is self-adjoint, that is, such that $J^*$ is in $\mathcal{J}$ for every $J \in \mathcal{J}$.

The quotient $\mathfrak{A}/\mathcal{J}$ of a $C^*$-algebra $\mathfrak{A}$ by an ideal $\mathcal{J}$ is the set of cosets $A + \mathcal{J} = \{A + J \mid J \in \mathcal{J}\}$ with the adjoint defined by $(A + \mathcal{J})^* = A^* + \mathcal{J}$ and the norm defined by $\|A + \mathcal{J}\| = \inf_{J \in \mathcal{J}} \|A + J\|$.

Since $\mathcal{J}$ is self-adjoint, we have $\|(A + \mathcal{J})^*\| = \|A + \mathcal{J}\|$. Actually, the $C^*$ norm condition is also satisfied and thus one has the following statement.

**Theorem 10.2.2** For every ideal $\mathcal{J}$ in a $C^*$-algebra $\mathfrak{A}$, the quotient algebra $\mathfrak{A}/\mathcal{J}$ is a $C^*$-algebra.

The following now shows that the basic isomorphism theorem for algebras still holds for $C^*$-algebras.

**Theorem 10.2.3** If $\mathcal{J}$ is an ideal in $C^*$-algebra $\mathfrak{A}$ and that $\mathfrak{B}$ is a $C^*$-subalgebra of $\mathfrak{A}$. Then $\mathfrak{B} + \mathcal{J}$ is a $C^*$-algebra and

$$
\mathfrak{B}/(\mathfrak{B} + \mathcal{J}) \cong (\mathfrak{B} + \mathcal{J})/\mathcal{J}.
$$

The following result is classical.

**Theorem 10.2.4** Every $C^*$-algebra is semisimple.

We give the proof for a finite dimensional $C^*$-algebra. Consider a $C^*$-algebra $\mathfrak{A}$ of matrices contained in $\mathcal{M}_n$. Suppose that $I$ is a nilpotent ideal of $\mathfrak{A}$. Up to a change of basis, all matrices in $I$ are upper diagonal with zeroes on the diagonal. But since $I$ is self adjoint, this forces $I = \{0\}$. 
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10.2.2 Finite dimensional C*-algebras

As a direct sum of C*-algebras, any algebra of the form (10.1.4) is itself a C*-algebra. Actually, any C*-algebra which is finite dimensional (as a vector space) is of this form.

Theorem 10.2.5 Every finite dimensional C*-algebra is *-isomorphic to a direct sum of full matrix algebras

\[ A = M_{n_1} \oplus M_{n_2} \oplus \ldots \oplus M_{n_k}. \]

Proof. Since a C*-algebra is semisimple by Theorem 10.2.4, the result follows from Wedderburn Theorem (see Appendix D).

An element \( U \) of a C*-algebra is unitary if \( UU^* = U^*U = I \). Two elements \( A, B \) are unitarily equivalent if there is a unitary element \( U \) in \( B \) such that \( \varphi = Ad(U) \circ \psi \) where \( Ad(U)(B) = UBU^* \) for every \( B \in B \).

The following result shows that the theory of finite dimensional C*-algebras is similar to that of ordinary semisimple algebras.

Proposition 10.2.6 Suppose that \( \varphi \) is a unital *-morphism of finite dimensional C*-algebras from \( A = M_{m_1} \oplus \ldots \oplus M_{m_t} \) into \( B = M_{n_1} \oplus \ldots \oplus M_{n_s} \). Then \( \varphi \) is determined up to unitarily equivalence by its \( s \times t \)-matrix of partial multiplicities.

Proof. By the analysis made in Appendix D, there is an invertible matrix \( U \) such that \( Ad(U) \circ \varphi = \varphi_1 \oplus \ldots \oplus \varphi_t \) is of the form \( \varphi_i = id_{m_i}^{(a)} \oplus \ldots \oplus id_{m_i}^{(a)} \) for \( 1 \leq i \leq t \).

There remains to show that if \( \varphi : M \mapsto UMU^{-1} \) is a C*-algebra morphism from \( M \) onto itself, then \( U \) is unitary. Consider the elementary matrices \( E_{ij} \in M_n \) having all entries equal to 0 except the entry \( i, j \) which is 1. We have

\[ \varphi(E_{ij}) = \ell_i r_j \]  

(10.2.4)

where \( \ell_i \) is the column of index \( i \) of \( U \) and \( r_j \) is the row of index \( j \) of \( U^{-1} \). Such a decomposition with vectors \( r_i, \ell_i \) such that \( r_i \ell_i = 1 \) is unique. Since \( E_{ij}^* = E_{ji} \) and since \( \varphi(E_{ij})^* = E_{ij} \), we have \( r_i^* = \ell_j \) and \( \ell_i^* = r_i \). Thus \( U^{-1} = U^* \) and \( U \) is unitary.

Thus, any sequence \( A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \ldots \) of *-morphisms between finite dimensional C*-algebras is determined by a Bratteli diagram. This fact plays a fundamental role in the sequel.
10.2.3 Direct limits of $C^*$-algebras

Let $\mathfrak{A}$ be a $*$-algebra. A $C^*$-seminorm on $\mathfrak{A}$ is a seminorm $\rho$ on $\mathfrak{A}$ such that for all $A, B \in \mathfrak{A}$,

$$\rho(AB) \leq \rho(A)\rho(B), \quad \rho(A^*) = \rho(A), \quad \text{and} \quad \rho(A^*A) = \rho(A)^2.$$  

If additionally, $\rho$ is in fact a norm, it is called a $C^*$-norm.

If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a $*$-morphism, the map from $\mathfrak{A}$ into $\mathbb{R}_+$ defined by $\rho(A) = \|\varphi(A)\|$ is a $*$-seminorm on $\mathfrak{A}$.

If $\rho$ is a $C^*$-seminorm on a $*$-algebra $\mathfrak{A}$, the set $\mathcal{N} = \rho^{-1}(0)$ is self-adjoint ideal of $\mathfrak{A}$ and we get a $C^*$-norm on the quotient $*$-algebra $\mathfrak{A}/\mathcal{N}$ by setting $\|A + \mathcal{N}\| = \rho(A)$. If $\mathfrak{B}$ denotes the Banach space completion of $\mathfrak{A}/\mathcal{N}$ with this norm, it can be checked that the operations on $\mathfrak{A}/\mathcal{N}$ extend uniquely to $\mathfrak{B}$ to make a $C^*$-algebra, called the enveloping $C^*$-algebra of $\mathfrak{A}$ with respect to $\rho$.

Let now $(\mathfrak{A}_k)$ be a sequence of $C^*$-algebras with morphisms $\varphi_{k,k+1} : \mathfrak{A}_k \to \mathfrak{A}_{k+1}$. The set

$$\mathfrak{B} = \{(A_k)_{k \geq 0} | A_k \in \mathfrak{A}_k, A_{k+1} = \varphi_{k,k+1}(A_k) \text{ for every } k \text{ large enough}\}$$

is a $*$-subalgebra of the direct product $\prod_{k \geq 0} \mathfrak{A}_k$. Since the $\varphi_{k,k+1}$ are $C^*$-algebra morphisms, they are norm decreasing and we can define a semi-norm on $\mathfrak{B}$ by setting $\rho(A) = \lim\|A_n\|$ for $A = (A_n)_{n \geq 0}$. Note that $\rho(A) = 0$ for every $A = (A_n)_{n \geq 0}$ such that $A_n = 0$ for $n$ large enough.

The enveloping $C^*$-algebra $\mathfrak{B}$ of $\mathfrak{B}$ with respect to $\rho$ is called the direct limit of the sequence $(\mathfrak{A}_n)$, denoted $\lim\mathfrak{A}_k$.

As in the case of a direct limit of groups, there is a natural morphism $\varphi_k$ from each $\mathfrak{A}_k$ into $\mathfrak{A}$ which sends $A \in \mathfrak{A}_k$ to the class of any sequence $(A_\ell)_{\ell \geq 0}$ such that $A_\ell = A$ and $A_{\ell+1} = \varphi_{\ell,\ell+1}(A_\ell)$ for all $\ell \geq k$.

We will see examples of direct limits of $C^*$-algebras in the next section.

10.2.4 Positive elements

An element $A$ of a $C^*$-algebra $\mathfrak{A}$ is positive if $A = A^*$ (that is, $A$ is self-adjoint) and its spectrum $\sigma(A)$ is contained in $\mathbb{R}_+$.

A useful property of positive elements is the existence, for every positive element $A \in \mathfrak{A}$ of a unique positive square root denoted $A^{1/2}$.

It can be shown that if $A, B \in \mathfrak{A}$ are positive, then $A + B$ is positive. As a consequence, the positive elements determine an order on the set of self-adjoint elements by $A \leq B$ if $B - A$ is positive. Indeed, if $A \leq B$ and $B \leq C$, then $C - A = (C - B) + (B - A)$ which is positive and thus $A \leq C$.

**Theorem 10.2.7** For every $A \in \mathfrak{A}$, the element $A^*A$ is positive.

A positive linear functional on $C^*$-algebra is a linear functional $f : \mathfrak{A} \to \mathbb{C}$ such that $f(A) \geq 0$ whenever $A \geq 0$. A state is a positive linear functional of norm 1.
For example, let $\mathcal{H}$ be a Hilbert space and let $\pi$ is a $*$-morphism from $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$. Then
\[ f(A) = \langle \pi(A)x, x \rangle \]
is a positive linear functional. Indeed, if $A \geq 0$, using the square root $A^{1/2}$ of $A$, we have
\[ f(A) = \langle \pi(A^{1/2})^2 x, x \rangle = \|\pi(A^{1/2})x\|^2 \geq 0. \]
It is a state if $\mathfrak{A}$ is unital and $\|x\| = 1$.

### 10.3 Approximately finite algebras

A $C^*$-algebra is *approximately finite dimensional* (or an AF algebra) if it is the closure of an increasing union of finite dimensional subalgebras $(\mathfrak{A}_k)_{k \geq 0}$. When $\mathfrak{A}$ is unital, we further stipulate that $\mathfrak{A}_0$ consists of scalar multiples of the identity element $1$.

As an equivalent definition, an AF algebra is a direct limit of finite dimensional $C^*$-algebras. Indeed, if $\varphi_{k,k+1} : \mathfrak{A}_k \to \mathfrak{A}_{k+1}$ are morphisms, then $\mathfrak{A} = \lim_{\to} \mathfrak{A}_k$ is an AF algebra since it is the closure of the images $\varphi_n(\mathfrak{A}_k)$ of the finite dimensional $C^*$-algebras $\mathfrak{A}_k$ by the natural morphisms $\varphi_k$.

Let $\mathfrak{A} = \bigcup_{k \geq 1} \mathfrak{A}_k$ be an AF algebra with embeddings $\alpha_k$ from $\mathfrak{A}_k$ into $\mathfrak{A}_{k+1}$. We associate to the sequence $(\mathfrak{A}_k)$ the Bratteli diagram defined by the sequence of matrices of partial multiplicities of the morphisms $\alpha_k$.

An AF algebra is separable. Indeed, every finite dimensional $C^*$-algebra is separable. The following statement, which we will admit, gives an equivalent definition of AF algebras which does not depend on a sequence of subalgebras.

**Theorem 10.3.1** A $C^*$-algebra is AF if and only if it is separable and for every $n \geq 1$, every $A_1, \ldots, A_n$ in $\mathfrak{A}$ and every $\varepsilon > 0$ there is a finite dimensional $C^*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ and $B_1, \ldots, B_n$ such that $\|A_i, B_i\| \leq \varepsilon$ for $1 \leq i \leq n$.

As a simple example of a $C^*$-algebra which is not an AF algebra, the algebra $C([0, 1])$ is not an AF algebra (Exercise 10.5).

**Example 10.3.2** Consider again the sequence of algebras $\mathfrak{A}_k = M_{n_k} \oplus M_{n_k-1}$ with the morphisms $\pi_k : \mathfrak{A}_k \to \mathfrak{A}_{k+1}$ where $n_k$ is the Fibonacci sequence (Example 10.1.3). The direct limit of the sequence of $C^*$-algebras $\mathfrak{A}_k$ is the Fibonacci algebra.

**Example 10.3.3** Consider the sequence $\mathfrak{A}_k = M_k \oplus \mathbb{C}$ of algebras with the embeddings $\pi_k(x, \lambda) = (x \oplus \lambda, \lambda)$, that is
\[ \pi_k(A, \lambda) = \begin{bmatrix} A & 0 \\ 0 & \lambda \end{bmatrix}, \lambda \]
of Example 10.1.4. The corresponding AF algebra is the $C^*$-algebra $\mathfrak{A} + CI$ where $\mathfrak{A}$ is the $C^*$-algebra of compact operators on the Hilbert space $l^2(\mathbb{C})$ of
sequences \( x = (x_n) \) such that \( \| x \|_2 < \infty \). The algebra \( \mathfrak{A} \oplus CI \) is the direct limit of the algebras \( \bigcup_{k \geq 1} \mathfrak{A}_k \). Thus \( \mathfrak{A} \oplus CI \) is a unital AF algebra. The \( C^* \)-algebra \( \mathfrak{A} \) itself (which is not unital) is obtained as the closure of the sequence of subalgebras \( \mathcal{M}_k \) as in Example 10.1.6.

**Example 10.3.4** Consider \( \mathfrak{A}_k = \mathcal{M}_{2^k} \) with the embedding of \( \mathfrak{A}_k \) into \( \mathfrak{A}_{k+1} \) being

\[
\varphi_k(A) = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}
\]

The corresponding AF algebra is called the CAR algebra. The corresponding Bratteli diagram is represented in Figure 10.3.1. We recognize of course the

![Figure 10.3.1: A Bratteli diagram for the CAR algebra.](image)

BV-representation of the \((2^n)\) odometer. Let \( \mathfrak{B}_k = \mathcal{M}_{2^k} \oplus \mathcal{M}_{2^k} \). Since

\[
\varphi_{k+1}(\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}) = \begin{bmatrix} B_1 & 0 & 0 & B_2 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix}
\]

the morphism \( \varphi_{k+1} \) embeds \( \mathfrak{B}_k \) into \( \mathfrak{B}_{k+1} \). This shows that \( \mathfrak{A} \) is also the algebra defined by the diagram of Figure 10.3.1.

![Figure 10.3.2: Another Bratteli diagram for the CAR algebra.](image)

We now give two examples of AF-algebras with non stationary diagram.

**Example 10.3.5** Let \( X = \{0,1\}^\mathbb{N} \) be the one-sided full shift on \( \{0,1\} \). The space \( \mathfrak{A} = C(X) \) of continuous complex valued functions on \( X \) is a commutative \( C^* \)-algebra. Let \( \mathfrak{A}_k \) be the subalgebra of functions which are constant on each cylinder \([w]\) with \( w \) of length \( k \). Then \( \mathfrak{A} = \bigcup \mathfrak{A}_k \) and thus \( \mathfrak{A} \) is an AF algebra. A Bratteli diagram for \( \mathfrak{A} \) is represented in Figure 10.3.3.

![Figure 10.3.3: A Bratteli diagram for the GICAR algebra.](image)

**Example 10.3.6** Consider the Bratteli diagram represented in Figure 10.3.4. The corresponding AF algebra \( \mathfrak{A} \) is called the GICAR algebra. We have \( \mathfrak{A} = \lim \mathfrak{A}_k \) where \( \mathfrak{A}_k \) is

\[
\mathfrak{A}_k = \mathcal{M}_{(k)} \oplus \mathcal{M}_{(k)} \oplus \ldots \oplus \mathcal{M}_{(k)}
\]
Figure 10.3.3: A diagram for the algebra $C(\{0,1\}^\mathbb{N})$.

Figure 10.3.4: The Pascal triangle.

where 

$$\binom{k}{p} = \frac{k!}{(k-p)!p!}$$

is the binomial coefficient. The embedding of $\mathfrak{A}_k$ into $\mathfrak{A}_{k+1}$ is

$$\varphi_k(A_0, A_1, \ldots, A_k) = (A_0, A_0 + A_1, \ldots, A_{k-1} + A_k, A_k).$$

The following result, which we will admit, is of fundamental importance.

**Theorem 10.3.7 (Bratteli)** Let $\mathfrak{A} = \bigcup \mathfrak{A}_m = \bigcup \mathfrak{B}_n$ be an AF algebra obtained from two chains of finite dimensional $C^*$-algebras $\mathfrak{A}_m, \mathfrak{B}_n$. There exists subsequences $m_k, n_k$ and an automorphism $\alpha$ of $\mathfrak{A}$ such that

$$\mathfrak{A}_{m_k} \subset \alpha(\mathfrak{B}_{n_k}) \subset \mathfrak{A}_{m_{k+1}}$$
for all $k \geq 1$.

We deduce the following characterisation of Bratteli diagrams defining isomorphic AF algebras. It uses the notion of intertwining of diagrams introduced in Chapter 6 and playing an essential role in the Strong Orbit Equivalence Theorem (Theorem 6.5.1).

**Corollary 10.3.8** Two Bratteli diagrams define isomorphic AF algebras if and only if they have a common intertwining.

**Proof.** This is a direct consequence of Theorem 10.3.7. Indeed, The Bratteli diagram corresponding to the sequence $A_m \subset \alpha(B_n) \subset A_m \subset \ldots$ is an intertwining of the diagrams corresponding to the sequences $(A_m)$ and $(\alpha(B_n))$, the latter being the same as the diagram corresponding to the sequence $(B_n)$.

---

**10.3.1 Simple AF algebras**

As for ordinary algebras, a $C^\ast$-algebra is *simple* if it has no nontrivial ideals. As well known every full matrix algebra is simple. The following result explains why the term of simple Bratteli diagram was chosen.

**Theorem 10.3.9** The AF algebra defined by a Bratteli diagram is simple if and only if the diagram is simple.

We first prove the following lemma. The first statement holds for general $C^\ast$-algebras because the hypothesis that the algebras $A_n$ are finite dimensional is not used in the proof.

**Lemma 10.3.10** If $\mathcal{J}$ is an ideal of an AF algebra $\mathfrak{A} = \bigcup_{k \geq 1} A_k$, then

$$\mathcal{J} = \bigcup_{k \geq 1} (\mathcal{J} \cap A_k) = \mathcal{J} \cap \bigcup_{k \geq 1} A_k.$$

In particular $\mathcal{J}$ is an AF algebra.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{J} \cap A_n & \longrightarrow & A_n \\
\downarrow & & \downarrow \pi \\
\mathcal{J} & \longrightarrow & A_n / (\mathcal{J} \cap A_n) \\
\end{array}
\]

where the unlabeled arrows are the canonical injections and $\pi$ is the canonical quotient map from $\mathfrak{A}$ to $\mathfrak{A}/\mathcal{J}$. The map $\alpha$ is an isomorphism by Theorem 10.2.3.

If $J \in \mathcal{J}$, since $\mathfrak{A} = \bigcup_{k \geq 1} A_k$, there is for every $\varepsilon > 0$ an element $A$ of some
\[ \mathfrak{A}_k \text{ such that } \| J - A \| < \varepsilon. \] Then, by definition of the norm in the quotient, \[ \| A + J \| < \varepsilon. \] Since \( \alpha \) is an isomorphism, we have also \( \| A + (J \cap \mathfrak{A}_k) \| < \varepsilon \) and thus there is \( J' \in J \cap \mathfrak{A}_k \) such that \( \| A - J' \| < \varepsilon \). Since \( \| J - J' \| < 2\varepsilon \), we conclude that \( J = \bigcap_{k \geq 1} (J \cap \mathfrak{A}_k) \).

We can now give the proof of Theorem 10.3.9.

**Proof of Theorem 10.3.9.** Let \( (V, E) \) be a Bratteli diagram and let \( \mathfrak{A} \) be the corresponding AF algebra. We shall establish a one-to-one correspondence between the ideals of \( \mathfrak{A} \) and the sets \( W \) of vertices which are both directed and hereditary.

By Proposition 6.1.1 \((V, E)\) is simple if and only if there is no nontrivial directed and hereditary subset of \( V \). Thus the above correspondence will prove the theorem.

Let first \( J \) be an ideal of \( \mathfrak{A} \). For each \( p \geq 1 \), the set \( J_p = J \cap \mathfrak{A}_p \) is an ideal of \( \mathfrak{A}_p = M_{(p, 1)} \oplus \ldots \oplus M_{(p, k)} \). Since every summand \( M_{(p, i)} \) is simple, the ideal \( J_p \) corresponds to a set \( W_p \) of vertices \((p, i)\). Let \( W = \bigcup_{p \geq 1} W_p \).

Suppose that \( v = (p, i) \) is in \( W_p \) and that there is an edge \((p, i) \rightarrow (p + 1, j)\). Then, denoting \( \alpha_p \) the injection of \( \mathfrak{A}_p \) into \( \mathfrak{A}_{p+1} \), we have

\[
J \cap M_{(p+1, i)} \supset \alpha_p(M_{(p, i)}) \cap M_{(p+1, j)} \neq \emptyset. \tag{10.3.1}
\]

Hence \( J \) contains \( M_{(p+1, j)} \) and thus \((p + 1, j)\) is in \( W_{p+1} \). This shows that \( W \) is directed.

Next, let \((p, i)\) be a vertex and let \( J = \{ j \mid (p, i) \rightarrow (p + 1, j) \} \). Assume that \( W \) contains all \((p + 1, j)\) such that \( j \in J \). Then

\[
\alpha_p(M_{(p, i)}) \subseteq \sum_{j \in J} M_{(p+1, j)} \subseteq J \tag{10.3.2}
\]

which implies that \((p, i) \in W \). Thus \( W \) is also hereditary. This allows us to associate to every ideal of \( J \) a directed and hereditary set \( W = \alpha(J) \) of vertices of the diagram. By Lemma 10.3.10 the ideal \( J \) can be recovered from \( W \) and thus the map \( \alpha \) is injective.

Conversely, let \( W \) be a directed and hereditary subset of \( W \). Let \( J_p \) be the ideal of \( \mathfrak{A}_p \) corresponding to the vertices of \( W \) at level \( p \). Since \( W \) is directed, the sequence \( J_p \) is increasing. The closure of the union is an ideal \( J \) of \( \mathfrak{A} \). Since \( W \) is hereditary, we have \( J_p = \mathfrak{A}_p \cap J \). Indeed, if \( M_{(p, i)} \) is in \( J \), there is a \( q \geq p \) such that \( M_{(p, i)} \in \mathfrak{A}_p \). All descendants of \((p, i)\) at level \( q \) are then in \( W_q \) and thus \((p, i)\) is in \( W \) because \( W \) is hereditary. Thus \( W = \alpha(J) \) and this completes the proof.

**Example 10.3.11** The algebra \( CI + \mathfrak{R} \) of Example 10.3.3 is not simple since \( \mathcal{G}_k \) is a proper ideal. Accordingly, the Bratteli diagram of Figure 10.1.2 is not simple. The vertices of the lower level form a directed hereditary set of vertices. The restriction of the diagram to this set of vertices is the partial Bratteli diagram of Figure 10.1.3 which represents the \( C^*\)-algebra \( \mathfrak{R} \).
10.3. APPROXIMATELY FINITE ALGEBRAS

10.3.2 Dimension groups of AF algebras

We define the dimension group of an AF algebra as follows. Let $\mathfrak{A} = \bigcup_{n \geq 1} \mathfrak{A}_n$ be an AF algebra where the sequence $(\mathfrak{A}_n)_{n \geq 1}$ is defined by the sequence of $k_{n+1} \times k_n$-matrices $A_n$. The dimension group $K_0(\mathfrak{A})$ of $\mathfrak{A}$ is the direct limit of the sequence

$$
\mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \mathbb{Z}^{k_3} \xrightarrow{A_3} \ldots
$$

Thus, the dimension group of an AF algebra is the dimension group of its Bratteli diagram (although the diagram is not unique, the group is well defined, see below). Note that if $\mathfrak{A} = M_{n_1} \oplus \ldots \oplus M_{n_k}$ is a finite dimensional AF algebra, one obtains $K_0(\mathfrak{A}) = (\mathbb{Z}^{k_1}, \mathbb{Z}^{k_2}, \ldots, \mathbb{Z}^{k_k}, (n_1, \ldots, n_k))$. Thus one can also write

$$
K_0(\mathfrak{A}) = \lim_{\rightarrow} K_0(\mathfrak{A}_n)
$$

with the connecting morphisms given by the matrices $A_n$.

It follows from Corollary 10.3.8 that the definition of the dimension group is independent of the sequence $\mathfrak{A}_n$ such that $\mathfrak{A} = \bigcup \mathfrak{A}_n$. Indeed, by Theorem 6.1.5, the dimension groups of Bratteli diagrams equivalent by intertwining are isomorphic.

Example 10.3.12 Consider the Fibonacci algebra $\mathfrak{A}$ of Example 10.3.2. The dimension group $K_0(\mathfrak{A})$ is the group $\mathbb{Z}[\alpha]$ where $\alpha = (1 + \sqrt{5})/2$. Indeed, it is the direct limit $\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \cdots$ with $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and thus the assertion results from Example 3.3.6.

Example 10.3.13 The dimension group of the algebra $CI \oplus \mathbb{R}$ (Example 10.3.3) is the ordered group $\mathbb{Z}^2$ with positive cone $\{ (x, y) \mid y > 0 \} \cup \{ (x, 0) \mid x \geq 0 \}$ (see Example 3.3.7).

Example 10.3.14 The dimension group of the CAR algebra is $\mathbb{Z}[1/2]$ (see Example 3.3.11).

The next example introduces an ordered group not seen before.

Example 10.3.15 Consider the GICAR algebra (Example 10.3.6). We have $K_0(\mathfrak{A}_n) = \mathbb{Z}^{n+1}$ with morphisms

$$
\varphi_n(a_0, a_1, \ldots, a_n) = (a_0, a_0 + a_1, \ldots, a_n).
$$

Let us represent the group $K_0(\mathfrak{A}_n)$ as the set of polynomials with integer coefficients of degree at most $n$ though the map

$$(a_0, a_1, \ldots, a_n) \mapsto a_0 + a_1 x + \ldots + a_n x^n.$$

Since

$$(1 + x)(a_0 + a_1 x + \ldots + a_n x^n) = a_0 + (a_0 + a_1)x + \ldots + (a_{n-1} + a_n x^n + a_n x^{n+1},$$
the morphisms $\varphi_n$ are now replaced by the multiplication by $1 + x$. Thus, the dimension group of the GICAR algebra is the group
\[ G = \{(1 + x)^{-n}p(x) \mid p \in \mathbb{Z}[x] \text{ of degree at most } n, n \geq 0\}. \]
The positive cone $G_+$ corresponds to the polynomials $p$ such that $(1+x)^Np(x) > 0$ for some $N \geq 1$. One can show that a polynomial is of this form if and only if $p(x) > 0$ for $x \in [0, \infty]$ (Exercise 10.6).

A trace in a $\mathbb{C}^*$-algebra $\mathfrak{A}$ is a state $\tau$ such that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathfrak{A}$. When $\mathfrak{A} = M_n$, there is a unique trace on $\mathfrak{A}$ which is
\[ \tau(A) = \frac{1}{n} \text{Tr}(A) \]
where $\text{Tr}(A)$ is the usual trace of the matrix $A$. More generally, when $\mathfrak{A} = M_{n_1} \oplus \ldots \oplus M_{n_k}$ is a finite dimensional $\mathbb{C}^*$-algebra, the traces on $\mathfrak{A}$ are the maps $\tau$ such that
\[ \tau(A_1 \oplus \ldots \oplus A_k) = \frac{t_1}{n_1} \text{Tr}(A_1) + \ldots + \frac{t_k}{n_k} \text{Tr}(A_k) \quad (10.3.3) \]
where $t_1, \ldots, t_k \geq 0$ are such that $\sum_{j=1}^k t_k = 1$.

**Theorem 10.3.16** The traces on an AF algebra $\mathfrak{A}$ are in one-to-one correspondence with the states on $K_0(\mathfrak{A})$.

**Proof.** Let first $\tau$ be a trace on the AF algebra $\mathfrak{A} = \overline{\mathfrak{A}_n}$ where $\mathfrak{A}_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$. The restriction of $\tau$ to $\mathfrak{A}_n$ is a trace on $\mathfrak{A}_n$ and thus it is given by Equation (10.3.3). Set $t_j = \tau(I_j)$ where $I_j$ is the identity of the $M_{n_j}$. Let $\tau_n$ be the map on $K_0(\mathfrak{A}_n)$ defined by
\[ \tau_n(x_1, \ldots, x_{k_n}) = t_1 x_1 + \ldots + t_k x_k. \]
Then $\tau_n$ is a state on $(\mathbb{Z}^{k_n}, \mathbb{Z}^{k_n}_+)$. Conversely, for every state $\sigma$ on $K_0(\mathfrak{A})$, we define a state on $\mathfrak{A}_n = M_{n_1} \oplus \ldots \oplus M_{n_k}$ by (10.3.3) with $k_i = \sigma(e_i)$ and $e_i$ the $i$-th basis vector of $\mathbb{Z}^{k_n}$.

**10.3.3 Elliott Theorem**

The following classification result is due to Elliott.

**Theorem 10.3.17 (Elliott)** Two unital AF algebras $\mathfrak{A}$ and $\mathfrak{B}$ are $\ast$-isomorphic if and only if their dimension groups are isomorphic.
We first prove two lemmas which treat particular cases.

Let \( \varphi \) be a unital \( \ast \)-morphism from \( \mathfrak{A} = \mathcal{M}_{m_1} \oplus \ldots \oplus \mathcal{M}_{m_k} \) into \( \mathfrak{B} = \mathcal{M}_{m_1} \oplus \ldots \oplus \mathcal{M}_{m_r} \) as in Proposition \[10.2.6\]. We denote by \( \varphi_* \) the morphism from \( K_0(\mathfrak{A}) = (\mathbb{Z}^k, \mathbb{Z}_+^k, (n_1, \ldots, n_k)^t) \) to \( K_0(\mathfrak{B}) = (\mathbb{Z}^k, \mathbb{Z}_+^k, (m_1, \ldots, m_k)^t) \) which corresponds to the multiplication by the matrix of partial multiplicities \( A \) of \( \varphi \). Since \( A \) has nonnegative coefficients the morphism \( \varphi_* \) is positive and since \( A \) satisfies \[10.1.1\], it is unital. Thus \( \varphi_* \) is a unital morphism of ordered groups.

The next result shows how to recover morphisms of finite dimensional AF algebras from morphisms of their dimension groups.

**Lemma 10.3.18** Suppose that \( \mathfrak{A}, \mathfrak{B} \) are finite dimensional \( C^* \)-algebras and that \( \psi \) is a unital morphism of ordered groups from \( K_0(\mathfrak{A}) \) into \( K_0(\mathfrak{B}) \). Then there is a unital \( \ast \)-morphism \( \varphi : \mathfrak{A} \to \mathfrak{B} \) such that \( \varphi_* = \psi \) and \( \varphi \) is unique up to unitary equivalence.

**Proof.** Set \( \mathfrak{A} = \mathcal{M}_{m_1} \oplus \ldots \oplus \mathcal{M}_{m_k} \) and \( \mathfrak{B} = \mathcal{M}_{m_1} \oplus \ldots \oplus \mathcal{M}_{m_r} \). Then \( K_0(\mathfrak{A}) = (\mathbb{Z}^k, \mathbb{Z}_+^k, (n_1, \ldots, n_k)^t) \) and \( K_0(\mathfrak{B}) = (\mathbb{Z}^k, \mathbb{Z}_+^k, (m_1, \ldots, m_k)^t) \). Let \( A \) be the matrix of the morphism \( \psi \), which is such that \( \psi(v) = Av \) for every \( v \in \mathbb{Z}^k \).

Since \( \psi \) is positive, the coefficients of \( A \) are nonnegative. And since \( \psi \) is unital, \[10.1.1\] is satisfied. Thus the result follows from Proposition \[10.2.6\].

We now improve the last result by replacing \( \mathfrak{B} \) by an AF algebra. For this, we introduce a notation. Let \( \mathfrak{B} = \bigcup_{m \geq 1} \mathfrak{B}_m \) be an AF algebra with embeddings \( \beta_m : \mathfrak{B}_m \to \mathfrak{B} \). Then \( K_0(\mathfrak{B}) = \lim_{\rightarrow} K_0(\mathfrak{B}_m) \) with connecting morphisms \( \beta_{m,n} \). We denote by \( \beta_{m,*} \) the natural morphism from \( K_0(\mathfrak{B}_m) \) into \( K_0(\mathfrak{B}) \) which corresponds to the connecting morphisms \( \beta_{m,n,*} \). Thus \( \beta_{m,*}(x) \) is, for \( x \in \mathfrak{B}_m \), the class of sequences \( (x_k) = \prod_{k \geq 1} K_0(\mathfrak{B}_k) \) such that \( x_n = x \) and \( x_{m+1} = \beta_{m+1,m,*}(x_m) \) for all \( m \geq n \).

**Lemma 10.3.19** Let \( \mathfrak{A} \) be a finite dimensional \( C^* \)-algebra. Let \( \mathfrak{B} = \bigcup_{m \geq 1} \mathfrak{B}_m \) be an AF algebra with the embeddings \( \beta_m : \mathfrak{B}_m \to \mathfrak{B} \). Let \( \psi : K_0(\mathfrak{A}) \to K_0(\mathfrak{B}) \) be a unital morphism of ordered groups. Then there is an integer \( m \) and a \( \ast \)-morphism \( \varphi \) from \( \mathfrak{A} \) into \( \mathfrak{B}_m \) such that \( \beta_{m,*}\varphi_* = \psi \). Moreover, \( \varphi \) is unique up to unitary equivalence.

**Proof.** Set \( \mathfrak{A} = \mathcal{M}_{n_1} \oplus \ldots \oplus \mathcal{M}_{n_k} \). Then, as we have seen, \( K_0(\mathfrak{A}) \) is the unital ordered group \( (\mathbb{Z}^k, \mathbb{Z}_+^k, (n_1, \ldots, n_k)^t) \). Let \( e_j \in \mathbb{Z}^k \) be the \( j \)-th basis vector of \( \mathbb{Z}^k \).

For each \( j = 1, \ldots, k \), since \( K_0(\mathfrak{B}) \) is the direct limit of the ordered groups \( K_0(\mathfrak{B}_m) \) with natural morphisms \( \beta_{m,*} \), there is, by definition of the direct limit, an integer \( m \) such that \( \psi(e_j) \) is in \( \beta_{m,*}(K_0^*(\mathfrak{B}_m)) \). Taking the maximum of these integers, we may assume that this holds with the same \( m \) for all \( j = 1, \ldots, k \). Set \( \psi(e_j) = \beta_{m, *}(v_j) \) for some \( v_j \in K_0^*(\mathfrak{B}_m) \).

We define a morphism \( \rho : \mathbb{Z}^k \to K_0(\mathfrak{B}_m) \) by \( \rho(e_j) = v_j \) for \( j = 1, \ldots, k \). Then \( \psi = \beta_{m,*}\rho \) and we have the commutative diagram below.
The morphism $\rho$ is actually a morphism of unital ordered groups. Indeed, $\rho$ is positive since $\rho(e_j)$ belongs to $\mathcal{K}_0^+(\mathcal{B}_m)$, and $\rho$ is unital since

$$
\rho(n_1, \ldots, n_k)^t = \sum_{j=1}^k n_j \rho(e_j) = \sum_{j=1}^k n_j \psi(e_j)
$$

$$= \psi(n_1, \ldots, n_k)^t = 1 \mathcal{K}_0(\mathcal{B})
$$

We now apply Lemma 10.3.18 to obtain a unital $\ast$-morphism $\varphi : \mathfrak{A} \to \mathcal{B}_m$ such that $\varphi^* = \rho$.

Assume that $\varphi' : \mathfrak{A} \to \mathcal{B}_m'$ is another map with the same properties. Taking the maximum of $m, m'$, we may assume that $m = m'$. Again by definition of the direct limit, there is an integer $p \geq m$ such that $\beta_{p,m}(\varphi) = \beta_{p,m}(\varphi')$.

We replace $\varphi, \varphi'$ by $\beta_{p,m}(\varphi)$ and $\beta_{p,m}(\varphi')$. Since $\beta_{p,m}(\varphi) = \beta_{p,m}(\varphi')$, applying Lemma 10.3.18 we obtain that $\varphi, \varphi'$ are unitarily equivalent.

**Proof of Theorem 10.3.17.** We will prove that given a unital isomorphism $\rho : \mathcal{K}_0(\mathfrak{A}) \to \mathcal{K}_0(\mathcal{B})$, there is a $\ast$-isomorphism $\varphi : \mathfrak{A} \to \mathcal{B}$ such that $\varphi^* = \rho$.

Let $\mathfrak{A} = \lim A_m$ and $\mathcal{B} = \lim B_n$ with natural maps $\alpha_m$ and $\beta_n$ respectively. We will build a commutative diagram with increasing sequences $m_1 < m_2 < \ldots$ and $n_1 < n_2 < \ldots$ as represented below so that $\varphi^* = \rho$ and $\psi^* = \rho^{-1}$.

![Diagram](image)

Figure 10.3.5: The sequences $m_1, m_2 \ldots$ and $n_1, n_2, \ldots$.

We start with $m_1 = 1$. Apply Lemma 10.3.19 to the map $\rho \circ \alpha_{m_1^*} : \mathcal{K}_0(\mathfrak{A}_{m_1}) \to \mathcal{K}_0(\mathcal{B})$ to obtain an integer $n_1$ and a morphism $\varphi_1 : \mathfrak{A}_{m_1} \to \mathcal{B}_{n_1}$ such that $\beta_{n_1, \cdot} \varphi_1^* = \rho \alpha_{m_1^*}$ (see Figure 10.3.6 on the left).

Now apply Lemma 10.3.19 to the map $\rho^{-1} \circ \beta_{n_1^*} : \mathcal{K}_0(\mathcal{B}_{n_1}) \to \mathcal{K}_0(\mathfrak{A})$ to obtain a positive integer $m$ and a morphism $\psi$ of $\mathcal{B}_{n_1}$ into $\mathfrak{A}_m$ such that
10.4 Exercises

Section 10.2

10.1 Show that the norm in a $C^*$-algebra is unique.
10.2 Show that if \( \pi : A \to B \) is a *-morphism, then \( \|\pi(A)\| \leq \|A\| \) for all \( A \in \mathfrak{A} \).

Section 10.3

10.3 Show that if \( \mathfrak{A} = \bigcup_{n \geq 1} \mathfrak{A}_n \) and \( \mathfrak{B} = \bigcup_{n \geq 1} \mathfrak{B}_n \) have the same Bratteli diagram, they are isomorphic.

10.4 A \( C^* \)-algebra is called uniformly hyperfinite of UHF if it is the increasing union of full matrix algebras \( \mathcal{M}_{k_n} \) with \( k_1 | k_2 | \ldots \). The supernatural number associated to such \( C^* \)-algebra \( \mathfrak{A} \) is \( \delta(\mathfrak{A}) = \prod p^{n_p} \) where the product is over all prime numbers \( p \) and \( n_p \in \mathbb{N} \cup \infty \) is the supremum of the exponents of powers of \( p \) which divide \( k_n \). Show that two UHF algebras are isomomorphic if and only if \( \delta(\mathfrak{A}) = \delta(\mathfrak{B}) \).

10.5 Show that \( C([0,1]) \) is not an AF algebra.

10.6 Show that for every \((a,b) \in \mathbb{R}^2 \) with \( b \neq 0 \) there is an \( N \geq 1 \) such that \((1+x)^N(x^2-2ax+a^2+b^2)\) has positive coefficients. Conclude that a polynomial \( p \) is such that \((1+x)^N p(x)\) has positive coefficients for some \( N \geq 1 \) if and only if \( p(x) > 0 \) for every \( x \in ]0,\infty[ \).

10.5 Solutions

Section 10.2

10.1 If \( A \) is self-adjoint, then \( \|A\| = \text{spr}(A) \) by Proposition 10.2.1. In the general case, we have
\[
\|A\|^2 = \|A^*A\| = \text{spr}(A^*A)
\]
since \( A^*A \) is self-adjoint.

10.2 The spectrum of \( \pi(A) \) is contained in the spectrum of \( A \). Thus, if \( A \) is self-adjoint, \( \pi(A) \) is also self-adjoint and we have by Proposition 10.2.1
\[
\|\pi(A)\| = \text{spr}(A) \leq \text{spr}(A) = \|A\|.
\]
In the general case,
\[
\|\pi(A)\|^2 = \|\pi(A)^*\pi(A)\| = \|\pi(A^*A)\| \leq \|A^*A\| = \|A\|^2.
\]
Section 10.3

Denote $\alpha_n$ the embedding of $\mathfrak{A}_n$ into $\mathfrak{A}_{n+1}$ and by $\beta_n$ the embedding of $\mathfrak{B}_n$ into $\mathfrak{B}_{n+1}$. Each $\mathfrak{A}_n$ is isomorphic to $\mathfrak{B}_n$, as one can prove easily by induction on $n$. Let $\varphi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ be such isomorphism. Then $\varphi_{n+1} \circ \alpha_n$ and $\beta_n \circ \varphi_n$ are embeddings of $\mathfrak{A}_n$ into $\mathfrak{B}_{n+1}$ with the same partial multiplicities. By Proposition 10.2.6 there is a unitary element $U_{n+1}$ of $\mathfrak{B}_{n+1}$ such that

$$\beta_n \circ \varphi_n = \text{Ad}(U_{n+1})(\varphi_{n+1} \circ \alpha_n).$$

Define recursively $V_n \in \mathfrak{B}_n$ and $\psi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ by $\psi_1 = \varphi_1$ and $V_1 = I$ and

$$V_{n+1} = \beta_n(V_n)U_{n+1} \text{ and } \psi_{n+1} = \text{Ad}(V_{n+1})\varphi_{n+1}$$

for $n \geq 1$. We then have

$$\beta_n \circ \psi_n = \beta_n \circ \text{Ad}(V_n)\varphi_n = \text{Ad}(\beta_n(V_n))\beta_n \circ \varphi_n$$

$$= \text{Ad}(\beta_n(V_n))\text{Ad}(U_{n+1})(\varphi_{n+1} \circ \alpha_n)$$

$$= \text{Ad}(\beta_n(V_n)U_{n+1})\varphi_{n+1} \circ \alpha_n = \psi_{n+1} \circ \alpha_n.$$

where in the first line, we have used the identity $\beta \circ \text{Ad}(V)\varphi = \text{Ad}(\beta(V))\beta \circ \varphi$ for $*$-morphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\varphi : \mathfrak{B} \to \mathfrak{B}'$. We therefore have the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{A}_1 & \xrightarrow{\alpha_1} & \mathfrak{A}_2 & \xrightarrow{\alpha_2} & \mathfrak{A}_3 & \xrightarrow{\alpha_3} & \cdots \\
\downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_3} & & \\
\mathfrak{B}_1 & \xrightarrow{\beta_1} & \mathfrak{B}_2 & \xrightarrow{\beta_2} & \mathfrak{B}_3 & \xrightarrow{\beta_3} & \cdots
\end{array}$$

showing that there is a map $\psi : \mathfrak{A} \to \mathfrak{B}$ which extends the maps $\psi_n$ and is an $*$-isomorphism from $\bigcup_{n \geq 1} \mathfrak{A}_n$ to $\bigcup_{n \geq 1} \mathfrak{B}_n$. Since $\psi$ is an isometry, it extends to a $*$-isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.

10.4 This is a consequence of Elliott Theorem. In fact, two UHF algebras $\mathfrak{A}, \mathfrak{B}$ are such that $\delta(\mathfrak{A}) = \delta(\mathfrak{B})$ if and only if the odometers associated to the Bratteli diagrams of $\mathfrak{A}$ and $\mathfrak{B}$ have the same supernatural number and thus have isomorphic dimension groups (as seen in Exercise 7.9).

10.5 Assume that $C([0,1]) = \overline{\mathfrak{A}_n}$ with $\mathfrak{A}_n$ finite dimensional. Let $\mathfrak{B}$ be the subalgebra of polynomials. Then $\mathfrak{B} = \overline{\mathfrak{A}_n \cap \mathfrak{B}}$ with $\mathfrak{A}_n \cap \mathfrak{B}$ being a finite dimensional subalgebra of $\mathfrak{B}$. But the only finite dimensional subalgebra of $\mathfrak{B}$ is formed by the constant functions, a contradiction.
Set $\alpha = 2a$ and $\beta = a^2 + b^2$. We may assume that $\alpha > 0$. We have

\[
(1 + x)^N(x^2 - \alpha x + \beta) = (x^2 - \alpha x + \beta) \sum_{k=0}^{N} \binom{N}{k} x^k
\]

\[
= \sum_{k=0}^{N+2} \left( \binom{N}{k} - \alpha \binom{N}{k-1} + \binom{N}{k-2} \right) x^k
\]

\[
= \sum_{k=0}^{N+2} \frac{N!}{k!(N+2-k)!} a_{N,k} x^k
\]

where

\[
a_{N,k} = (1 + \alpha + \beta)(N + 2 - k)(N + 1 - k) - \alpha k(N + 2 - k) + k(k - 1)
\]

\[
= (1 + \alpha + \beta)k^2 - (2\beta + \alpha)Nk - (3\beta + 2\alpha + 1)k + \beta(N^2 + 3N + 2)
\]

\[
= (1 + \alpha + \beta)(k - \frac{\beta + \alpha/2}{1 + \alpha + \beta}N^2) + \beta(3N - 3k + 2) - (2\alpha + 1)k
\]

\[
+ (1 + \alpha + \beta)^{-1}N^2((1 + \alpha + \beta) - (\beta + \alpha/2)^2).
\]

Since $1 + \alpha + \beta > 0$ and $k \leq N$, we obtain

\[
a_{N,k} \geq (1 + \alpha + \beta)^{-1}N^2(\beta - \frac{\alpha^2}{4}) + 2\beta - (2\alpha + 1)N
\]

which is positive for $N$ large enough, given that $4\beta > \alpha^2$.

The condition is clearly necessary. Conversely, a polynomial satisfying the condition has positive leading coefficient and no positive real root. Thus it factors as

\[
p(x) = c \prod_i (x + \lambda_i) \prod_j (x^2 - 2a_jx + a_j^2 + b_j^2)
\]

with $\lambda_i \geq 0$ and $b_j > 0$. By what we have seen previously, there are integers $N_j$ such that $(1 + x)^{N_j}(x^2 - 2a_jx + a_j^2 + b_j^2)$ has positive coefficients. Thus $N = \sum_j N_j$ is a solution.

### 10.6 Notes

The reader is referred for a more detailed presentation to the numerous monographs on the subject, including Davidson (1996) which we follow here and also Pedersen (2018) which we occasionally follow.

For a proof of Proposition 10.2.6 see Davidson (1996, Corollary III.2.1).

The proof that for every ideal $J$ in a $C^*$-algebra $A$, the quotient algebra $A/J$ is a $C^*$-algebra (Theorem 10.2.2) is in Davidson (1996, Theorem I.5.4).

The definition of the dimension group of an AF algebra given here is not the usual one. The standard presentation involves a development of the K-theory of AF algebras. Indeed, $K_0$ is a functor which assigns an ordered abelian group to
each ring based on the structure of idempotents in the matrix algebra over the
ring. It occurs that for AF algebras, this group coincides with the dimension
group of a Bratteli diagram defining the algebra, a result due to [Elliott (1976)].

The CAR algebra (Example [10.3.4]) comes from quantum mechanics. It is
named for the *Canonical Anticommutation Relations algebra*. If \( V \) is a vector
space with a nonsingular symmetric bilinear form, the unital \(*\)-algebra generated
by the elements of \( V \) subject to the relations

\[
fg + gf = \langle f, g \rangle \\
f^* = f
\]

for every \( f, g \in V \) is the CAR algebra on \( V \). It can be shown that the CAR alge-
bra as defined in Example [10.3.4] is isomorphic the CAR algebra on a separable
Hilbert space (see [Davidson (1996)]).

The name of the GICAR algebra (Example [10.3.6]) stands for the *Gauge
Invariant CAR* and is also called the *current algebra* (see [Davidson (1996)]). On
positive polynomials (Exercise [10.6]), see [Handelman (1985)].

Elliott’s theorem (Theorem [10.3.17]) appeared in [Elliott (1976)]. The proof
follows that of ([Davidson (1996], Theorem IV.5.3)).
Appendix A

Topological, metric and normed spaces

We give in this appendix a list of the main notions and results of general topology and elementary analysis used in the book. This appendix (so as the following ones) is intended to serve as a memento for notions that some of the readers may have missed in part or forgotten long ago.

A.1 Topological spaces

A topological space is a set $X$ with a family of subsets, called open sets, containing $X$ and $\emptyset$ and closed by union and finite intersection. The complement of an open set is a closed set. The sets that are both open and closed are called clopen sets.

The topology for which all subsets are open is called the discrete topology. The closure of a subset $S$ of a topological space $X$ is the intersection of the closed sets containing $S$. It is also the smallest closed set containing $S$.

The interior of a set $S$ is the union of all open sets contained in $S$.

A neighborhood of a point $x$ is a set $U$ containing an open set $V$ which contains $x$. Thus $U$ is a neighborhood of $x$ if $x$ belongs to the interior of $U$.

Any subset $S$ of a topological space inherits the topology of $X$ by considering as open sets the family of $S \cap U$ for $U \subset X$ open. This is called the induced topology on $S$ and $U$ is called a subspace of $S$.

A topological space $X$ is a Hausdorff space if for every distinct $x, y \in X$ there are disjoint open sets $U, V$ such that $x \in U$ and $y \in V$.

An isolated point in a topological space is a point $x$ such that $\{x\}$ is open. A sequence $(x_n)$ of points in $X$ is convergent to a limit $x \in X$ if every open set $U$ containing $x$ contains all $x_n$ for $n$ large enough. When $X$ is Hausdorff, the limit is unique. We denote $\lim x_n$ the limit.

A family $\mathcal{F}$ of subsets of a topological space $X$ is a basis of the topology if every open set is a union of elements of $\mathcal{F}$. One also says that $\mathcal{F}$ generates
the topology of $X$. For $\mathcal{F}$ to be a basis of some topology, it is necessary and sufficient that it satisfies the two following conditions.

(i) Every point belongs to some $U \in \mathcal{F}$.

(ii) For every $U, V \in \mathcal{F}$, there is some $W \subset U \cap V$ such that $W \in \mathcal{F}$.

Any direct product $\prod_{i \in I} X_i$ of topological spaces $X_i$ is a topological space for the topology (called the product topology) with basis of open sets formed by the sets $\prod_{i \in I} U_i$ with $U_i \subset X_i$ open sets such that $U_i = X_i$ for all but a finite number of indices $i \in I$.

As a particular case, the set $Y^X$ of functions $f : X \to Y$ between topological spaces $X, Y$ is a subset of the product $\prod_{x \in X} Y_x$. The topology induced by the product topology on $Y^X$ is the topology of pointwise convergence. A sequence $(f_n)$ converges in this topology to a function $f$ if $\lim f_n(x) = f(x)$ for every $x \in X$. This notion is weaker than the notion of uniform convergence (see below).

A function $f : X \to Y$ between topological spaces $X, Y$ is continuous if $f^{-1}(U)$ is open for every open set $U \subset Y$. A continuous invertible function with a continuous inverse is called a homeomorphism.

A topological space is separable if it has a countable dense subset. For example, the real line is separable with the usual topology but not separable with the discrete topology.

### A.2 Metric spaces

A distance on a set $X$ is a map $d : X \to \mathbb{R}_+$ such that

(i) $d(x, y) = d(y, x)$, for every $x, y \in X$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in X$ (triangular inequality).

A set $X$ with a distance $d$ is called a metric space.

For $x \in X$ and $\varepsilon > 0$, the open ball centered at $x$ with radius $\varepsilon$ is

$$ B(x, \varepsilon) = \{ y \in X \mid d(x, y) < \varepsilon \}. $$

The topology with basis of open sets the open balls is the topology defined by the distance $d$. In this topology, a sequence $(x_n)$ converges to a limit $x$ if for every $\varepsilon$ there is an $N$ such that $d(x_n, x) \leq \varepsilon$ for all $n \geq N$.

A subset $S$ of a metric space $X$ is bounded if it is contained in some open ball or, equivalently if the set of distances $d(x, y)$ for $x, y \in S$ is bounded.

A topological space is metrizable if its topology can be defined by a distance. Not all topological spaces are metrizable, even if all topological spaces met in this book are metrizable spaces. It is however useful to use the terminology of general topological spaces, even when working with metric spaces.
A metric space is Hausdorff. Indeed, if \( x \neq y \), the balls \( B(x, \varepsilon) \) and \( B(y, \varepsilon) \) are disjoint if \( \varepsilon < d(x, y) / 2 \) by the triangular inequality.

In a metric space, many notions become easier to handle. For example, a set \( U \) in a metric space \( X \) is closed if and only if \( U \) contains the limits of all converging sequences of elements of \( X \).

A sequence of functions \( f_n : X \rightarrow Y \) between metric spaces \( X, Y \) is uniformly convergent to \( f : X \rightarrow Y \) if for every \( \varepsilon > 0 \) there is an \( N \geq 1 \) such that for all \( n \geq N \), one has
\[
d(f_n(x), f(x)) \leq \varepsilon
\]
for every \( x \in X \). Uniform convergence implies pointwise convergence but the converse is false.

The limit of a uniformly convergent sequence of continuous function is continuous.

A sequence \((x_n)\) in a metric space \((X, d)\) is a Cauchy sequence if for every \( \varepsilon > 0 \) there is an integer \( n \geq 1 \) such that for all \( i, j \geq n \) one has \( d(x_i, x_j) \leq \varepsilon \). Every convergent sequence is a Cauchy sequence but the converse is not true in general.

A metric space \((X, d)\) is complete if every Cauchy sequence converges. Every metric space \( X \) can be embedded in a complete metric space \( \hat{X} \), called its completion such that \( X \) is dense in \( \hat{X} \). The space \( \hat{X} \) is build as the set of classes of Cauchy sequences which are equivalent, in the sense that \( \lim d(x_n, y_n) = 0 \). The distance on \( \hat{X} \) is defined by continuity.

### A.3 Normed spaces

When \( X \) is a complex vector space, a norm on \( X \) is a map \( x \rightarrow ||x|| \) from \( X \) to \( \mathbb{R}_+ \) such that

1. \( ||x|| = 0 \) if and only if \( x = 0 \),
2. \( ||\alpha x|| = ||\alpha|| ||x|| \) for \( \alpha \in \mathbb{C} \) and \( x \in X \), and
3. \( ||x + y|| \leq ||x|| + ||y|| \) for every \( x, y \in X \).

The space \( X \) is called a normed space. If the separation property \( ||x|| = 0 \Rightarrow x = 0 \) is missing, one speaks of a seminorm. A norm defines a distance by \( d(x, y) = ||x - y|| \). Indeed, using (iii), we have
\[
d(x, y) + d(y, z) = |x - y| + |y - z| \leq |x - z| = d(x, z).
\]

The completion of a normed space \( X \) is again a normed space. The addition \( x + y \) is defined as follows. If \( (x_n) \) and \( (y_n) \) are Cauchy sequences in \( X \), then \( (x_n + y_n) \) is a Cauchy sequence and if \( (x'_n) \) is equivalent to \( (x_n) \), then \( (x'_n + y_n) \) is equivalent to \( (x_n + y_n) \). Thus the addition is well-defined on the classes.

A complex algebra \( \mathfrak{A} \) is a normed algebra if it has a norm \( || \cdot || \) such that \( ||AB|| \leq ||A|| ||B|| \) for all \( A, B \in \mathfrak{A} \). The completion of a normed algebra \( X \) is again a normed algebra. One defines the product of two Cauchy sequences \( (x_n) \) and \( (y_n) \)
and \((y_n)\) as the sequence \((x_n y_n)\). The result is, as for the sum, compatible with the equivalence.

If \(X, Y\) are normed spaces, a linear map \(f : X \to Y\) (also called a linear operator) is continuous if and only if there is a constant \(c \geq 0\) such that \(\|f(x)\|/\|x\| \leq c\). The least such \(c\) is called the norm of \(f\). This turns the space \(L(X, Y)\) of bounded linear maps from \(X\) to \(Y\) into a normed space.

An inner product on a vector space \(X\) over \(\mathbb{C}\) is a map \((x, y) \in X \times X \mapsto \langle x, y \rangle \in \mathbb{C}\) such that

(i) \(\langle x, y \rangle = \langle y, x \rangle\) (conjugate symmetry)

(ii) \(\langle x + y, z \rangle = \langle x, y \rangle + \langle y, z \rangle\) and \(\langle \alpha x, y \rangle = \alpha \langle x, y \rangle\) (sesquilinearity)

(iii) \(\langle x, x \rangle \geq 0\) and \(\langle x, x \rangle = 0\) only if \(x = 0\) (definite positivity)

for every \(x, y, z \in X\) and \(\alpha \in \mathbb{C}\). By (iii), we can define a nonnegative real number \(\|x\|\) by \(\|x\|^2 = \langle x, x \rangle\). By Cauchy-Schwarz inequality, we have

\[|\langle x, y \rangle| \leq \|x\|\|y\|\]

It follows from this inequality that \(\|x\|\) satisfies the triangular inequality and thus is a norm.

A Hilbert space is a complex vector space with an inner product which is complete for the topology induced by the norm.

The space \(\mathbb{C}^n\) is a Hilbert space for the inner product

\[\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i\]

The corresponding norm \(\|x\|_2\) is called the Hermitian norm This extends to the sequences \(x = (x_i)\) such that \(\|x\|_2 < \infty\) which form the Hilbert space \(\ell_2(\mathbb{C})\).

A Banach algebra is a complete normed algebra. The algebra \(M_n\) of \(n \times n\)-matrices with complex elements is a Banach algebra (for the norm induced by some norm on \(\mathbb{C}^n\)).

If \(\mathfrak{A}\) is a Banach algebra, the spectrum of \(A \in \mathfrak{A}\) is

\[\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda I - A \text{ not invertible}\}\]

It is a nonempty closed set. The spectral radius of \(A \in \mathfrak{A}\) is

\[\text{spr}(A) = \sup_{\lambda \in \sigma(A)} |\lambda|\]

The spectral radius satisfies \(\text{spr}(A) \leq \|A\|\) and it is given by the formula

\[\text{spr}(A) = \lim_{n \to \infty} \|A^n\|^{1/n}.\quad (A.3.1)\]

The set of bounded linear operators from a Hilbert space \(H\) into itself, denoted \(\mathfrak{B}(H)\), is a Banach algebra. The adjoint of \(T \in \mathfrak{B}(H)\) is the unique operator \(T^*\) such that

\[\langle Tx, y \rangle = \langle x, T^* y \rangle\]

for every \(x, y \in H\). The operator \(T \in \mathcal{G}(H)\) is said to be unitary if \(TT^* = T^*T = I\) that is, if \(T\) and \(T^*\) are mutually inverse.
A.4 Compact spaces

A Hausdorff space $X$ is compact if for every family $(U_i)_{i \in I}$ of open sets with union $X$ there is a finite subfamily with union $X$.

As a reformulation of the definition, say that a family $(U_i)_{i \in I}$ of subsets of $X$ is an cover of $X$ if $X = \cup_{i \in I} U_i$ and an open cover if additionally the $U_i$ are open. Then $X$ is compact if and only if from every open cover one may extract a finite cover.

Here a few important properties of compact spaces.

1. A subset of a compact space is compact (for the induced topology) if and only if it is closed.
2. A metric space is compact if and only if every sequence has a converging subsequence.
3. The image of a compact space by a continuous map is compact.

It follows easily from the second property that every compact metric space is complete.

A subset of a topological space is relatively compact if its closure is compact. Thus any subset of a compact space is relatively compact.

**Theorem A.4.1 (Tychonov)** Any product of compact spaces is compact.

This implies in particular that the set $A^\mathbb{Z}$ of sequences on a finite set $A$ is compact. This particular case can of course be proved directly using König’s Lemma: every sequence of infinite words on a finite alphabet has a converging subsequence.

We state below (in one of its equivalent forms) Baire Category Theorem.

**Theorem A.4.2 (Baire)** In compact metric space, a countable intersection of dense open sets is dense.

In particular, in a nonempty compact metric space, a countable intersection of dense open sets is nonempty. Taking the complements, the space is not the countable union of closed sets with empty interior.

A function $f : X \to Y$ between metric spaces $X,Y$ is said to be uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x,y \in X$, $d(x,y) < \varepsilon$ implies $d(f(x), f(y)) < \delta$. Any continuous function on a compact metric space is uniformly continuous.

The space $C(X, \mathbb{R})$ of continuous real valued functions on a compact metric space $X$ is a metric space for the norm $\|f\| = \sup_{x \in X} |f(x)|$. The corresponding topology is the topology of uniform convergence.

A linear map from a Banach space $X$ to a Banach space $Y$ is compact if the image of a bounded subset is relatively compact. A compact operator is continuous.
A.5 Connected spaces

A topological space is *disconnected* if it is a union of disjoint nonempty open sets. Otherwise, the space is *connected*.

The following are important properties of connected sets.

1. A continuous image of a connected set is connected.
2. Any union of connected sets is connected.

Every topological space decomposes as a union of disjoint connected subsets, called its *connected components*. The connected component of $x$ is the union of all connected sets containing $x$.

A topological space is *totally disconnected* if its connected components are singletons.

Any product of totally disconnected spaces is totally disconnected and every subspace of a totally disconnected space is totally disconnected.

A topological space is *zero-dimensional* if it admits a basis consisting of clopen sets. A compact space is zero-dimensional if and only if it is totally disconnected.

A.6 Notes

These notes follow mostly Willard (2004) but a similar content can be found in most textbooks on topology. Most authors (but all of them, see for example Rudin (1987)) assume as we do a compact space to be Hausdorff.
Appendix B

Measure and Integration

We give below a short review of some definitions and concepts concerning measure theory.

B.1 Measures

A family $\mathcal{F}$ of subsets of a set $X$ is a $\sigma$-algebra if it contains $X$ and is closed under complement and countable union.

Let $X$ be a topological space. The family of Borel subsets of $X$ is the closure of the family of open sets under countable union and complement. It is thus the $\sigma$-algebra generated by the family of open sets.

A measure on a $\sigma$-algebra $\mathcal{F}$ is a map $\mu : \mathcal{F} \to \mathbb{R}_+ \cup \{\infty\}$ which is countably additive, that is such that $\mu(\bigcup_n X_n) = \sum_n \mu(X_n)$ for any family $X_n$ of pairwise disjoint elements of $\mathcal{F}$. The measure $\mu$ is finite if $\mu(X) < \infty$.

The set $X$ is then called a measurable space and the elements of $\mathcal{F}$ are the measurable subsets of $X$. We also say that $(X, \mathcal{F}, \mu)$ is a measure space.

A property of points in a measurable space is said to hold almost everywhere (often abbreviated a.e.) if the set of points for which it is false has measure 0.

A probability measure on $X$ is a measure $\mu$ such that $\mu(X) = 1$.

Note the following important properties of measures.

1. A measure is monotone, that is, if $U \subset V$, then $\mu(U) \leq \mu(V)$.

2. A measure is subadditive, that is, $\mu(\bigcup_n U_n) \leq \sum_n \mu(U_n)$ for every family $X_n$ of measurable sets.

A function $f : X \to Y$ between measurable spaces $X, Y$ is measurable if $f^{-1}(U)$ is a measurable set for every measurable subset $U$ in $Y$.

A Borel measure on a topological space $X$ is a measure on the family of Borel sets of $X$.

As an example, in a Hausdorff space, for every $x \in X$, the Dirac measure $\delta_x$, defined by $\delta_x(U) = 1$ if $x \in U$ and 0 otherwise, is a Borel probability measure on $X$. 

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As another example, the usual Borel measure on \( \mathbb{R}^n \) is called the Lebesgue measure.

The Carathéodory Extension Theorem below shows that one may extend measures defined on a Boolean algebra to one on a \( \sigma \)-algebra. Recall that a family \( R \) of subsets of a set \( X \) is a Boolean algebra if for every \( U, V \in R \) one has \( U \cup V, U \cap V, X \setminus U \in R \).

**Theorem B.1.1 (Carathéodory)** Any probability measure on a Boolean algebra \( R \) of subsets of a topological space has a unique extension to a probability measure on the \( \sigma \)-algebra generated by \( R \).

Note that the hypothesis that \( \mu \) is a measure on \( R \) means that whenever the union of disjoint sets \( U_n \in R \) is in \( R \), then \( \mu(\bigcup U_n) = \sum \mu(U_n) \).

### B.2 Integration

Let \( \mu \) be a positive measure. A **simple function** is of the form

\[
s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}
\]

where the \( A_i \) are measurable sets and \( \alpha_i \in \mathbb{R} \). When \( s \) is a simple function, we define for a measurable set \( U \),

\[
\int_U s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap U).
\]

with the convention \( 0 \cdot \infty = 0 \).

If \( f : X \to [0, \infty] \) is measurable, the **integral** (or **Lebesgue integral**) of \( f \) over \( U \) is defined as

\[
\int_U f \, d\mu = \sup \int_U s \, d\mu
\]

the supremum being taken over the simple measurable functions \( s \) such that \( 0 \leq s \leq f \). For \( U = X \), we omit the subscript \( U \).

For an arbitrary measurable function \( f : X \to \mathbb{R} \), we define

\[
\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu
\]

where \( f^+ = \max\{f, 0\} \) and \( f^- = -\min\{f, 0\} \). A measurable function \( f : X \to \mathbb{R} \) is **integrable** if

\[
\int |f| \, d\mu < \infty.
\]

The set of integrable functions is denoted \( L^1(X, \mu) \).

The following result is known as the **Monotone Convergence Theorem**.

Theorem B.2.1 Let \((f_n)\) be a sequence of measurable functions on \(X\) such that \(0 \leq f_1(x) \leq f_2(x) \leq \cdots\) and \(f_n(x) \to f(x)\) for every \(x \in X\). Then \(f\) is measurable and
\[
\int f_n \, d\mu \to \int f \, d\mu
\]
as \(n \to \infty\).

It follows from Theorem B.2.1 that if \(f_n : X \to [0, \infty]\) is measurable for \(n \geq 1\), then
\[
\int \sum_{n \geq 1} f_n \, d\mu = \sum_{n \geq 1} \int f_n \, d\mu
\]

The next result is known as the Dominated Convergence Theorem.

Theorem B.2.2 (Lebesgue) Suppose \((f_n)\) is a sequence of measurable functions from \(X\) to \(\mathbb{R}\) such that \(f(x) = \lim_{n \to \infty} f_n(x)\) exists for every \(x \in X\). If there is a function \(g \in L^1(X)\) such that \(|f_n(x)| \leq g(x)\) for all \(x \in X\), then
\[
\lim_{n \to \infty} \int f_n \, d\mu = \int f \, d\mu.
\]

Let \((X, \mathcal{F}, \mu)\) be a measure space and let \(T : X \to X\) be a measurable map. The map \(\mu \circ T^{-1}\) is a measure on \(\mathcal{F}\). The following formula is called the change of variable formula. Its name relates to the usual change of variable used to compute integrals of real functions of one variable. We have, for every real-valued measurable function \(f\) on \(X\),
\[
\int (f \circ T) \, d\mu = \int f \, d(\mu \circ T^{-1}). \tag{B.2.1}
\]

Formula (B.2.1) is easy to verify in the case of a simple function \(s = \sum_{i=1}^n \alpha_i \chi_{A_i}\). Indeed, we have \(\int (s \circ T) \, d\mu = \sum_{i=1}^n \alpha_i \mu(T^{-1} A_i) = \int s \, d(\mu \circ T^{-1})\).

Every Borel measure \(\mu\) on a topological space \(X\) defines a bounded linear map from \(C(X, \mathbb{R})\) to \(\mathbb{R}\) by \(f \mapsto \int f \, d\mu\). By the Riesz representation Theorem the converse is true when \(X\) is a compact metric space. Thus the space of measures on a compact metric space \(X\) can be identified with the (topological) dual of the space \(C(X, \mathbb{R})\).

Let \(\mu, \nu\) be two Borel probability measures on \(X\). Then \(\nu\) is absolutely continuous with respect to \(\mu\), denoted \(\nu \ll \mu\), if for every Borel set \(U \subset X\) such that \(\mu(U) = 0\) one has \(\nu(U) = 0\).

Theorem B.2.3 (Radon-Nikodym) If \(\nu \ll \mu\), there is a nonnegative \(\mu\)-integrable function \(f\) such that \(\nu(U) = \int_U f \, d\mu\) for every measurable set \(U \subset X\).

A subset \(S\) of a real vector space is convex if for every \(x, y \in S\) and \(t \in [0, 1]\), one has \(tx + (1-t)y \in S\).

The set \(\mathcal{M}(X)\) of Borel probability measures on a compact metric space \(X\) is a convex subspace of the dual space of \(C(X, \mathbb{R})\). The weak-star topology on
\( \mathcal{M}(X) \) is the topology for which a sequence \((\mu_n)\) converges to \(\mu \in \mathcal{M}(X)\) if for all \(f \in C(X, \mathbb{R})\)
\[
\int f \, d\mu_n \to \int f \, d\mu.
\]

**Theorem B.2.4 (Banach-Alaoglu)** For any compact metric space \(X\), the space \(\mathcal{M}(X)\) is metrizable and compact for the weak-star topology.

An extreme point of a convex set \(K\) is a point which does not belong to any open line segment in \(K\). The closed convex hull of a set \(K\) is the closure of the intersection of all closed convex subsets containing \(K\).

**Theorem B.2.5 (Krein-Milman)** Every compact convex set in a normed space is the closed convex hull of its extreme points.

### B.3 Notes

We have mostly followed the classical [Rudin (1987)] for this summary of notions on measures and integration. We restricted the presentation to positive measures to simplify the picture. For the Carathéodory Extension Theorem (also known as Kolmogorov extension Theorem), see [Halmos (1974)]. On the weak-star topology (or weak*-topology) and the Banach-Alaoglu Theorem, see [Rudin (1991)]. The Krein-Milman Theorem is proved in [Rudin (1991), Theorem 3.23].
Appendix C

Algebraic Number Theory

In this appendix, we give an introduction to the basic concepts and results of algebraic number theory. We assume the reader to know the basic concepts of algebra, as the notion of ideal in a ring and of a principal ring.

C.1 Algebraic numbers

An algebraic number is a complex number $x$ solution of an equation

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0 \quad (C.1.1)$$

with coefficients in $\mathbb{Q}$. It is an algebraic integer if the coefficients are in $\mathbb{Z}$. We denote by $\mathbb{Q}[x]$ (resp. $\mathbb{Z}[x]$) the subring generated by $\mathbb{Q}$ (resp. $\mathbb{Z}$) and $x$.

**Theorem C.1.1** For $x \in \mathbb{C}$, the following conditions are equivalent.

(i) $x$ is an algebraic number (resp. and algebraic integer).

(ii) The $\mathbb{Q}$-vector space $\mathbb{Q}[x]$ (resp. the $\mathbb{Z}$-module $\mathbb{Z}[x]$) is finitely generated.

As a consequence, all elements of $\mathbb{Q}[x]$ (resp. $\mathbb{Z}[x]$) are algebraic numbers (resp. algebraic integers). Moreover, $\mathbb{Q}[x]$ is a field.

Let $K$ be a field containing $\mathbb{Q}$. It is called an algebraic extension of $\mathbb{Q}$, or also a number field, if all its elements are algebraic over $\mathbb{Q}$.

When $x$ is an algebraic number, the set of polynomials $p(X)$ such that $p(x) = 0$ is a nonzero ideal of the ring $\mathbb{Q}[X]$. Since $\mathbb{Q}[X]$ is a principal ring, this ideal is generated by a unique polynomial of the form $\text{(C.1.1)}$ (that is, with leading coefficient 1). This polynomial is called the minimal polynomial of $x$.

The degree over $\mathbb{Q}$ of an extension $K$, denoted $[K : \mathbb{Q}]$ is the dimension of the $\mathbb{Q}$-vector space $K$. By Theorem [C.1.1] every extension of finite degree is algebraic (the converse is not true).
C.2 Quadratic fields

A quadratic field is an extension of degree 2 of $\mathbb{Q}$. Every quadratic field is of the form $\mathbb{Q}[\sqrt{d}]$ where $d$ is an integer without square factor.

**Theorem C.2.1** Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field.
1. If $d \equiv 1, 3 \mod 4$ the ring of integers of $K$ is $\mathbb{Z} + \sqrt{d}\mathbb{Z}$.
2. If $d \equiv 1 \mod 4$, the ring of integers of $K$ is $\mathbb{Z} + \lambda\mathbb{Z}$ with $\lambda = (1 + \sqrt{d})/2$.

**Norm and trace** Let $K$ be a number field. For $x \in K$, the multiplication by $x$ in $K$ is a $\mathbb{Q}$-linear map $\rho(x)$. The **norm** of $x$, denoted $N(x)$, is the determinant of $\rho(x)$. Its **trace**, denoted $\text{Tr}(x)$, is the trace of $\rho(x)$.

If $x$ is an algebraic integer, then $N(x)$ and $\text{Tr}(x)$ are integers.

For example, if $K = \mathbb{Q}(\sqrt{d})$ and $x = a + b\sqrt{d}$, then $N(x) = a^2 - db^2$ and $\text{Tr}(x) = 2a$.

**Discriminant** Let $K$ be a number field of degree $n$ over $\mathbb{Q}$. For $x_1, x_2, ..., x_n \in K$, the **discriminant** of $(x_1, x_2, ..., x_n)$ is

$$D(x_1, x_2, ..., x_n) = \det(\text{Tr}(x_i x_j)).$$

One has $D(x_1, ..., x_n) = 0$ if and only if $x_1, x_2, ..., x_n$ is a basis of $K$.

If $y_1, y_2, ..., y_n$ is such that $y_i = \sum_{j=1}^{n} a_{ij} x_j$, then $D(y_1, y_2, ..., y_n) = \det(a_{ij})^2 D(x_1, ..., x_n)$. Thus the set of discriminants of the bases of the $\mathbb{Z}$-module of algebraic integers of $K$ is an ideal of $\mathbb{Z}$. The generator of this ideal is called the **discriminant** of $K$.

For example, the discriminant of $\mathbb{Q}(\sqrt{d})$ is $4d$ if $d \equiv 2, 3 \mod 4$ and $d$ if $d \equiv 1 \mod 4$.

C.3 Classes of ideals

Let $K$ be a number field and let $A$ be its ring of algebraic integers. Two ideals $I, J$ of $A$ are **equivalent** if there are nonzero $\alpha, \beta \in A$ such that $\alpha I = \beta J$.

The equivalence classes of ideals form a group with respect to the product, called the **class group** of $F$. The neutral element is the class of the ideal $I = A$, which can be shown to be formed of the principal ideals of $F$. Thus, when $A$ is a principal ring, the class group has only one element.

The ring of integers of a number field may fail to be principal and thus to have unique factorisation. For example, in $\mathbb{Q}[\sqrt{-5}]$, we have

$$(1 + \sqrt{5})(1 - \sqrt{5}) = 2 \cdot 3$$

although $1 + \sqrt{5}$ has no nontrivial divisor. However, one has the following result.

**Theorem C.3.1 (Dirichlet)** For every number field, the class group is finite.
C.4. CONTINUED FRACTIONS

Units A unit in a number field $K$ is an invertible element of the ring $A$ of integers of $K$. The set of units forms a multiplicative group, called the group of units of $A$.

An algebraic integer in a number field $K$ is a unit if and only if $N(x) = \pm 1$.

For example, the group of units in a real quadratic field $\mathbb{Q}[\sqrt{d}]$ with $d \geq 2$, is formed of the $a + b\sqrt{d}$ with $a, b \in \mathbb{Q}$ solution of

$$a^2 - db^2 = \pm 1,$$

which is known as Pell’s equation.

The following result is known as Dirichlet Unit Theorem.

Theorem C.3.2 (Dirichlet) The group of units of a number field $K$ is of the form $\mathbb{Z}^r \times G$ where $r \geq 0$ and $G$ is a finite cyclic group formed by the roots of unity contained in $K$.

The integer $r$ is $r = r_1 + r_2 - 1$ where $r_1$ is the number of real embeddings of $K$ in $\mathbb{C}$ and $2r_2$ the number of complex embeddings. One has $n = r_1 + 2r_2$ where $n$ is the degree of $K$ over $\mathbb{Q}$.

For example, in a real quadratic field $\mathbb{Q}[\sqrt{d}]$, the group of units is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Every unit is of the form $\pm u^n$ where $u$ is a fundamental unit.

For $d = 2$, the positive fundamental unit is $1 + \sqrt{2}$. For $d = 5$, it is $(1 + \sqrt{5})/2$.

As another example, if $K = \mathbb{Q}[i]$, one has $r = 0$. The group of roots of unity is of order 4 and is generated by $-i$. The corresponding ring of integers is called the ring of Gaussian integers.

C.4 Continued fractions

Every irrational real number $\alpha > 0$ has a unique continued fraction expansion of $\alpha$

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$$

where $a_0, a_1, \ldots$ are integers with $a_0 \geq 0$ and $a_n > 0$ for $n \geq 1$. We denote $\alpha = [a_0, a_1, \ldots]$. The integers $a_i$ are the coefficients of the continued fraction.

Theorem C.4.1 (Lagrange) A real number $\alpha > 0$ is quadratic if and only if its continued fraction expansion is eventually periodic.

For example, we have $[0, 1, 1, \ldots] = \frac{\sqrt{5} - 1}{2}$ and $[0, 2, 2, \ldots] = \sqrt{2} - 1$.

Notes For this brief introduction to algebraic number theory, we have followed Hardy and Wright (2008) and Samuel (1970) to which the reader is referred for a complete presentation.
Appendix D

Groups graphs and algebras

We recall in this appendix some of the basic notions of algebra concerning free groups, fundamental graphs and simple algebras. We begin with some general notions on groups.

D.1 Groups

Let $H$ be a subgroup of a group $G$. A right coset of $H$ in $G$ is a set of the form $Hg = \{hg \mid h \in H\}$ for some $g \in G$. Two cosets are equal or disjoint. The index of $H$ in $G$, denoted $[G : H]$, is the number of distinct cosets of $H$ in $G$.

The free abelian group on a set $A$, denoted $\mathbb{Z}(A)$, is the additive group formed of linear combinations $\sum_{a \in A} n_a a$ with $n_a \in \mathbb{Z}$. When $A$ is finite with $n$ elements, it is isomorphic with $\mathbb{Z}^n$.

The following statement is known as the Fundamental Theorem of abelian groups. A group is cyclic $A$ finite cyclic group is primary if its order is a power of a prime.

**Theorem D.1.1** Every finitely generated abelian group, is in a unique way a direct product of primary cyclic groups and infinite cyclic groups.

Thus every finitely generated abelian group $G$ can be written uniquely as

$$G = \mathbb{Z}^n \times \mathbb{Z}/q_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/q_m \mathbb{Z}$$

with $n, m \geq 0$ and all $q_i$ powers of primes.

D.2 Free groups

Recall that the free group on the alphabet $A$ is formed of the words on $A \cup A^{-1}$ which are reduced (that is, contain no $aa^{-1}$ or $a^{-1}a$ for $a \in A$). The product of two reduced words $u, v$ is the unique reduced word obtained by reduction of $uv$. 
We denote by $F(A)$ the free group on $A$. For $U \subseteq F(A)$, we denote by $\langle U \rangle$ the subgroup generated by $U$.

The following result is known as the Nielsen-Schreier Theorem,

**Theorem D.2.1 (Nielsen-Schreier)** Every subgroup of the free group is free.

Thus, every subgroup $H$ of the free group has a generating set $U$, called a basis of $H$, such that $H$ is isomorphic to the free group on $U$. Two basis of a subgroup have the same number of elements, called the rank of the subgroup.

Let $H$ be a subgroup of the free group $F(A)$. Let $Q$ be a set of reduced words on $A$ which is a prefix-closed set of representatives of the right cosets $Hg$ of $H$. Such a set is traditionally called a Schreier transversal for $H$. Let

$$U = \{pa^{-1}q \mid a \in A, p, q \in Q, pa \notin Q, pa \in Hq\}. \quad \text{(D.2.1)}$$

Each word $x$ of $U$ has a unique factorization $pa^{-1}q$ with $p, q \in Q$ and $a \in A$. The letter $a$ is called the central part of $x$. The set $U$ is a basis of $H$, called the Schreier basis relative to $Q$. It is clear that $U$ is closed by taking inverses. Indeed, if $x = pa^{-1}q \in U$, then $x^{-1} = qa^{-1}p^{-1}$. We cannot have $qa^{-1} \in Q$ since otherwise $p \in Hqa^{-1}$ implies $p = qa^{-1}$ by uniqueness of the coset representative and finally $pa \in Q$. It generates $H$ as a monoid because if $x = a_1a_2 \cdots a_m \in H$ with $a_i \in A$, then $x = (a_1p_1^{-1})(p_1a_2p_2^{-1}) \cdots (p_{m-1}a_m)$. We cannot have $1 \leq k \leq m - 1$ is a factorization of $x$ in elements of $X \cup \{1\}$. Finally, if a product $x_1x_2 \cdots x_m$ of elements of $U$ is equal to 1, then $x_kx_{k+1} = 1$ for some index $k$ since the central part $a$ never cancels in a product of two elements of $U$.

If $A$ has $k$ elements and $H$ is a subgroup of finite index $n$ in the free group on $A$, the rank $r$ of $H$ satisfies the equality

$$r - 1 = n(k - 1)$$

called Schreier’s Formula.

A group $G$ is called residually finite if for every element $g \neq 1$ of $G$, there is a morphism $\varphi$ from $G$ onto a finite group such that $\varphi(g) \neq 1$.

The following can be proved directly without much difficulty by associating to $g \neq 1$ a map $\varphi : F(A)$ into the symmetric group on $\{1, 2, \ldots, n\}$ such that $\varphi(g) \neq 1$.

**Theorem D.2.2** A finitely generated free group is residually finite.

A group $G$ is said to be Hopfian if any surjective morphism from $G$ onto $G$ is also injective.

**Theorem D.2.3 (Malcev)** Any finitely generated residually finite group is Hopfian.

In particular, by Theorem D.2.2, a finitely generated free group is Hopfian. This can be proved directly as follows. Let $\alpha : F(A) \to F(A)$ be a surjective morphism. Since $\alpha(A)$ generates $F(A)$, it cannot have less than $\text{Card}(A)$ elements. Thus $\text{Card}(\alpha(A)) = \text{Card}(A)$ and consequently $\alpha(A)$ is a basis of $F(A)$. This implies that $\alpha$ is an automorphism.
D.3 Group of a graph

Let \( \Gamma = (V, E) \) be a finite oriented graph. Each edge \( e \in E \) has an origin \( \alpha(e) \) and an end \( \omega(e) \) (also called its source and range). We allow multiple edges and thus we are considering oriented multigraphs.

Two edges \( e \) and \( f \) are consecutive whenever \( \omega(e) = \alpha(f) \). A path in the graph is, as usual, a sequence \( (e_1, \ldots, e_n) \) of consecutive edges. Its origin is \( \alpha(e_1) \), its end \( \omega(e_n) \) and the integer \( n \) is its length. By convention, there is, for each vertex \( v \) a path of length 0 of origin \( v \) and end \( v \). The path is a cycle if its origin and end are equal. If \( (e_1, \ldots, e_n) \) and \( (f_1, \ldots, f_m) \) are two paths such that \( \omega(e_n) = \alpha(f_1) \), then the concatenation of these paths is the path \( (e_1, \ldots, e_n, f_1, \ldots, f_m) \).

For each edge \( e \in E \), we consider an inverse edge \( e^{-1} \) from \( \omega(e) \) to \( \alpha(e) \). By convention \( (e^{-1})^{-1} \) is \( e \). A generalized path in \( G \) is a sequence \( (e_1, \ldots, e_n) \) of edges or their inverses such that, for all \( i \in [1, n - 1] \), the edges \( e_i \) and \( e_{i+1} \) are consecutive. The generalized path is reduced whenever \( e_i \) is different from \( e_{i+1}^{-1} \) for all \( i \). Two paths are equivalent whenever they can be obtain one from an other by a sequence of insertions or deletions of a sequence \( (e, e^{-1}) \). Every generalized path is equivalent to a unique reduced generalized path. It is a generalized cycle if its origin and end are equal.

Let \( v \in V \) be a vertex of the graph \( \Gamma = (V, E) \). The fundamental group \( G(\Gamma, v) \) is the group formed by the generalized cycles around \( v \). When \( G \) is connected, its isomorphism class does not depend on \( v \) and we denote \( G(\Gamma) \) the fundamental group of \( \Gamma \).

As well known, the fundamental group of a connected graph \( G \) is a free group and a basis can be obtained as follows. Let \( T \) be a spanning tree of \( \Gamma \) rooted at \( v \), that is, a set of edges such that for every \( w \in V \) there is a unique path \( p_w \) from \( v \) to \( w \) using the edges in \( T \). Then, the set

\[
\{ p_{\alpha(e)}e p_{\omega(e)}^{-1} \mid e \in E \setminus T \}
\]  

is a basis of \( G(\Gamma, v) \).

**Example D.3.1** Let \( \Gamma \) be the graph represented in Figure D.3.1 with \( V = \{1, 2\} \) and \( E = \{e, f, g, h\} \).

![Figure D.3.1: A connected graph.](image)

The set \( T = \{f\} \) is a spanning tree rooted at 1. The corresponding basis of \( G(\Gamma, 1) \) is \( \{e, fg, fhf^{-1}\} \).
D.4 Stallings graph

For a graph $\Gamma = (V,E)$ labeled by an alphabet $A$, the group defined by $\Gamma$ with respect to a vertex $v$ is the subgroup of the free group on $A$ formed by the labels of generalized paths from $v$ to itself. Thus, if all labels are distinct, the graph defined by $G$ is the fundamental graph of $G$.

A Stallings folding of a labeled graph is the following transformation. Suppose that two vertices $p,p'$ of a graph $G$ have edges (or inverse edges) going to $q$ with the same label $a$. Then we change $G$ to $G'$ by merging $p$ and $p'$. A Stallings folding does not change the subgroup defined by the graph. Indeed, any generalized path in $G'$ can be obtained from a generalized path in $G$ by insertion of paths of length 2 labeled $aa^{-1}$. A graph on which no Stallings folding can be performed is called Stallings reduced.

Given a finitely generated subgroup $H$ of the free group, there is a unique Stallings reduced graph which defines $H$. This graph is called the Stallings graph of the subgroup $H$.

Example D.4.1 Let $A = \{a,b\}$ and let $H$ be subgroup of $F(A)$ generated by $\{a, bab, bb\}$.

![Figure D.4.1: A Stallings folding.](image)

A graph defining $H$ is represented in Figure [D.4.1] on the left. The Stallings folding merging 2 and 3 gives the graph represented on the right, which is the Stallings graph of $H$.

D.5 Free products

Given two groups $G$ and $H$, the free product of $G$ and $H$, denoted $G \ast H$ is the set of all $g_1h_1 \cdots g_nh_n$ with $n \geq 1$, $g_i \in G$ and $h_i \in H$ for $1 \leq i \leq n$. Thus the free group on $A$ is the free product of the infinite cyclic groups generated by the $a \in A$.

The following result is known as the Kurosh Subgroup Theorem,

**Theorem D.5.1 (Kurosh)** Any subgroup of a free product $G_1 \ast G_2 \ast \cdots \ast G_n$ is itself a free product of a free group and of groups conjugate to subgroups of the $G_i$. 
D.6 Semisimple algebras

Let $A$ be an algebra over the field $\mathbb{C}$ of complex numbers. It is said to be simple if it has no proper nonzero two-sided ideal. As an equivalent definition, an algebra is semisimple if it does not contain nonzero nilpotent ideals. The algebra $M_n$ of $n \times n$-matrices with coefficients in $\mathbb{C}$ is simple. Any automorphism of $M_n$ is an inner automorphism, that is, of the form $M \mapsto AMA^{-1}$ for some invertible matrix $A \in M_n$.

A direct sum $\mathfrak{A}_1 \oplus \ldots \oplus \mathfrak{A}_k$ of simple algebras $\mathfrak{A}_i$ is called a semisimple algebra.

Theorem D.6.1 (Wedderburn) A finite dimensional algebra $\mathfrak{A}$ over $\mathbb{C}$ is semisimple if and only if $\mathfrak{A} = M_{n_1} \oplus \ldots \oplus M_{n_t}$.

Moreover, if $\mathfrak{A} = M_{n_1}^{(a_1)} \oplus \ldots \oplus M_{n_k}^{(a_k)}$ where the $n_i$ are distinct and each $M_{n_i}$ is repeated $a_i$ times, the $n_i$ and the $a_i$ are determined uniquely up to a permutation. Consequently any embedding of an algebra $\mathfrak{A}_1 = M_{n_1} \oplus \ldots \oplus M_{n_s}$ into an algebra $\mathfrak{A}_1 = M_{n_1} \oplus \ldots \oplus M_{n_s}$ is, up to conjugacy, of the type $\varphi = \varphi_1 \oplus \ldots \oplus \varphi_t$ with

$$\varphi_i = id_{n_1}^{(a_1)} \oplus \ldots \oplus id_{n_k}^{(a_k)}$$

where $id_n$ is the identity of $M_n$ and $id_n^{(a)} : M_n \to M_{an}$ is the morphism $x \mapsto (x, \ldots, x)$ ($a$ times). In other terms, each morphism $\varphi_i : \mathfrak{A}_1 \to M_{n_i}$ has the form

$$\varphi_i(M_1, \ldots, M_t) = \begin{bmatrix}
M_1 & & \\
& \ddots & \\
& & M_1
\end{bmatrix}
\begin{bmatrix}
a_{i1} & & \\
& \ddots & \\
& & a_{it}
\end{bmatrix}
\begin{bmatrix}
M_1 & & \\
& \ddots & \\
& & M_t
\end{bmatrix}
\begin{bmatrix}
a_{i1} & & \\
& \ddots & \\
& & a_{it}
\end{bmatrix}
\begin{bmatrix}
M_1 & & \\
& \ddots & \\
& & M_t
\end{bmatrix}$$

D.7 Notes

We have only briefly recalled the basic definitions and properties of free groups. For a systematic exposition (in particular of the Nielsen-Schreier Theorem), see (Lyndon and Schupp, 2001) or (Magnus et al., 2004).

See Lyndon and Schupp (2001) (p.197) for a proof of Malcev Theorem D.2.3.

The Stallings graphs are from Stallings (1983) (see also Kapovich and Myasnikov 2002).
The elementary properties of semisimple algebras (serving as a preparation of the properties of finite dimensional $C^*$-algebras in Chapter 10), in particular Wedderburn Theorem, can be found in Lang (2002).
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